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MULTIPLE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR EQUATIONS WITH CRITICAL POINTS*

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Abstract. We consider first two the second order autonomous differential equations with critical points, which allow for detecting an exact number of solutions to the Dirichlet boundary value problem. Then non-autonomous equations with similar behavior of solutions are considered. Estimations from below of the number of solutions to the Dirichlet boundary value problem are given.

Key words. Critical points, multiple solutions, heteroclinic solutions, homoclinic solutions

AMS subject classifications. 34B15, 34C23, 34C37

1. Introduction and examples.

In the work [2, Ch. 15] estimations of the number of solutions to the boundary value problem (BVP)

$$x' = h(t, x, y), \qquad y' = f(t, x, y) \tag{1.1}$$

$$\begin{aligned} a_1 x(a) - b_1 x'(a) &= 0, \\ a_2 x(b) - b_2 x'(b) &= 0 \end{aligned} \tag{1.2}$$

were obtained. These estimations were based on comparison of the behavior of solutions in some neighborhood of the zero solution and at infinity. Notice that the zero solution exists since $h(t, 0, 0) = f(t, 0, 0) = 0$. It is convenient to explain the result of A. Perov in terms of

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the angular function $\varphi(t)$, which can be introduced by the relations

$$x = \rho \sin\varphi, \quad y = \rho\cos\varphi, \quad \rho^2 = x^2 + y^2. \tag{1.3}$$

One gets the following equations for the functions φ and ρ :

$$\begin{cases} \varphi' &= \dfrac{1}{\rho}\cdot[h\,\cos\varphi - f\,\sin\varphi], \\ \rho' &= h\,\sin\varphi + f\,\cos\varphi. \end{cases} \tag{1.4}$$

Let φ_0 and φ_1 be the angles which relate respectively to the first and the second of the boundary conditions (1.2).

Set

$$\rho_0 = \sqrt{x^2(a) + y^2(a)}. \tag{1.5}$$

Suppose that a solution $\varphi(t)$ of the system (1.4), which is defined by the initial condition $\varphi(a) = \varphi_0$ for $\rho_0 \sim 0$, takes exactly m values of the form $\varphi_1 (\operatorname{mod}\pi)$. Moreover, assume that a solution $\varphi(t)$, which is defined by the initial condition $\varphi(a) = \varphi_0$ and which relates to values $\rho_0 \sim +\infty$, takes n values of the form $\varphi_1(\operatorname{mod}\pi)$. Then there exist at least $2|n-m|$ nontrivial solutions of the problem.

The Fig. 1.1 visualizes the case of $n = 0$ and $m = 1$. Two possible solutions of the BVP are represented by two semicircles.

Due to different rates of whirling of solutions near the zero and at infinity multiple solutions of the problem appear.

The above mentioned result by A. Perov is much more general than that described by Fig. 1.1, since equations in (1.1) are non-autonomous.

Our aim in this paper is the following. We consider the second order equations, which are equivalent to two-dimensional systems, which are similar to those treated by A. Perov and which, moreover, can have hetero- and homoclinic type solutions.

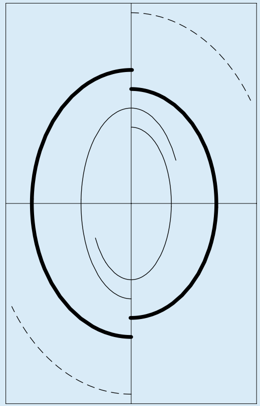

FIG. 1.1. *Perov's result ($n = 0$, $m = 1$), bold – orbits of solutions to BVP; normal – orbits at zero and dashed – orbits at infinity.*

Our plan is to consider first autonomous equations which have critical points of the type saddle – center – saddle. This equation has a heteroclinic solution and it may have multiple solutions of the Dirichlet problem.

The results are then generalized to the case of non-autonomous equation, which has a solution, defined on a finite interval and which possesses some properties of a heteroclinic solution.

Similar situation is considered for autonomous equations which have critical points of the type focus – saddle. This equation has a homoclinic solution and it may also have multiple solutions of the Dirichlet problem.

2. Autonomous equations, I. Consider the problem

$$x'' = -\alpha x + x^3, \tag{2.1}$$

$$x(0) = 0, \qquad x(1) = 0, \tag{2.2}$$

where the parameter α is positive.

The equivalent system

$$\begin{cases} x' = y, \\ y' = -\alpha x + x^3 \end{cases} \tag{2.3}$$

has a center at $(0;0)$ and two saddle points at $(-\sqrt{\alpha};0)$ and $(\sqrt{\alpha};0)$. The heteroclinic orbit connects two saddle points. The respective heteroclinic solution has "an infinite" period [5].

PROPOSITION 2.1. *Let the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{2.4}$$

hold, where n is a non-negative integer. Then the problem (2.1), (2.2) has exactly $2n$ non-trivial solutions.

Proof. Consider solutions $x(t;\gamma)$ of equation (2.1), which satisfy the initial conditions

$$x(0) = 0, \quad x'(0) = \gamma. \tag{2.5}$$

Equation (2.1), linearized at zero, takes the form

$$y'' = -\alpha y. \tag{2.6}$$

Solutions of the problem (2.1), (2.5) have exactly n zeros in the interval $(0;1)$ and do not vanish at $t = 1$ for small $\gamma > 0$. The parameter γ varies from zero value to γ_0, where γ_0

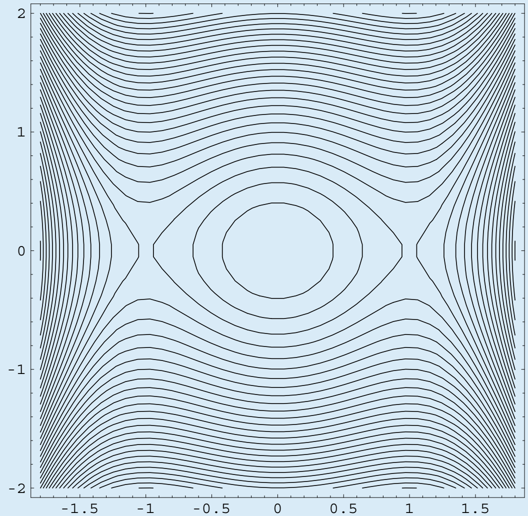

FIG. 2.1. *Phase portrait of solutions of equation* $x'' = -x + x^3$.

defines a heteroclinic orbit. The zeros of solutions $x(t;\gamma)$ monotonically increase and leave the interval $(0;1]$ passing through the right end point as $\gamma \to \gamma_0$. If for some γ a solution $x(t;\gamma)$ has a zero at $t = 1$, then $x(t;\gamma)$ is a solution to the boundary value problem (2.1), (2.2). Thus n solutions of the problem.

Similarly n solutions to the boundary value problem appear for $\gamma \in (0, -\gamma_0)$.

Solutions $x(t,\gamma)$ of the initial value problem (2.1), (2.5) do not vanish for $|\gamma| > |\gamma_0|$.

Hence we considered 4 segments on a line $x(0) = 0$ of the phase plane: two of them inside the heteroclinic orbit, (we discovered $2n$ solutions of the problem (2.1), (2.2) there) and another two segments outside the heteroclinic orbit, which do not contain a solution. Therefore the problem (2.1), (2.2) has exactly $2n$ nontrivial solutions. □

3. Non-autonomous equations, I. Consider the problem

$$x'' = f(t, x), \tag{3.1}$$

$$x(0) = x(1) = 0, \tag{3.2}$$

where function f satisfies the conditions:

(A1) f and f_x are $C(I \times \mathbb{R})$-functions;
(A2) $f(t, 0) \equiv 0$;
(A3) there exists a solution $\eta(t)$ of the problem (3.1), $\eta(0) = 0, \eta'(0) > 0$ such that $\eta(t)$ does not vanish in the interval $(0; 1]$;
(A4) there exists a solution $\xi(t)$ of the problem (3.1), $\xi(0) = 0, \xi'(0) < 0$ such that $\xi(t)$ does not vanish in the interval $(0; 1]$;
(A5) solutions of equation (3.1) extend to the interval $(0; 1]$.

THEOREM 3.1. *Let the conditions* **(A1)–(A5)** *hold. Assume also that a solution $y(t)$ of the Cauchy problem*

$$y'' = f_x(t, 0)y, \tag{3.3}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{3.4}$$

has exactly n zeros in the interval $(0, 1)$ and $y(1) \neq 0$.

Then the problem (3.1), (3.2) has at least $2n$ nontrivial solutions.

Proof. Consider a set S of solutions to the Cauchy problem (3.1), (2.5). Since all solutions are extendable **(A5)** this set is compact in $C^1(I)$ [2, Theorem 15.1].

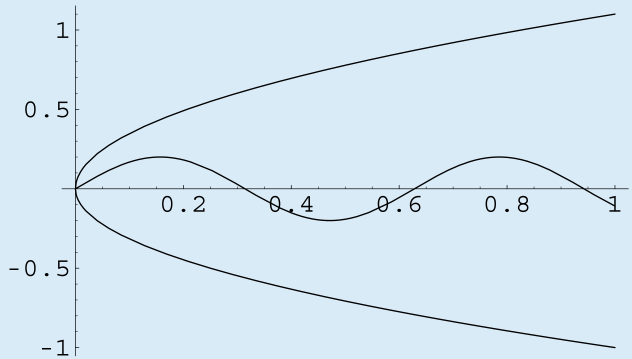

FIG. 3.1. *Visualization of* THEOREM 3.1.

Due to properties of the equation of variations (3.3) along the trivial solution of (3.1) solutions $x(t;\gamma)$ of the problem (3.1), (2.5) have exactly n zeros if $\gamma := x'(0)$ is small. Denote these zeros by $t_1(\gamma), \dots, t_n(\gamma)$. It follows from compactness of S and Valle-Poussin's Theorem [6, p. 122] that the distance between two consecutive zeros cannot be smaller than some number $\delta = \delta(S) > 0$ [4, Lemma 3; Latv. Mat. Ezheg.]. Since these zeros move continuously with respect to γ they have to leave the interval $(0,1]$ if γ varies from 0 to γ_η, γ_η standing for $\eta'(0)$. Hence at least n solutions of the boundary value problem (3.1), (3.2) for $\gamma > 0$.

Similarly the existence of at least n solutions of the problem (3.1), (3.2) can be obtained for $\xi'(0) < \gamma < 0$. Totally at least $2n$ nontrivial solutions of the problem. ∎

4. Autonomous equations, II. Consider the problem

$$x'' = -\alpha x + x^2, \tag{4.1}$$

(2.2), where the parameter α is positive.

The equivalent to equation (4.1) system

$$\begin{cases} x' &= y, \\ y' &= -\alpha x + x^2 \end{cases} \tag{4.2}$$

has a focus at $(0;0)$ and a saddle point at $(\alpha;0)$. The homoclinic orbit connects the saddle point to itself. It has "an infinite" period.

LEMMA 4.1. *Consider the Cauchy problem* (4.1), $x(0) = 0$, $x'(0) = -\gamma$, $\gamma > 0$. *Denote by* $t_1(\gamma)$ *the first zero of* $x(t, \gamma)$. *The function* $t_1(\gamma)$ *strictly decreases.*

Proof. Consider the phase portrait of solutions to the equation $x'' = -\alpha x + x^2$, where $\alpha > 0$. One has for phase orbits crossing the axis $x = 0$ that

$$x'^2 + \alpha x^2 = \frac{2}{3}x^3 + \gamma^2, \tag{4.3}$$

where $\gamma = x'(0)$. Consider a phase orbit passing through the point $(-m, 0)$. One has from (4.3) that γ and m relate as $\alpha m^2 = -\frac{2}{3}m^3 + \gamma^2$, or $\alpha m^2 + \frac{2}{3}m^3 = \gamma^2$.

Then

$$x'^2 + \alpha x^2 = \frac{2}{3}x^3 + \alpha m^2 + \frac{2}{3}m^3 \tag{4.4}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm\sqrt{\alpha m^2 + \frac{2}{3}m^3 - \alpha x^2 + \frac{2}{3}x^3}. \tag{4.5}$$

Let T_1 be time needed for a point to move from the point $(0, -\gamma)$ to the point $(-m, 0)$ along the phase orbit. Let also T_2 be time needed for a point to move from $(-m, 0)$ to $(0, \gamma)$. It can be shown easily by integration of (4.5) that $T_1 = T_2 =: T_m$.

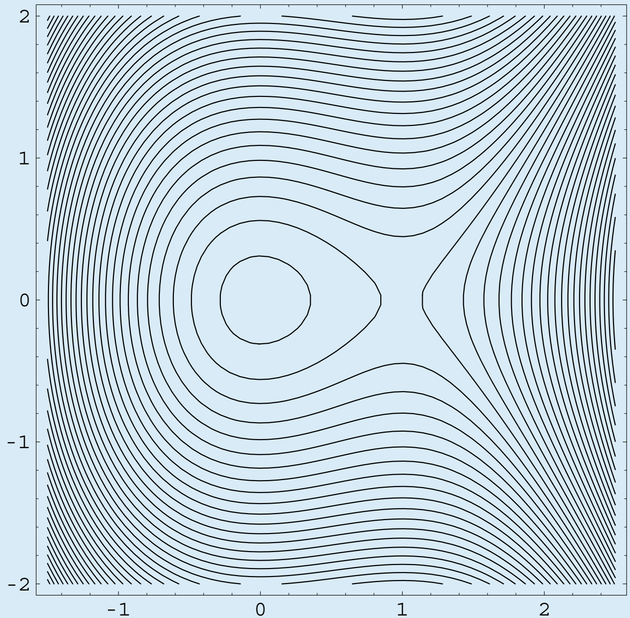

Fig. 4.1. *Phase portrait of solutions of equation* $x'' = -x + x^2$.

Let us compute the value of T_m.

One has that

$$
\begin{aligned}
T_m &= \int_{-m}^{0} \frac{\mathrm{d}x}{\sqrt{\alpha m^2 + \frac{2}{3}m^3 - \alpha x^2 + \frac{2}{3}x^3}} \\
(z=-x) &= \int_{0}^{m} \frac{\mathrm{d}x}{\sqrt{\alpha m^2 + \frac{2}{3}m^3 - \alpha z^2 - \frac{2}{3}z^3}} \\
(\xi = \frac{z}{m}) &= \int_{0}^{1} \frac{\mathrm{d}\xi}{\sqrt{\alpha + \frac{2}{3}m - \frac{2}{3}\xi^2 m\xi}} \\
&= \int_{0}^{1} \frac{\delta\,\mathrm{d}\xi}{\sqrt{\alpha(1-\xi^2) + \frac{2}{3}m(1-\xi^3)}}.
\end{aligned} \tag{4.6}
$$

Thus T_m monotonically decreases as a function of m. Since m increases together with γ the function $t_1(\gamma)$ is decreasing also. □

Thus the following statement is true.

PROPOSITION 4.2. *Suppose that the condition*

$$\pi^2 n^2 < \alpha < \pi^2 (n+1)^2 \tag{4.7}$$

holds, where n is a positive integer. Then the problem

$$x'' = -\alpha x + x^2, \quad , \tag{4.8}$$

$$x(0) = 0, \quad x(1) = 0 \tag{4.9}$$

has exactly $2n-1$ nontrivial solutions.

Proof. Consider solutions $x(t;\gamma)$ of the Cauchy problem (4.1), (2.5). Equation (4.1), linearized at zero, takes the form (2.6). In view of the condition (4.7) solutions of the problem

(4.1), (2.5) have exactly n zeros in the interval $(0;1)$ and does not vanish at $t=1$ for small $\gamma>0$.

Let γ be in the interval $(0,\gamma_H)$, where γ_H is $x'(0)>0$ of the homoclinic orbit. The zeros of solutions $x(t;\gamma)$ monotonically increase and leave the interval $(0;1]$ passing through the right end point as $\gamma\to\gamma_H$. If for some γ a solution $x(t;\gamma)$ has a zero at $t=1$, then $x(t;\gamma)$ is a solution to the boundary value problem (4.1), (2.2). Thus n solutions of the problem (4.8) with $x'(0)=\gamma>0$.

Consider solutions $x(t;\gamma)$ for $\gamma\in(-\gamma_H,0)$. If $\gamma\sim 0$ then $x(t;\gamma)$ has exactly n zeros in $(0,1)$ and $t=1$ is not a zero. If $\gamma\sim-\gamma_H$ then the respective solutions behave like the homoclinic one. They have exactly one zero $t_1(\gamma)\in(0,1)$. Other $n-1$ zeros leave the interval if $\gamma\to-\gamma_H$. Thus $n-1$ solutions of the problem (4.8) with $x'(0)<0$.

Therefore the problem (4.1), (2.2) has exactly $2n-1$ nontrivial solutions. □

5. Non-autonomous equations, II.

Consider the problem

$$x''=f(t,x), \tag{5.1}$$

$$x(0)=x(1)=0, \tag{5.2}$$

where function f satisfies the conditions:

(B1) f and f_x are $C(I\times R)$-functions;
(B2) $f(t,0)\equiv 0$;
(B3) $f(t,x)>c|x|^p$ for $t\in I$, $|x|>M$, where $c>0,\ p>1,\ M>0$ are constants;
(B4) solutions of equation (5.1) extend to the interval $(0;1]$.

Theorem 5.1. *Let the conditions* **(B1)** *to* **(B4)** *hold. Suppose that a solution $y(t)$ of the Cauchy problem*

$$y''=f_x(t,0)y, \tag{5.3}$$

$$y(0)=0,\quad y'(0)=1 \tag{5.4}$$

has exactly $n\geq 1$ zeros in the interval $(0,1)$ and $y(1)\neq 0$.

Then the problem (5.1), (5.2) *has at least* $2n-1$ *solutions.*

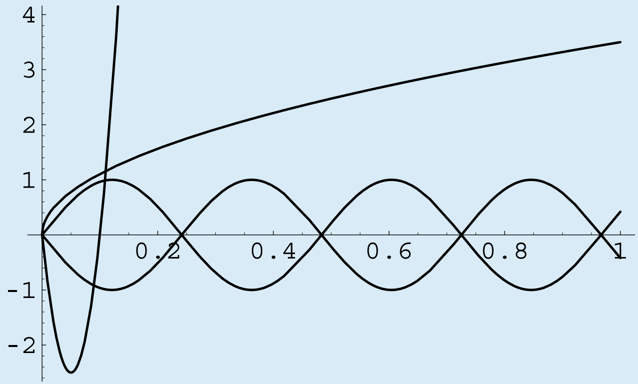

FIG. 5.1. *Visualization of Theorem 2.??*

LEMMA 5.2. *Suppose that conditions* **(B1)**, **(B3)** *and* **(B4)** *are satisfied.*

Then there exist solutions $\eta(t)$ *and* $\xi(t)$ *of equation* (5.1) *with the following properties:* $\eta(0)=0,\ \eta'(0)>0$ *and* $\eta(t)>0\ \forall t\in((0;1]$*;*

$\xi(0)=0,\ \xi'(0)<0$ *and there exists* $\tau\in(0,1)$ *such that* $\eta(t)<0\ \forall t\in(0,\tau)$ *and* $\eta(t)>0$ $\forall t\in(\tau,1]$.

Proof. Let us prove first that there exists $\eta(t)$. Choose $\eta'(0)>0$ so large that $\eta(t_*)=M$ and $\eta'(t_*)>0$ for some $t_*\in(0,1)$. Since $\eta''(t)=f(t,\eta(t))\geq c|\eta(t)|^p$, $\eta(t)$ increases also for $t\in(t_*,1)$. Existence of $\eta(t)$ as described in the conditions of lemma is proved.

Let us prove now that there exists a solution $\xi(t)$.

Consider the Cauchy problem

$$x'' = c|x|^p, \quad x(0) = -M, \quad x'(0) = -\beta, \quad \beta > 0 \tag{5.5}$$

and prove that for β large enough $x(t)$ has the first zero in the interval $(0, 1)$.

In order to avoid difficulties manipulating with negative numbers we consider a symmetric problem

$$y'' = -cy^p, \quad y(0) = M, \quad y'(0) = \beta, \quad \beta > 0, \tag{5.6}$$

looking for solutions with a point of maximum. Multiplying both sides of the equation in (5.6) by $2y'$ and integrating, one gets

$$y'^2(t) - y'^2(0) = -\frac{2c}{p+1}\left(y^{p+1}(t) - y^{p+1}(0)\right) \tag{5.7}$$

or

$$y'^2(t) - \beta^2 = -\frac{2c}{p+1}\left(y^{p+1}(t) - M^{p+1}\right). \tag{5.8}$$

At a point of maximum $T > 0$ the relations $y'^2(T) = 0$ and, therefore,

$$y_{\max} = y(T) = \left[\frac{p+1}{2c}\beta^2 + M^{p+1}\right]^{\frac{1}{p+1}} \tag{5.9}$$

hold. Then

$$y'^2(t) = -\frac{2c}{p+1}\left(y^{p+1}(t) - M^{p+1}\right) + \beta^2 \tag{5.10}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\beta^2 - \frac{2c}{p+1}\left[y^{p+1} - M^{p+1}\right]}. \tag{5.11}$$

Integrating the above expression one gets that

$$\int_0^{y_{max}} \frac{\mathrm{d}y}{\sqrt{\frac{2c}{p+1}(-M)^{p+1} + \beta^2 - \frac{2c}{p+1} y^{p+1}}} = \int_0^T \mathrm{d}t = T. \tag{5.12}$$

Transformation of the left side yields

$$\begin{aligned} &\int_0^{y_{max}} \frac{\mathrm{d}y}{\sqrt{\frac{2cM^{p+1}+(p+1)\beta^2}{p+1}} \cdot \sqrt{1 - \frac{2c}{2cM^{p+1} + (p+1)\beta^2} \cdot y^{p+1}}} \\ &= \sqrt{\frac{p+1}{2cM^{p+1} + (p+1)\beta^2}} \cdot \int_0^{y_{max}} \frac{\mathrm{d}y}{\sqrt{1 - \frac{2c}{2cM^{p+1}+(p+1)\beta^2} \cdot y^{p+1}}} \end{aligned} \tag{5.13}$$

$$= \left| \begin{array}{l} \frac{2c}{2cM^{p+1}+(p+1)\beta^2} \cdot y^{p+1} = \xi^{p+1} \\ \xi = \sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}}\, y \\ \mathrm{d}\xi = \sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}}\, dy \end{array} \right| \tag{5.14}$$

$$\begin{aligned} &= \frac{(p+1)^{\frac{1}{2}}}{(2cM^{p+1} + (p+1)\beta^2)^{\frac{1}{2}}} \cdot \int_0^1 \frac{\delta\,\mathrm{d}\xi}{\sqrt[p+1]{\frac{2c}{2cM^{p+1}+(p+1)\beta^2}} \cdot \sqrt{1-\xi^{p+1}}} \\ &= \frac{(p+1)^{\frac{1}{2}}}{(2cM^{p+1} + (p+1)\beta^2)^{\frac{1}{2}}} \cdot \frac{(2cM^{p+1} + (p+1)\beta^2)^{\frac{1}{p+1}}}{(2c)^{\frac{1}{p+1}}} \cdot \int_0^1 \frac{\delta\,\mathrm{d}\xi}{\sqrt{1-\xi^{p+1}}} \\ &= \frac{(p+1)^{\frac{1}{2}}}{(2c)^{\frac{1}{p+1}}} \cdot \frac{1}{[2cM^{p+1} + (p+1)\beta^2]^{\frac{p}{2(p+1)}}} \cdot \int_0^1 \frac{\delta\,\mathrm{d}\xi}{\sqrt{1-\xi^{p+1}}}. \end{aligned} \tag{5.15}$$

Since the integral in the last line is finite for $p > 1$, the expression in the last line tends to zero as β goes to infinity.

Thus the distance from the initial point $t = 0$ to the point of minimum T of a solution $x(t)$ of the problem (5.5) can be made arbitrarily small by choosing of appropriate $\beta = -x'(0)$. Since $x(\mathrm{const} - t)$ is also a solution of equation in (5.5), a solution $x(t)$ is symmetric with respect to $t = T$ and $x(2T) = -M$. Since equation in (5.5) is autonomous, a function $x(t + \mathrm{const})$ is a solution also.

Suppose that β is so large that a solution $x(t)$ of the problem (5.5) attains the minimal value x_{min} at the point $T(\beta) < \frac{1}{4}$. Then $x(2T) = -M$ and $x'(2T) = \beta$.

A solution $\xi(t)$ of equation (5.1) with the initial data $\xi(0) = 0$, $\xi'(0) = -\gamma$, $\gamma > 0$, can be shown to have a unique zero $\tau \in (0, 1)$ and to be positive for $t \in (\tau, 1]$ using comparison technique and the condition **(B3)**. ∎

Proof of THEOREM 5.1. Consider a set S of solutions to the Cauchy problem (5.1), (2.5), where $\gamma \in [\xi'(0), \eta'(0)]$ Since all solutions are extendable **(B4)** this set is compact in $C^1(I)$ [2, Theorem 15.1].

Due to properties of the equation of variations along the trivial solution of (5.1) solutions $x(t; \gamma)$ of the problem (5.1), (2.5) have exactly n zeros if $\gamma := x'(0)$ is small. Denote these zeros by $t_1(\gamma), \ldots, t_n(\gamma)$. Repeating the argument using in the proof of THEOREM 3.1 one concludes that there exist at least n solutions of the boundary value problem (5.1), (5.2) for $\gamma > 0$.

Making use of properties of a solution $\xi(t)$ one concludes that there exist at least $n - 1$ solutions of the problem (5.1), (5.2) with $\gamma \in (\xi'(0), 0)$. Totally at least $2n - 1$ nontrivial solutions of the problem. □

EXAMPLE. Consider the problem

$$x'' = -40(t^2 + 1)x + (t + 1)x^2, \quad x(0) = x(1) = 0. \tag{5.16}$$

FIGURE 5.2 shows solutions of the equation satisfying the initial conditions $x(0) = 0$, $x'(0) = \gamma$, where γ is small and positive. This solution has 2 zeros.

FIGURE 5.3 shows that zeros move to the right as γ increases.

FIGURE 5.4 shows that all zeros have escaped the interval.

Therefore there exist two solutions of the BVP if $\gamma > 0$ changes from zero to large enough values.

Similarly the existence of one solution can be shown for $\gamma < 0$ and changing from zero to $-\infty$.

Then the problem (5.2) has 3 nontrivial solutions.

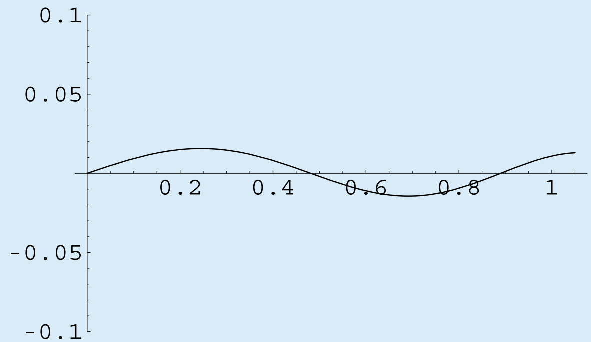

FIG. 5.2. $x'(0) = 0.1$.

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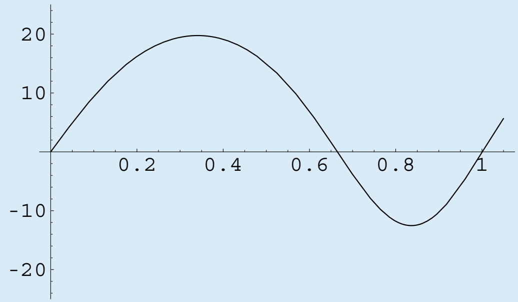

FIG. 5.3. $x'(0) = 100$.

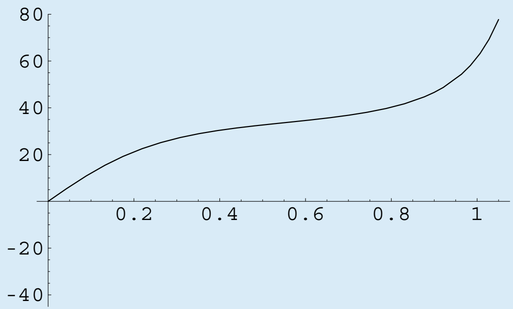

FIG. 5.4. $x'(0) = 128$.

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