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ASYMPTOTIC ANALYSIS OF A NONLINEAR PARTIAL DIFFERENTIAL EQUATION IN A SEMICYLINDER

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Abstract. Small solutions of a nonlinear partial differential equation in a semi-infinite cylinder will be studied. We consider the asymptotic behaviour of these solutions at infinity under Neumann boundary condition as well as Dirichlet boundary condition. In the Neumann case it can be shown that any solution small enough either vanishes at infinity or tends to a nonzero periodic solution of a nonlinear ordinary differential equation. In the Dirichlet case every solution small enough vanishes.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^{n-1}$ with C^2-boundary. We define the semi-infinite cylinder $\mathcal{C}_+ = \{x = (x', x_n) : x' \in \Omega, x_n > 0\}$. In Section 2 we consider bounded solutions of the equation

$$\Delta U + q(U)U = H \quad \text{ in } \mathcal{C}_+ \tag{1.1}$$

under the boundary condition

$$\frac{\partial U}{\partial \nu} = 0 \quad \text{ on } \partial\Omega \times (0, \infty). \tag{1.2}$$

The subject is to describe the asymptotic behaviour as $x_n \to \infty$ of solutions U of problem (1.1), (1.2) subject to

$$|U(x)| \leq \Lambda \quad \text{ for } x \in \mathcal{C}_+, \tag{1.3}$$

where Λ is a positive constant.

We assume that $q(u) > 0$ if $u \neq 0$. Moreover, q is continuous and

$$|s|, |t| \leq \Lambda \Rightarrow |q(s)s - q(t)t| \leq C_\Lambda |s - t| \tag{1.4}$$

with $C_\Lambda < \lambda_1$. Here, λ_1 is the first positive eigenvalue of the Neumann problem for the operator

$$-\Delta' = -\sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2}$$

in Ω.

We set $\mathcal{C}_t = \Omega \times (t, t+1)$ and define $L^r_{\text{loc}}(\mathcal{C}_+)$, $1 \leq r \leq \infty$, as the space of functions which belong to $L^r(\mathcal{C}_t)$ for every $t \geq 0$. We also suppose $H \in L^p_{\text{loc}}(\mathcal{C}_+)$ and

$$\int_0^\infty (1+s) \|H\|_{L^p(\mathcal{C}_s)}\, ds < \infty, \tag{1.5}$$

where

$$\begin{cases} p > n/2 & \text{if } n \geq 4 \\ p = 2 & \text{if } n = 2, 3. \end{cases} \tag{1.6}$$

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The main result of Section 2 is THEOREM 2.1, which states that one of two alternatives is valid:

1. U admits the asymptotic representation

$$U(x) = u_h(x_n) + w(x) \quad \text{ as } x_n \to +\infty,$$

where u_h is a nonzero periodic solution of

$$u_h'' + q(u_h)u_h = 0$$

and $w \to 0$ as $x_n \to \infty$. An estimate for the remainder term w is given.

2. $U \to 0$ as $x_n \to \infty$. An estimate for U is given in the theorem.

In Section 3 we consider solutions U of (1.1) subject to (1.3) under the Dirichlet boundary condition

$$U = 0 \qquad \text{on } \partial\Omega \times (0, \infty). \tag{1.7}$$

Now we suppose that q is continuous and that

$$|q(v)| \le C_\Lambda \quad \text{ if } |v| \le \Lambda,$$

where $C_\Lambda < \lambda_D$. Here, λ_D is the first eigenvalue of the Dirichlet problem for $-\Delta'$ in Ω. We assume also that $\|H\|_{L^p(\mathcal{C}_t)}$, with p as in (1.6), is a bounded function of t for $t \ge 0$. We get THEOREM 3.1 which gives an explicit bound for $\|U\|_{L^\infty(\mathcal{C}_t)}$ in terms of the function $\|H\|_{L^p(\mathcal{C}_t)}$. This implies in particular that $\|U\|_{L^\infty(\mathcal{C}_t)} \to 0$ as $t \to \infty$ if the same is valid for $\|H\|_{L^p(\mathcal{C}_t)}$. If $H = 0$ and $q(0) = 0$, then the estimate from THEOREM 3.1 implies that

$$|U(x', x_n)| \le C_\varepsilon \, \mathrm{e}^{-\sqrt{\lambda_D - \varepsilon}\, x_n}, \tag{1.8}$$

where $\varepsilon > 0$ is arbitrary.

In Section 4 we study all bounded solutions of (1.1), (1.7). Here we have some additional assumptions on Ω and q, among other things that Ω is star-shaped with respect to the origin. We present a theorem which states that, under these assumptions, every bounded solution of (1.1), (1.7) with $H = 0$ satisfies (1.8). Some examples of functions q satisfying the additional assumptions are given.

The work considered in this paper is a brief review of the first part of the author's doctoral thesis [12]. Complete proofs of the mentioned theorems can be found there.

The problem (1.1) under the boundary conditions (1.2) or (1.7) with $q(U) = |U|^{p-1}$, $p > 1$, has been studied in Kozlov [11]. There it is shown that the restriction (1.3) is essential for THEOREMS 2.1 and 3.1. One of the goals of this work is to extend some results from [11] to the equation (1.1).

The equation

$$\Delta u - a|u|^{q-1}u = 0 \quad \text{ in } \mathcal{C}_+, \tag{1.9}$$

where $q > 1, a > 0$ and with the boundary condition (1.2) is considered in Kondratiev [8]. Furthermore, the problem

$$\begin{cases} Lu = 0 & \text{in } \mathcal{C}_+ \\ \dfrac{\partial u}{\partial \nu} + a|u|^{q-1}u = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

where L is an elliptic partial differential operator, $a > 0$ and $q > 1$ are constants, is studied in Kondratiev [9]. In both these cases it is proved that the solutions of these problems have asymptotics of the form $u(x', x_n) = Cx_n^{-\sigma}$ with $\sigma > 0$. This shows that the minus sign in (1.9) essentially changes the asymptotic behaviour of solutions at infinity.

There is a lot of research on positive solutions of nonlinear problems in an infinite cylinder and other unbounded domains. We direct the reader to Bandle and Essén [2], Berestycki [3], Berestycki, Caffarelli and Nirenberg [4], Berestycki, Larrouturou and Roquejoffre [5], Berestycki and Nirenberg [6] and Kondratiev [10] where also further references can be found.

Small global solutions of the equation

$$\Delta u + \lambda u + f(u, u_x, u_y) = 0$$

in a two-dimensional strip with homogeneous Dirichlet boundary conditions are studied in Amick, Toland [1] and Kirchgässner, Scheurle [7].

2. The Neumann problem. Assume that p is subject to (1.6). We study the asymptotic behaviour as $x_n \to \infty$ of solutions $U \in W^{2,p}_{\mathrm{loc}}(\mathcal{C}_+)$ of the problem

$$\begin{cases} \Delta U + q(U)U = H & \text{in } \mathcal{C}_+ \\ \dfrac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \tag{2.1}$$

satisfying (1.3). Here Δ denotes the Laplace operator in $\mathbb{R}^n$ and ν denotes the outward unit normal to the curved part of $\partial\mathcal{C}_+$.

We assume that q is continuous and positive for $u \neq 0$ and satisfies (1.4). We suppose further that $H \in L^p_{\mathrm{loc}}(\mathcal{C}_+)$ is subject to (1.5). We have the following theorem concerning the asymptotic behaviour of solutions of (2.1) subject to (1.3):

Theorem 2.1. *Suppose that $U \in W^{2,p}_{\mathrm{loc}}(\mathcal{C}_+)$, where p satisfies* (1.6)*, is a solution of* (2.1) *subject to* (1.3)*. Suppose also that q is continuous, $q(u) > 0$ if $u \neq 0$ and that the Lipschitz condition* (1.4) *is fulfilled. Finally, assume that $H \in L^p_{\mathrm{loc}}(\mathcal{C}_+)$ satisfies* (1.5)*. Then one of the following alternatives is valid:*

1. *$U(x) = u_h(x_n) + w(x)$, where u_h is a nonzero periodic solution of*

$$u_h'' + q(u_h)u_h = 0$$

and

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C\biggl(\int_t^\infty s\|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + t\int_0^t \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\,\|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + t\,\mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}\,t}\biggr)$$

for $t \ge 1$. The right-hand side tends to 0 as $t \to \infty$.

2. *$\|U(\cdot, x_n)\|_{L^\infty(\Omega)} \to 0$ as $x_n \to \infty$. Furthermore, $U(x) = u_0(x_n) + w(x)$, where*

$$\frac{(u_0'(t))^2}{2} + \int_0^{u_0(t)} q(v)v\,dv \le C\biggl(\int_t^\infty \|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + \int_0^t \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\,\|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}\,t}\biggr)$$

and

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C\biggl(\int_0^\infty \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|}\,\|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}\,t}\biggr)$$

for $t \ge 1$.

Since the proof of this theorem is long, it is just outlined here. We begin by working in the cylinder

$$\mathcal{C} = \{(x', x_n) : x' \in \Omega \text{ and } x_n \in \mathbb{R}\}$$

and use then a smooth cut-off function in order to revert to the original problem in $\mathcal{C}_+$. So let us study a solution $u \in W^{2,p}_{\mathrm{loc}}(\mathcal{C})$ of the problem

$$\begin{cases} \Delta u + q(u)u = h & \text{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{C} \end{cases} \tag{2.2}$$

satisfying

$$\sup_{x \in \mathcal{C}} |u(x)| \leq \Lambda, \tag{2.3}$$

where Λ is the same constant as in (1.3) and p is subject to (1.6). We assume that $h \in L^p_{\mathrm{loc}}(\mathcal{C})$ and that

$$\int_{-\infty}^{\infty} (1 + |s|)\|h\|_{L^p(\mathcal{C}_s)} \,\mathrm{d}s < \infty.$$

We see immediately that 0 is an eigenvalue of the Neumann problem for $-\Delta'$ in Ω with the constant $\phi_0 = |\Omega|^{-1/2}$ as normalized eigenfunction in $L^2(\Omega)$. We set $\overline{u}$ to the orthogonal projection of u onto the subspace of $L^2(\Omega)$ spanned by ϕ_0, that is

$$\overline{u}(x_n) = \frac{1}{|\Omega|} \int_\Omega u(x', x_n) \,\mathrm{d}x',$$

and define $v(x)$ by the equality

$$u(x) = \overline{u}(x_n) + v(x).$$

After a few calculations we obtain the equation

$$\overline{u}''(x_n) + f(\overline{u}(x_n)) = \overline{h}(x_n) + f(\overline{u}(x_n)) - \overline{f(u)}(x_n), \tag{2.4}$$

where

$$f(u) = q(u)u,$$

$$\overline{h}(x_n) = \frac{1}{|\Omega|} \int_\Omega h(x', x_n) \,\mathrm{d}x'$$

and

$$\overline{f(u)}(x_n) = \frac{1}{|\Omega|} \int_\Omega f(u(x', x_n)) \,\mathrm{d}x'$$

together with the problem

$$\begin{cases} \Delta v = \overline{f(u)} - f(u) + h - \overline{h} & \text{in } \mathcal{C} \\ \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{C}. \end{cases} \tag{2.5}$$

We have thus splitted the problem (2.2) into the equation (2.4) and the problem (2.5). It is possible to prove the following two nontrivial lemmas.

Lemma 2.2. *Let ξ be a bounded solution of the equation*

$$\xi''(t) + q(\xi(t))\xi(t) = g(t),$$

where g is subject to

$$\int_1^\infty |sg(s)| \,\mathrm{d}s < \infty. \tag{2.6}$$

Then one of the two following alternatives occurs:

1. *$\xi(t) = \xi_h(t) + w(t)$, where $\xi_h(t)$ is a nonzero periodic solution of the equation*

$$\xi''(t) + q(\xi(t))\xi(t) = 0$$

and

$$|w(t)| + |w'(t)| = O\left(\int_t^\infty |sg(s)| \,\mathrm{d}s\right)$$

as $t \to \infty$.

2. *Both $\xi(t)$ and $\xi'(t)$ tend to 0 as $t \to \infty$ and*

$$\frac{(\xi'(t))^2}{2} + \int_0^{\xi(t)} q(v)v\,dv = O\left(\int_t^\infty |g(s)| \,\mathrm{d}s\right).$$

Lemma 2.3. *The function v in (2.5) satisfies the estimate*

$$\|v\|_{L^\infty(\mathcal{C}_t)} \le C \int_{-\infty}^\infty \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|} \, \|h\|_{L^p(\mathcal{C}_s)} \,\mathrm{d}s,$$

where C depends on p, n, Ω, Λ and C_Λ.

After checking that the right-hand side of (2.4) satisfies condition (2.6), we obtain the following result from Lemma 2.2 and Lemma 2.3.

Lemma 2.4. *Let $u \in W^{2,p}_{\mathrm{loc}}(\mathcal{C})$ be a solution of (2.2) subject to (2.3). Then either*

1. *$u(x) = u_h(x_n) + w(x)$, where u_h is a nonzero periodic solution of*

$$u_h'' + q(u_h)u_h = 0$$

and

$$\begin{aligned}\|w\|_{L^\infty(\mathcal{C}_t)} \le C \Bigg(&\int_t^\infty s\|h\|_{L^p(\mathcal{C}_s)} \,\mathrm{d}s \\ &+ t\int_{-\infty}^t \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)} \, \|h\|_{L^p(\mathcal{C}_s)} \,\mathrm{d}s\Bigg)\end{aligned} \tag{2.7}$$

for $t \ge 1$

or

2. $\|u(\cdot, x_n)\|_{L^\infty(\Omega)} \to 0$ *as* $x_n \to \infty$. *If* $u = \overline{u} + v$ *as before, then*

$$\begin{aligned}
&\frac{(\overline{u}'(t))^2}{2} + \int_0^{\overline{u}(t)} q(v)v\,dv \\
&\le C\left(\int_t^\infty \|h\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + \int_{-\infty}^t \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\,\|h\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s\right)
\end{aligned} \tag{2.8}$$

and

$$\|v\|_{L^\infty(\mathcal{C}_t)} \le C\int_{-\infty}^\infty \mathrm{e}^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|}\,\|h\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s. \tag{2.9}$$

The right-hand sides of (2.7), (2.8) *and* (2.9) *tend to* 0 *as* $t \to \infty$.

Finally, by setting $u(x) = \eta(x_n)U(x)$, where η is a smooth cut-off function equal to 0 for $x_n \le 1$ and 1 for $x_n \ge 2$, we get Theorem 2.1.

3. The Dirichlet problem. Here we study bounded solutions of the Dirichlet problem

$$\begin{cases} \Delta U + q(U)U = H & \text{in } \mathcal{C}_+ \\ U = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.1}$$

We assume that

$$\sup_{x\in\mathcal{C}_+} |U(x)| \le \Lambda \tag{3.2}$$

for some positive constant Λ and that q is continuous. Let λ_D be the first eigenvalue of the Dirichlet problem for $-\Delta'$ in Ω. We suppose that there exists a constant $C_\Lambda < \lambda_D$ such that

$$|v| \le \Lambda \Rightarrow |q(v)| \le C_\Lambda. \tag{3.3}$$

Finally, we assume that $H \in L^p_{\mathrm{loc}}(\mathcal{C}_+)$, where p satisfies (1.6), and that $\|H\|_{L^p(\mathcal{C}_t)}$ is a bounded function of t for $t \ge 0$.

We get the following theorem concerning the asymptotic behaviour of solutions U of (3.1).

Theorem 3.1. *Assume that* $U \in W^{2,p}_{\mathrm{loc}}(\mathcal{C}_+)$, *where* p *satisfies* (1.6), *is a solution of* (3.1) *subject to* (3.2). *Assume further that* q *is continuous and that* (3.3) *is satisfied. Also, assume that* $H \in L^p_{\mathrm{loc}}(\mathcal{C}_+)$ *and that* $\|H\|_{L^p(\mathcal{C}_t)}$ *is a bounded function of* t *for* $t \ge 0$. *Then*

$$\|U\|_{L^\infty(\mathcal{C}_t)} \le C\left(\int_0^\infty \mathrm{e}^{-\sqrt{\lambda_D - C_\Lambda}|t-s|}\,\|H\|_{L^p(\mathcal{C}_s)}\,\mathrm{d}s + \mathrm{e}^{-\sqrt{\lambda_D - C_\Lambda}\,t}\right), \tag{3.4}$$

where C *is independent of* t. *In particular, if* $H = 0$ *and* $q(0) = 0$, *then*

$$\|U\|_{L^\infty(\mathcal{C}_t)} = O\left(\mathrm{e}^{-\sqrt{\lambda_D - \varepsilon}\,t}\right), \tag{3.5}$$

where $\varepsilon > 0$ *is arbitrary.*

Remark. If $\|H\|_{L^p(\mathcal{C}_t)} \to 0$ as $t \to \infty$, then it follows that the right-hand side of (3.4) tends to 0 as $t \to \infty$.

As for the Neumann problem in Section 2, we study the problem corresponding to (3.1) in the infinite cylinder $\mathcal{C}$ when proving Theorem 3.1. However, a crucial difference is that 0 is not an eigenvalue of the Dirichlet problem for $-\Delta'$ in Ω. This means that we do not need to make the spectral splitting $u = \overline{u} + v$ as in Section 2 but can immediately apply a result similar to Lemma 2.3 in order to obtain Theorem 3.1.

The equality (3.5) is merely a consequence of (3.4). Namely, for T large enough, we consider (3.1) in $\Omega \times (T, \infty)$ instead of $\mathcal{C}_+$.

4. The case of a star-shaped cross-section. We present a theorem which states that under some special assumptions on Ω and q, every bounded solution of (3.1) with $H = 0$ will satisfy (3.5). This is a generalization of [11, Theorem 2(iii)] where the case $q(U) = |U|^{p-1}$ is studied.

Theorem 4.1. *Suppose that $n \geq 4$ and Ω is star-shaped with respect to the origin and has C^2-boundary. Also assume that q is continuous with $q(0) = 0$, and $q(u) > 0$ otherwise, and that*

$$\frac{n-3}{2} q(u)u^2 - (n-1)\int_0^u q(v)v\,dv \geq \varepsilon\, q(u)u^2 \tag{4.1}$$

for some $\varepsilon > 0$. Then every bounded solution of (3.1) *with $H = 0$ is subject to* (3.5).

We do not prove the theorem here but check instead that all functions of the form

$$q(u) = f(|u|)|u|^{a+\delta} \tag{4.2}$$

with $a = 4/(n-3)$, $\delta > 0$ and f being a nondecreasing function satisfy (4.1). Obviously, the function q in (4.2) is even. Therefore also both sides of the inequality (4.1) are even and we can assume that $u \geq 0$. We have

$$\int_0^u q(v)v\,dv \leq \frac{f(u)u^{a+2+\delta}}{a+2+\delta}$$

and by using this inequality we obtain

$$\frac{n-3}{2} q(u)u^2 - (n-1)\int_0^u q(v)v\,dv \geq \varepsilon f(u)u^{a+2+\delta},$$

where

$$\varepsilon = \frac{\delta(n-3)}{2(a+2+\delta)}.$$

Hence (4.1) is fulfilled.

Here are some examples of functions satisfying (4.2):

(i) $q(u) = |u|^p, \quad p > \frac{4}{n-3}$.

(ii) $q(u) = |u|^p e^{|u|}, \quad p > \frac{4}{n-3}$.

(iii) $q(u) = |u|^p (e^{|u|} - 1), \quad p > \frac{7-n}{n-3}$.

(iv) Linear combinations with positive coefficients of functions from (i)–(iii).

The important point of Theorem 4.1 is that the solution is just supposed to be bounded, without any dependence of λ_D.

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