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NEW ROSENBROCK METHODS OF ORDER 3 FOR PDAES OF INDEX 2

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Abstract. In this note new Rosenbrock-methods for index 2 PDAEs are presented. These solvers are of order 3, have 4 internal stages, and satisfy certain order conditions to improve the convergence properties if inexact Jacobians and approximations of $\frac{\partial f}{\partial t}$ are used. A comparison with other Rosenbrock solvers shows the advantages of the new methods.

Key words. Nonlinear parabolic equations, Rosenbrock-methods, order reduction

AMS subject classifications. 34 A 09, 65 L 80.

1. Introduction.

In the papers [11] and [4] the Navier–Stokes equations

$$\begin{aligned} \dot{u} - Re^{-1}\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } J\times\Omega,\\ \nabla\cdot u &= 0 && \text{in } J\times\Omega,\\ u &= g && \text{on } J\times\partial\Omega,\\ u(0,x) &= u_0 && x\in\Omega, \end{aligned}$$

(Re denotes the positive Reynolds number) is solved numerically by the help of Rosenbrock methods. It is well known that on the one hand the semi-discretized Navier–Stokes equations form a MOL-DAE of index 2 (see [1] or [15]) and that on the other hand Rosenbrock methods

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have to satisfy certain conditions for DAEs of index 2 and for PDEs (see [8] and [7]). The Rosenbrock methods considered in [11] and [4] satisfy only one of the two conditions. The method ROWDA2IND (see [8]) is a method for DAEs of index 2 and most of the other methods, for example ROS3P (see [5, 6]) ROS3Pw, ROS34PW2 (see [11]) or ROSDAP (see [13]), are schemes for solving PDAEs of index 1. Moreover, the numerical examples in the above-mentioned papers have shown that Rosenbrock W-methods yield very good results. So the motivation for this paper was to create some new Rosenbrock methods for PDAEs of index 2. The new methods are of order 3 and have 4 internal stages.

The numerical comparisons presented at the end of the paper illustrate the good qualities of the methods in both academic and more practical problems.

2. Rosenbrock methods. An s-stage *Rosenbrock-method* for the implicit ODE

$$M\dot{u} = f(t,u), \qquad u(t_0) = u_0 \tag{2.1}$$

is given by

$$\begin{aligned} Mk_i &:= \tau\, f\left(t_{old} + \alpha_i\, \tau, u_{old} + \sum_{j=1}^{i-1} \alpha_{ij}\, k_j\right) \\ &\quad + \tau\, W \sum_{j=1}^{i} \gamma_{ij}\, k_j + \tau^2 \gamma_i T, \qquad i = 1, \dots, s \qquad (2.2) \\ u_{new} &:= u_{old} + \sum_{i=1}^{s} b_i\, k_i \end{aligned}$$

where s is the number of internal stages, τ is the step length, α_{ij}, γ_{ij}, b_i are the parameters of the method, $W := f'(t_{old}, u_{old})$, $T := \dot{f}(t_{old}, u_{old})$, $\alpha_i := \sum_{j=1}^{i-1} \alpha_{ij}$, and $\gamma_i := \sum_{j=1}^{i-1} \gamma_{ij}$. The values k_i are unknown. By "$\dot{}$" and "$'$" we denote differentiation with respect to the time t and the phase space variable, respectively.

The parameters α_{ij}, γ_{ij}, and b_i should be chosen in such a way that some order conditions are fulfilled to obtain a sufficient consistency order. A derivation of these conditions with Butcher series can be found in [2]. Here we only summarize the conditions up to order 3:

$$\left\{\begin{array}{rrcl} \text{(A1)} & \sum b_i & = & 1 \\ \text{(A2)} & \sum b_i\beta_i & = & \frac{1}{2}-\gamma \\ \text{(A3a)} & \sum b_i\alpha_i^2 & = & \frac{1}{3} \\ \text{(A3b)} & \sum b_i\beta_{ij}\beta_j & = & \frac{1}{6}-\gamma+\gamma^2 \end{array}\right. , \qquad (2.3)$$

where we use the abbreviations $\beta_{ij} := \alpha_{ij} + \gamma_{ij}$ and $\beta_i := \sum_{j=1}^{i-1} \beta_{ij}$. We get an additional consistency condition if we set $W := f'(t_{old}, u_{old}) + \mathcal{O}(h)$ (see [14]):

$$\text{(B2)} \quad \sum b_i\alpha_i \; = \; \frac{1}{2} \; . \qquad (2.4)$$

For arbitrary matrices W, we get the following order conditions (see [14]):

$$\left\{\begin{array}{rrcl} \text{(C3a)} & \sum b_i\alpha_{ij}\alpha_j & = & \frac{1}{6} \\ \text{(C3b)} & \sum b_i\alpha_{ij}\beta_j & = & \frac{1}{6}-\frac{\gamma}{2} \\ \text{(C3c)} & \sum b_i\beta_{ij}\alpha_j & = & \frac{1}{6}-\frac{\gamma}{2} \end{array}\right. . \qquad (2.5)$$

If a Rosenbrock-method is applied to semidiscretized PDAEs and PDEs, resp., the following condition should be satisfied to avoid order reduction (see [7]):

$$b^\top B^j(2B^2e - \alpha^2) = 0, \qquad 1 \le j \le 2 \qquad (2.6)$$

with $B := (\beta_{ij})_{i,j=1}^s$, $\alpha^2 := (\alpha_1^2, \ldots, \alpha_s^2)^\top$, and $e := (1, \ldots, 1)^\top \in \mathbb{R}^s$. To obtain convergence, the Rosenbrock-method should fulfill certain order conditions for both the ODE and the algebraic part. These consistency properties can be derived again via Butcher series technique (see [2] and [12]). For a third-order method we get the condition

$$\text{(E3)} \quad \sum b_i \omega_{ij} \alpha_j^2 = 1, \tag{2.7}$$

where $(\omega_{ij})_{i,j=1}^s = B^{-1}$.

From [8] we know that a Rosenbrock method should satisfy certain order conditions if the method is applied on an index-2 DAE, i.e.

$$\begin{aligned} &\text{(F3a)} & \sum b_i \omega_{ij} \omega_{jk} \alpha_k^2 &= 2 \\ &\text{(F3b)} & \sum b_i \alpha_i \alpha_{ij} \omega_{jk} \omega_{kl} \alpha_l^2 &= \tfrac{2}{3} \\ &\text{(F3c)} & \sum b_i \omega_{ij} \alpha_j \alpha_{jk} \omega_{kl} \omega_{lm} \alpha_m^2 &= 2 \end{aligned} \tag{2.8}$$

If u_2 appears non-linearly in the semi-explicit DAE

$$\begin{aligned} \dot{u}_1 &= f_1(u_1, u_2), \\ 0 &= f_2(u_1), \end{aligned}$$

then the condition

$$\text{(G3)} \quad \sum b_i \alpha_{ij} \omega_{jk} \omega_{kl} \alpha_l^2 \alpha_{lm} \omega_{mn} \omega_{nr} \alpha_r^2 = \tfrac{4}{3} \tag{2.9}$$

has to be satisfied (see [8]).

The stability function of (2.2) is given by

$$R_0(z) = 1 + z b^\top (I - zB)^{-1} e,$$

where $b = (b_1, \ldots, b_s)^\top$ and $e = (1, \ldots, 1)^\top$.

3. Construction of methods. We start with the following result.

Lemma 3.1. *There exists no Rosenbrock method of order 3 with 3 internal stages which satisfies* (2.3), (2.6), (F3b), *and* (F3c).

Proof. This result can be shown by an easy calculation. □

Let us now consider Rosenbrock methods with 4 internal stages. The order conditions in this case read as (see [2])

$$\left\{\begin{array}{rrcl}
\text{(A1)} & b_1+b_2+b_3+b_4 & = & 1\\
\text{(A2)} & b_2\beta_2+b_3\beta_3+b_4\beta_4 & = & \frac{1}{2}-\gamma\\
\text{(A3a)} & b_2\alpha_2^2+b_3\alpha_3^2+b_4\alpha_4^2 & = & \frac{1}{3}\\
\text{(A3b)} & b_3\beta_{32}\beta_2+b_4(\beta_{42}\beta_2+\beta_{43}\beta_3) & = & \frac{1}{6}-\gamma+\gamma^2\\
\text{(B2)} & b_2\alpha_2+b_3\alpha_3+b_4\alpha_4 & = & \frac{1}{2}\\
\text{(C3a)} & b_3\alpha_{32}\alpha_2+b_4(\alpha_{42}\alpha_2+\alpha_{43}\alpha_3) & = & \frac{1}{6}\\
\text{(C3b)} & b_3\alpha_{32}\beta_2+b_4(\alpha_{42}\beta_2+\alpha_{43}\beta_3) & = & \frac{1}{6}-\frac{\gamma}{2}\\
\text{(C3c)} & b_3\beta_{32}\alpha_2+b_4(\beta_{42}\alpha_2+\beta_{43}\alpha_3) & = & \frac{1}{6}-\frac{\gamma}{2}
\end{array}\right.$$

Lemma 3.2. *(see [11]) The conditions for PDEs* (2.6) *can be simplified by the help of* (A1), (A2), (A3a), *and* (A3b) *to*

$$\left\{\begin{array}{rrcl}
\text{(D3a)} & b_4\beta_{32}\beta_{43}\alpha_2^2 & = & 2\gamma^4-2\gamma^3+\frac{1}{3}\gamma^2\\
\text{(D3b)} & b_3\beta_{32}\alpha_2^2+b_4(\beta_{42}\alpha_2^2+\beta_{43}\alpha_3^2) & = & 2\gamma^3-3\gamma^2+\frac{2}{3}\gamma\\
\text{(D3c)} & b_4\beta_{43}\beta_{32}\beta_{21} & = & 0
\end{array}\right.$$

Remark: The expressions $b_3\beta_{32}\alpha_2^2 + b_4(\beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2)$ and $b_4\beta_{43}\beta_{32}\beta_{21}$ are known as part of the order-conditions for 4th-order Rosenbrock-methods (see [2]).

The algebraic order condition reads as (see [2])

$$\text{(E3)} \quad b_2\omega_{22}\alpha_2^2 + b_3(\omega_{32}\alpha_2^2 + \omega_{33}\alpha_3^2) + b_4(\omega_{42}\alpha_2^2 + \omega_{43}\alpha_3^2 + \omega_{44}\alpha_4^2) = 1 \ .$$

Lemma 3.3. *A Rosenbrock-method which satisfies* (A1)–(A3b) *and* (D3a)–(D3c) *fulfills* (E3), *too.*

Proof. See [11]). □

Lemma 3.4. *A Rosenbrock-method which satisfies* (A1)–(A3b) *and* (D3a)–(D3c) *fulfils* (F3a), *too.*

Proof. see [9]. □

The conditions (F3b) and (F3c) can be written as follows

$$\begin{cases} \text{(F3b)} \quad \gamma(b_3\alpha_3\alpha_{32}\alpha_2^2 + b_4\alpha_4(\alpha_{42}\alpha_2^2 + \alpha_{43}\alpha_3^2)) - 2b_4\alpha_4\alpha_{43}\beta_{32}\alpha_2^2 = \frac{2}{3}\gamma^3 \\ \text{(F3c)} \quad b_4\beta_{43}\alpha_3\alpha_{32}\alpha_2^2 = \frac{2}{3}\gamma^3 - 2\gamma^4 \end{cases}$$

The embedded methods should be L-stable, too. Therefore we need the following result from [11].

Lemma 3.5. *Let a Rosenbrock method which satisfies* (A1)–(A3b) *and* (D3a)–(D3c) *be given. The embedded method satisfying* (A1) *and* (A2) *is L-stable, too, if*

$$\hat{b}_4 = \frac{1}{\beta_3\beta_{43}}\left[\gamma^3 - 2\gamma^2 + \frac{1}{2}\gamma\right]. \tag{3.1}$$

Proof. See [11]. □

TABLE 3.1
Set of coefficients for ROSI2P1

γ	=	$4.3586652150845900e-01$			
α_{21}	=	$5.0000000000000000e-01$	γ_{21}	=	$-5.0000000000000000e-01$
α_{31}	=	$5.5729261836499822e-01$	γ_{31}	=	$-6.4492162993321323e-01$
α_{32}	=	$1.9270738163500176e-01$	γ_{32}	=	$6.3491801247597734e-02$
α_{41}	=	$-3.0084516445435860e-01$	γ_{41}	=	$9.3606009252719842e-03$
α_{42}	=	$1.8995581939026787e+00$	γ_{42}	=	$-2.5462058718013519e-01$
α_{43}	=	$-5.9871302944832006e-01$	γ_{43}	=	$-3.2645441930944352e-01$
b_1	=	$5.2900072579103834e-02$	$\hat{b}_1$	=	$1.4974465479289098e-01$
b_2	=	$1.3492662311920438e+00$	$\hat{b}_2$	=	$7.0051069041421810e-01$
b_3	=	$-9.1013275270050265e-01$	$\hat{b}_3$	=	$0.0000000000000000e+00$
b_4	=	$5.0796644892935516e-01$	$\hat{b}_4$	=	$1.4974465479289098e-01$

3.1. An *L*-stable Rosenbrock method. Our first method is *L*-stable and satisfies the conditions (A1)–(A3b), (B2), (C3a)–(C3c), (D3a)–(D3c), (F3b), and (F3c). We call the method ROSI2P1, where ROS stands for Rosenbrock, I2 for index 2 problems, P for semi-discretized PDE problems, and 1 is an internal number. To find a solution of the equations given above, we have used the software environment "MAPLE". We choose the free variables as follows: $\alpha_2 = 1/2$, $\alpha_3 = 3/4$, and $\alpha_4 = 1$. The coefficients of ROSI2P1 are given in TABLE 3.1. The embedded method satisfies the conditions (A1), (A2), and (3.1). Moreover we set $\hat{b}_3 = 0$. The resulting system of equations can be solved easily.

3.2. A stiffly accurate Rosenbrock method. A Rosenbrock method satisfying

$$\beta_{si} = b_i, \quad i = 1, \dots, s, \quad \text{and} \quad \alpha_s = 1 \tag{3.2}$$

is called *stiffly accurate*. Methods which satisfy (3.2) yield asymptotically exact results for the problem $\dot{u} = \lambda(u - \varphi(t)) + \dot{\varphi}(t)$. A stiffly accurate Rosenbrock method is *L*-stable, i.e. $\gamma \approx 0.4358665$ (see [2] or [11]).

Our conditions simplify to (see [11] and [8])

$$\begin{cases}
(\text{A1'}) & b_1 + b_2 + b_3 = 1 - \gamma \\
(\text{A2'}) & b_2\beta_2 + b_3\beta_3 = \frac{1}{2} - 2\gamma + \gamma^2 \\
(\text{A3a'}) & b_2\alpha_2^2 + b_3\alpha_3^2 = \frac{1}{3} - \gamma \\
(\text{A3b'}) & b_3\beta_{32}\beta_2 = \frac{1}{6} - \frac{3}{2}\gamma + 3\gamma^2 - \gamma^3 \\
(\text{B2'}) & b_2\alpha_2 + b_3\alpha_3 = \frac{1}{2} - \gamma \\
(\text{C3a'}) & b_3\alpha_{32}\alpha_2 + \gamma(\alpha_{42}\alpha_2 + \alpha_{43}\alpha_3) = \frac{1}{6} \\
(\text{C3b'}) & b_3\alpha_{32}\beta_2 + \gamma(\alpha_{42}\beta_2 + \alpha_{43}\beta_3) = \frac{1}{6} - \frac{\gamma}{2} \\
(\text{C3c'}) & b_3\beta_{32}\alpha_2 = \frac{1}{6} - \gamma + \gamma^2 \\
(\text{D3a'}) & b_3\beta_{32}\alpha_2^2 = 2\gamma^3 - 2\gamma^2 + \frac{1}{3}\gamma \\
(\text{D3c'}) & b_3\beta_{32}\beta_2 = 0 \\
(\text{F3b'}) & \gamma(\alpha_{42}\alpha_2^2 + \alpha_{43}\alpha_3^2) - 2\alpha_{43}\beta_{32}\alpha_2^2 = 2\gamma^3 \\
(\text{F3c'}) & b_3\alpha_3\alpha_{32}\alpha_2^2 = \frac{2}{3}\gamma^2 - 2\gamma^3
\end{cases}$$

Our new method should satisfy the conditions (A1')–(A3b'), (D3a'), (D3c'), (F3b'), and (F3c'). Moreover we set $\alpha_2 = 1/2$, $\alpha_{41} = \alpha_{31}$, $\alpha_{42} = \alpha_{32}$, and $\alpha_{43} = 0$, i.e. the method needs only three function evaluations. First we note that $\beta_2 = 0$. This follows from (D3a') and (D3c'). With (F3b') we get $\alpha_{42} = 8\gamma^2$. Inserting this result into (F3c') yields $b_3 = 1/3 - \gamma$. Using (D3a') we obtain

$$\beta_{32} = 4\frac{2\gamma^3 - 2\gamma^2 + \frac{1}{3}\gamma}{1/3 - \gamma}.$$

The remaining coefficients can be computed by the help of (A1'), (A2'), and (A3a'). The new method is called ROSI2P2 and its coefficients are given in TABLE 3.2. The embedded

TABLE 3.2
Set of coefficients for ROSI2P2

γ	=	$4.3586652150845900e-01$			
α_{21}	=	$5.0000000000000000e-01$	γ_{21}	=	$-5.0000000000000000e-01$
α_{31}	=	$-5.1983699657507165e-01$	γ_{31}	=	$-4.0164172503011392e-01$
α_{32}	=	$1.5198369965750715e+00$	γ_{32}	=	$1.1742718526976650e+00$
α_{41}	=	$-5.1983699657507165e-01$	γ_{41}	=	$1.1865036632417383e+00$
α_{42}	=	$1.5198369965750715e+00$	γ_{42}	=	$-1.5198369965750715e+00$
α_{43}	=	$0.0000000000000000e+00$	γ_{43}	=	$-1.0253318817512568e-01$
b_1	=	$6.6666666666666663e-01$	$\hat{b}_1$	=	$-9.5742384859111473e-01$
b_2	=	$-5.4847955522165341e-32$	$\hat{b}_2$	=	$2.9148476971822297e+00$
b_3	=	$-1.0253318817512568e-01$	$\hat{b}_3$	=	$5.0000000000000000e-01$
b_4	=	$4.3586652150845900e-01$	$\hat{b}_4$	=	$-1.4574238485911146e+00$

method satisfies the conditions (A1), (A2), and (3.1). Moreover we set $\hat{b}_3 = 1/2$. This system of equations can be solved easily.

3.3. A stiffly accurate Rosenbrock method with $W = f_u + \mathcal{O}(h)$ and $T = 0$. Our new method should satisfy the conditions (A1')–(A3b'), (B2'), (C3c'), (D3a'), (D3c'), (F3b'), (F3c'), and (G3). The condition (G3) can be simplified to

$$\alpha_{43}\alpha_3^2\alpha_{32}\alpha_2^2 = \frac{4}{3}\gamma^3.$$

As the free variable we choose $\alpha_3 = 3/4$. As in the previous section we have $\beta_2 = 0$. The variable $\alpha_2 = 2\gamma$ can be determined by (D3a') and (C3c'). The equations (A3a) and (B2) form a linear system of equations in the variables b_2 and b_3. Then the remaining coefficients can be determined easily. The method is called ROSI2Pw and the coefficients are given in TABLE 3.3. The embedded method satisfies the conditions (A1), (A2), and (3.1). Moreover we set $\hat{b}_3 = 0$. This system of equations can be solved easily.

TABLE 3.3
Set of coefficients for ROSI2Pw

γ	$=$	$4.3586652150845900e-01$			
α_{21}	$=$	$8.7173304301691801e-01$	γ_{21}	$=$	$-8.7173304301691801e-01$
α_{31}	$=$	$7.8938917169345013e-01$	γ_{31}	$=$	$-8.4175599602920992e-01$
α_{32}	$=$	$-3.9389171693450180e-02$	γ_{32}	$=$	$-1.2977652642309580e-02$
α_{41}	$=$	$6.2787416864263046e-01$	γ_{41}	$=$	$-3.7964867148089526e-01$
α_{42}	$=$	$6.9295440480994763e+00$	γ_{42}	$=$	$-8.3490231248017537e+00$
α_{43}	$=$	$-6.5574182167421071e+00$	γ_{43}	$=$	$8.2928052747741905e+00$
b_1	$=$	$2.4822549716173517e-01$	$\hat{b}_1$	$=$	$4.4315753191688778e-01$
b_2	$=$	$-1.4194790767022774e+00$	$\hat{b}_2$	$=$	$4.4315753191688778e-01$
b_3	$=$	$1.7353870580320832e+00$	$\hat{b}_3$	$=$	$0.0000000000000000e+00$
b_4	$=$	$4.3586652150845900e-01$	$\hat{b}_4$	$=$	$1.1368493616622447e-01$

3.4. A stiffly accurate Rosenbrock W-method. In the following a Rosenbrock method is constructed which satisfies the conditions (A1')–(A3b'), (B2'), (C3a')–(C3c'), (D3a'), (D3c'), (F3b'), and (F3c'). We have 12 equations and 12 unknowns. Note that 5 unknowns are determined by (3.2). There are no free variables. The coefficients $\alpha_2 = 2\gamma$ and $\beta_2 = 0$ can be computed as the in the previous section. Let us assume that we know the coefficient α_3. Then (A3a') and (B2') form a linear system of equations in the unknowns b_2 and b_3. The solution depends on α_3 and is given by

$$b_2 = \frac{1}{12}\frac{6\alpha_3\gamma + 2 - 6\gamma - 3\alpha_3}{\gamma(2\gamma - \alpha_3)}, \qquad b_3 = -\frac{1}{3}\frac{6\gamma^2 - 6\gamma + 1}{\alpha 3(2\gamma - \alpha_3)}.$$

An esay computation shows that

$$\alpha_3 = 6\frac{\gamma^2(-14\gamma^2 + 6\gamma^3 - 1 + 7\gamma)}{-12\gamma + 36\gamma^2 - 6\gamma^3 - 72\gamma^4 + 36\gamma^5 + 1} \approx -1.55 < 0.$$

The method is called ROSI2PW and the coefficients are given by Table 3.4. The embedded method satisfies the conditions (A1), (A2), and (3.1). Moreover we set $\hat{b}_3 = 0$. This system of equations can be solved easily.

Table 3.4
Set of coefficients for ROSI2PW

γ	=	$4.3586652150845900e-01$			
α_{21}	=	$8.7173304301691801e-01$	γ_{21}	=	$-8.7173304301691801e-01$
α_{31}	=	$-7.9937335839852708e-01$	γ_{31}	=	$3.0647867418622479e+00$
α_{32}	=	$-7.9937335839852708e-01$	γ_{32}	=	$3.0647867418622479e+00$
α_{41}	=	$7.0849664917601007e-01$	γ_{41}	=	$-1.0424832458800504e-01$
α_{42}	=	$3.1746327955312481e-01$	γ_{42}	=	$-3.1746327955312481e-01$
α_{43}	=	$-2.5959928729134892e-02$	γ_{43}	=	$-1.4154917367329144e-02$
b_1	=	$6.0424832458800504e-01$	$\hat{b}_1$	=	$4.4315753191688778e-01$
b_2	=	$-3.6210810811598324e-32$	$\hat{b}_2$	=	$4.4315753191688778e-01$
b_3	=	$-4.0114846096464034e-02$	$\hat{b}_3$	=	$0.0000000000000000e+00$
b_4	=	$4.3586652150845900e-01$	$\hat{b}_4$	=	$1.1368493616622447e-01$

4. Comparison of Rosenbrock methods and numerical results. All examples are solved numerically by the help of the FEM-package MooNMD3.0 (see [3]) on a uniform spatial grid consisting of 1024 quadrangles, i.e. $h = 2^{-5}$. We compare our new methods with other well-known Rosenbrock methods such as ROS3P, ROS3Pw, ROS34PW2, and RODASP. An overview of the selected Rosenbrock methods can be found in Table 4.1.

We apply these schemes to a PDAE of index 2 and to the Navier–Stokes equations with different right-hand sides. For the definition of the index of linear PDAEs we refer to the paper [10].

The global error $\underline{\epsilon}$ is measured in the discrete L_2-norm

$$\|\underline{\epsilon}\|_{l_2(J,V)} := \left(\tau_N \sum_{n=0}^{N} \|\mathbf{u}_n - \mathbf{u}(t_n)\|_V^2 \right)^{1/2},$$

where $V := L_2(\Omega)$ or $H^1(\Omega)$ and τ_N is a time-step depending on $N \in \mathbb{N}$. In this section, the letter J is used to denote a time interval.

Table 4.1
Properties of the selected Rosenbrock methods

Name	s	p	Index 1	Index 2	PDEs	$R(\infty)$	stiffly acc.	reference
ROS3P	3	3	yes	no	yes	0.73	no	[6]
ROWDAIND2	4	3	yes	yes	no	0	yes	[8]
ROS3Pw	3	3	yes	no	yes	0.73	no	[11]
ROS34PW2	4	3	yes	no	yes	0	yes	[11]
RODASP	6	4	yes	no	yes	0	yes	[13]
ROSI2P1	4	3	yes	yes	yes	0	no	see Section 3.1
ROSI2P2	4	3	yes	yes	yes	0	yes	see Section 3.2
ROSI2Pw	4	3	yes	yes	yes	0	yes	see Section 3.3
ROSI2PW	4	3	yes	yes	yes	0	yes	see Section 3.4

Example 4.1. Let $J := (0,1)$ and $\Omega := (0,1)$. We consider the following nonlinear PDAE

$$\begin{aligned} \dot{u}_1 - \Delta u_1 - u_3\dot{u}_2 + \dot{u}_2 u_3 &= 2\varepsilon^2\omega t && \text{in } J\times\Omega, \\ \Delta u_2 &= 0 && \text{in } J\times\Omega, \\ \Delta u_3 &= 0 && \text{in } J\times\Omega, \\ \dot{u}_4 - \Delta u_4 - \lambda\Delta u_1 &= -e^{-t}(x^2+2) - 2\lambda\varepsilon^2\omega t && \text{in } J\times\Omega \end{aligned} \tag{4.1}$$

where λ, ε and ω are free parameters. The right-hand side f, the initial conditions and the non-homogeneous Dirichlet boundary conditions are chosen such that

$$\begin{aligned} u_1(t,x,y) &= x^2\varepsilon^2\omega t, \\ u_2(t,x,y) &= x\varepsilon\sin\omega t, \\ u_3(t,x,y) &= x\varepsilon\cos\omega t, \\ u_4(t,x,y) &= 1 + e^{-t}x^2 \end{aligned}$$

is the solution of (4.2). Moreover we set $\varepsilon = \omega = \lambda = 1$. For the semi-discretization in space we used central finite differences with step length $h = 1/100$. The computations were carried out with time steps $\tau_N = \frac{1}{10N}$ with $N = 1, 2, 4, 8, 16, 32, 64, 128$. The Jacobian is computed exactly. Note that all occurring discretization errors will results from the temporal discretization. FIGURE 4.1 shows the results of the calculation.

The most inaccurate results were obtained with the methods for PDAEs of index 1, namely ROS3P, ROS3Pw and ROS34PW2. This is due to the fact that these methods do not satisfy the conditions (F3b) and (F3c). The method ROWDAIND2 satisfies these conditions, but it has order reduction because a semidiscretized PDAE is solved. The best results were obtained with the fourth order method RODASP and the solvers ROSI2P1, ROSI2P2, ROSI2Pw, and ROSI2PW.

EXAMPLE 4.2. Let $J := (0, 1)$ and $\Omega := (0, 1)^2$. We consider the Navier–Stokes equations

$$\begin{aligned} \dot{u} - Re^{-1}\Delta u + (u\cdot\nabla)u + \nabla p &= f && \text{in } J\times\Omega, \\ \nabla\cdot u &= 0 && \text{in } J\times\Omega, \\ u &= g && \text{on } J\times\partial\Omega, \\ u(0,x) &= u_0 && x\in\Omega, \end{aligned} \tag{4.2}$$

where Re denotes the positive Reynolds number. The right-hand side f, the initial condition

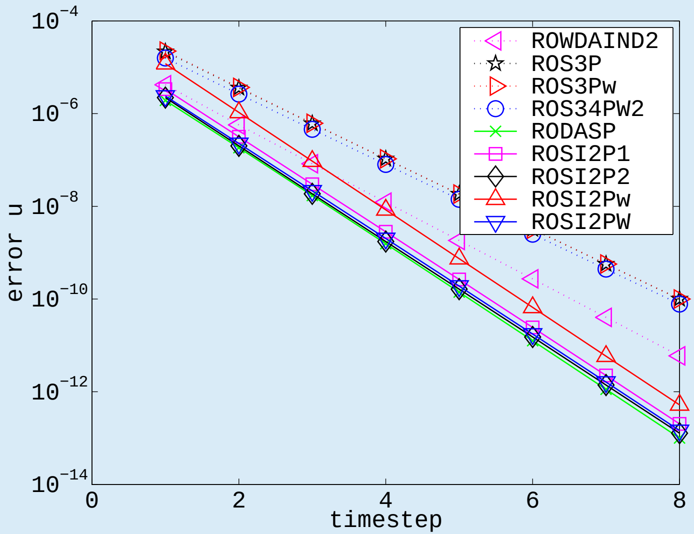

FIG. 4.1. *Example 4.1, results*

u_0 and the non-homogeneous Dirichlet boundary conditions are chosen such that

$$\begin{aligned} u_1(t,x,y) &= t^3y^2, \\ u_2(t,x,y) &= t^2x, \\ p(t,x,y) &= tx + y - (t+1)/2 \end{aligned}$$

is the solution of (4.2). Moreover we set $Re = 1$. We used the Q_2/P_1^{disc} discretization on a square mesh with an edge length $h = 1/64$ and solve the problem with variable time step sizes. The Jacobian is computed exactly. Note that for any t the solution can be represented exactly by the discrete functions. Hence, all occurring discretization errors will results from the temporal discretization. During the calculations we have to deal with 33 282 d.o.f. for the velocity and 11 288 d.o.f. for the pressure. FIGURE 4.2 shows the results of the calculation.

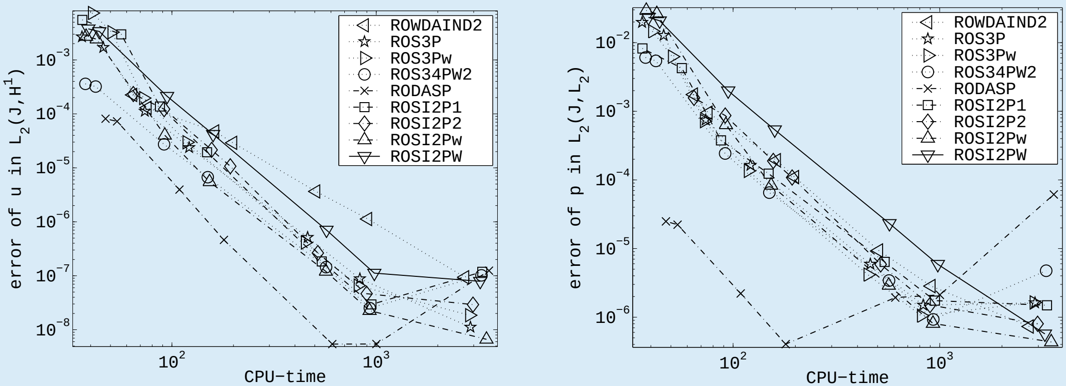

FIG. 4.2. *Example 4.2, results*

Considering the velocity error it can be observed that the fourth order method RODASP gave the best results. All other schemes gave good results, too. A similar observation can be made for the pressure error.

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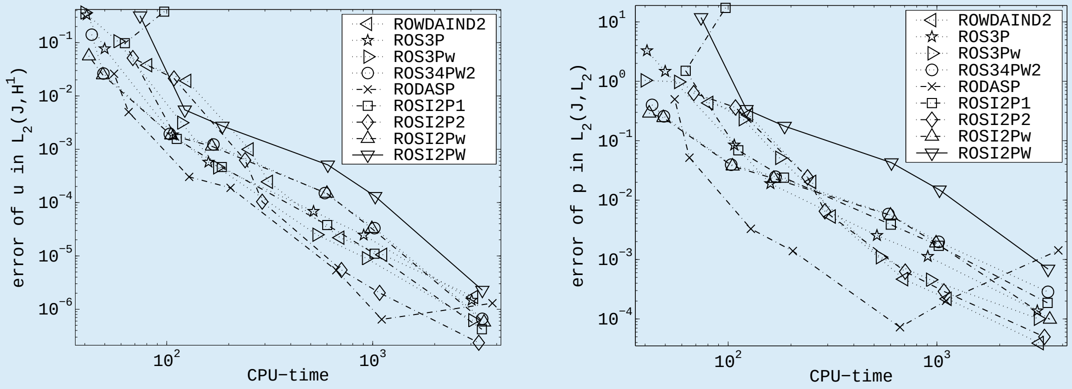

FIG. 4.3. *Example 4.3, results*

EXAMPLE 4.3. We consider the Navier-Stokes equations (4.2) with Dirichlet boundary conditions on the whole boundary and with the solution

$$\begin{aligned} u_1(t,x,y) &= t^3 y^2, \\ u_2(t,x,y) &= \exp(-50t)x, \\ p(t,x,y) &= (10+t)\exp(-t)(x+y-1). \end{aligned}$$

The computations were carried out with $Re = 1000$, a spatial grid consisting of squares of edge length $h = 1/32$, and variable time step sizes. This gives 8 450 velocity d.o.f. and 3 072 pressure d.o.f. for the Q_2/P_1^{disc} finite element discretization.

All methods gave good results. The differences between the fourth order method RODASP and the other third order methods is much smaller than in the previous example.

REFERENCES

[1] K. E. Brenan, S. L. Campbell and L. R. Petzold, *Numerical Solution of Initial-Value Problems in DAEs*, volume 14 of Classics In Applied Mathematics, SIAM, Philadelphia, 1996.

[2] E. Hairer and G. Wanner, *Solving ordinary differential equations II: Stiff and differential-algebraic problems*, volume 14 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2nd edition, 1996.

[3] V. John, and G. Matthies. MooNMD – a program package based on mapped finite element methods. Comput. Visual. Sci., **6** (2004), 163–170.

[4] V. John, G. Matthies and J. Rang, A comparison of time-discretization/linearization approaches for the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg., **195** (2006), 5995–6010.

[5] J. Lang, *Adaptive Multilevel Solution of Nonlinear Parabolic PDE Systems*, volume 16 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, 2001.

[6] J. Lang and J. Verwer, *ROS3P – an Accurate Third-Order Rosenbrock Solver Designed for Parabolic Problems*, BIT, **41** (2001), 730–737.

[7] C. Lubich and A. Ostermann, *Linearly implicit time discretization of non-linear parabolic equations*, IMA J. Numer. Anal., **15**(4) (1995), 555–583.

[8] C. Lubich and M. Roche, *Rosenbrock methods for differential-algebraic systems with solution-dependent singular matrix multiplying the derivative*. Computing, **43**(4) (1990), 325–342.

[9] J. Rang and L. Angermann, *New Rosenbrock W-methods of order 3 for PDAEs of Index 2*. Mathematik-Bericht Nr. 2005/2, Institut für Mathematik, Technische Universität Clausthal, 2005.

[10] J. Rang and L. Angermann, *Perturbation index of linear partial differential algebraic equations*. Appl. Num. Math., **53**(2/4) (2005), 437–456.

[11] J. Rang and L. Angermann, *New Rosenbrock W-methods of order 3 for PDAEs of Index 1*. BIT, **45** (2005), 761–787.

[12] M. Roche, *Rosenbrock methods for differential algebraic equations*. Numerische Mathematik, **52** (1988), 45–63.

[13] G. Steinebach, *Order-reduction of ROW-methods for DAEs and method of lines applications*. Preprint 1741, Technische Universität Darmstadt, Darmstadt, 1995.

[14] K. Strehmel and R. Weiner, *Linear-implizite Runge–Kutta-Methoden und ihre Anwendung*, volume 127 of Teubner-Texte zur Mathematik. Teubner, Stuttgart, 1992.

[15] J. Weickert, *Applications of the theory of differential-algebraic equations to partial differential equations of fluid dynamics*. PhD thesis, Technische Universität Chemnitz–Zwickau, Chemnitz, 1997.