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DISSIPATIVE DYNAMICS OF REACTION DIFFUSION EQUATIONS IN $\mathbb{R}^N$*

JOSE M. ARRIETA†, NANCY MOYA‡, AND ANIBAL RODRIGUEZ-BERNAL§

Abstract. In this note we present some recent results about the dissipative behavior of a broad class of reaction diffusion equations in $\mathbb{R}^N$. We show that in large spaces of initial data, not having any decay at infinity, these equations define asymptotically compact semigroups and have a locally compact global attractor.

The classes of equations considered include some type of logistic–like equations which may have travelling wave solutions.

Under some additional conditions we also show that the attractors have finite Hausdorff and fractal dimensions.

Key words. locally uniform spaces, unbounded domains, dissipative equations, attractors, finite dimension

1. Introduction. Understanding the global dynamics of a reaction diffusion equation in $\mathbb{R}^N$ or in an unbounded domain is not a trivial task. The fact that the domain is unbounded introduces important difficulties in the analysis of the solutions and, more important, of their asymptotic behavior. The choice of the functional setting where the initial conditions are taken and where the solutions live is not straitghforward. Once this choice is made the compactness properties of the semiflow have to be analyzed with extreme care.

To accomplish this, several solutions have been proposed in different articles. For instance in [6] an approach using weighted Sobolev spaces has been carried out, while in [2] the analysis was done in standard Sobolev spaces $L^p(\mathbb{R}^N)$ and $W^{s,p}(\mathbb{R}^N)$ spaces.

In this paper we present an approach based in *locally uniform* spaces, of the type $L^p_U(\mathbb{R}^N)$, which are characterized by the fact that $\phi \in L^p_U(\mathbb{R}^N)$ if $\phi \in L^p_{loc}(\mathbb{R}^N)$ and $\|\phi\|_{L^p(B(x,1))} \leq C$ with C independent of $x \in \mathbb{R}^N$, see Section 2. It turns out that for these spaces there is a nice linear theory for both, linear heat equations and the Schrödinger equations. This theory includes the generation of analytic semigroups, characterization of fractional power spaces (or of interpolation spaces), regularization properties of the semigroup, $L^p_U - -L^q_U$ estimates, etc.. This linear theory, which is presented in Section 3 is a first step towards understanding the dynamic behavior of nonlinear equations.

Once the linear theory is obtained we proceed to analyze nonlinear equations of the type $u_t - \Delta u = f(x,u)$, where the nonlinearity has the general form given by $f(x,u) = m(x)u + f_0(x,u) + g(x)$ whith $f_0(x,0) \equiv \frac{\partial f_0}{\partial u}(x,0) \equiv 0$ and the potential $m(\cdot) \in L^\sigma_U(\mathbb{R}^N)$ for appropriate σ and it may change sign. We start with conditions that garantee local existence of solutions in $L^q_U(\mathbb{R}^N)$. This conditions include the critical growth condition on the nonlinearity f. Afterwards, we obtain conditions on the nonlinearities that garantee that the flow is dissipative, that is, the existence of a bounded set in $L^q_U(\mathbb{R}^N)$ such that all solutions eventually enter this bounded set. We end up proving the existence of a bounded

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†Depto. de Matemática Aplicada, U. Complutense de Madrid, Spain (jose.arrieta@mat.ucm.es)
‡Depto. de Matemática Aplicada, U. Complutense de Madrid, Spain
§Depto. de Matemática Aplicada, U. Complutense de Madrid, Spain (arober@mat.ucm.es)

invariant set $\mathcal{A}$ that attracts bounded sets of $L^p_U(\mathbb{R}^N)$ in the topology of $C_{loc}(\mathbb{R}^N)$, see Section 4.

Finally in Section 5, we show that for appropriate conditions on the nonlinearity $f(x,u) = m(x)u + f_0(x,u) + g(x)$, including the case where m changes sign (unlike the results in references [1, 6, 7, 9]), the equation has an attractor in $L^2(\mathbb{R}^N)$ and that this attractor has finite Hausdorf and fractal dimension.

2. Locally uniform spaces. We define, for $1 \le p < \infty$, the *uniform space* $L^p_U(\mathbb{R}^N)$ as the set of functions $\phi \in L^p_{loc}(\mathbb{R}^N)$ such that

$$\sup_{x\in\mathbb{R}^N} \int_{B(x,1)} |\phi(y)|^p \, dy < \infty \tag{2.1}$$

with norm

$$\|\phi\|_{L^p_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^p(B(x,1))}.$$

Observe that for $p = \infty$, using the analogous definition, we have $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ with norm $\|\phi\|_{L^\infty_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^\infty(B(x,1))} = \|\phi\|_{L^\infty(\mathbb{R}^N)}$. Observe that $L^p_U(\mathbb{R}^N)$ contains $L^\infty(\mathbb{R}^N)$, $L^r(\mathbb{R}^N)$ and $L^r_U(\mathbb{R}^N)$ for any $r \ge p$.

Also denote by $\dot{L}^p_U(\mathbb{R}^N)$ a subspace of $L^p_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^p_U(\mathbb{R}^N)}$, that is

$$\|\tau_y\phi - \phi\|_{L^p_U(\mathbb{R}^N)} \to 0 \quad \text{as} \quad |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations. Note that $L^p(\mathbb{R}^N) \subset \dot{L}^p_U(\mathbb{R}^N)$ for $1 \le p < \infty$ and for $p = \infty$ we get $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$.

Finally we also define, for $k \in \mathbb{N}$, the *uniform Sobolev space* $W^{k,p}_U(\mathbb{R}^N)$ as the set of functions $\phi \in W^{k,p}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{W^{k,p}_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{W^{k,p}(B(x,1))} < \infty. \tag{2.2}$$

3. Linear parabolic problems. Now we consider the heat equation in $L^q_U(\mathbb{R}^N)$

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^N, \quad t > 0 \\ u(x,0) = u_0(x) \in L^q_U(\mathbb{R}^N) \end{cases} \tag{3.1}$$

whose solution is given by the convolution with the heat kernel

$$u(x,t) = T(t)u_0 = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \, u_0(y) \, \mathrm{d}y. \tag{3.2}$$

Now we review the results obtained in [3]. First we have

Theorem 3.1.

i) *There exists $M_0 = M_0(N)$ such that for $1 \le q \le p \le \infty$*

$$\|T(t)u_0\|_{L^p_U(\mathbb{R}^N)} \le M_0\Big(t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} + 1\Big)\|u_0\|_{L^q_U(\mathbb{R}^N)},$$

$$\|D^\beta T(t)u_0\|_{L^p_U(\mathbb{R}^N)} \le M_0 t^{-\frac{1}{2}(k-\frac{N}{p}+\frac{N}{q})}\|u_0\|_{L^q_U(\mathbb{R}^N)},$$

for any $1 \le |\beta| = k$.

ii) *For each bounded set* $B \subset \mathbb{R}^N$ *and* $u_0 \in L^q_U(\mathbb{R}^N)$

$$\|T(t)u_0 - u_0\|_{L^q(B)} \to 0, \quad \text{as } t \to 0.$$

If $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$, $1 \le q \le \infty$, *then* $T(t)u_0 \in \dot{L}^p_U(\mathbb{R}^N)$ *for* $t > 0$ *and any* $1 \le p \le \infty$ *and*

$$\|T(t)u_0 - u_0\|_{L^q_U(\mathbb{R}^N)} \to 0, \quad \text{as } t \to 0.$$

With this we can get

Theorem 3.2. *The heat equation defines an order preserving analytic semigroup in* $L^q_U(\mathbb{R}^N)$, *for* $1 \le q \le \infty$, *which is continuous even at* $t = 0$ *if* $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$. *Moreover for* $u_0 \in L^q_U(\mathbb{R}^N)$, $1 \le q \le \infty$

$$(0, \infty) \ni t \longmapsto T(t)u_0 \in \dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$$

is analytic.

Now we consider Schrödinger equations of the form

$$\begin{cases} u_t - \Delta u = V(x)u, & x \in \mathbb{R}^N \quad t > 0, \\ u(0) = u_0 \in L^q_U(\mathbb{R}^N) \end{cases}$$

where a large class of potential functions are allowed. Then we have,

Proposition 3.3. *Assume* $V \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > N/2$

i) *For* $1 \le q \le p \le \infty$

$$\| \mathrm{e}^{(\Delta+V)t} u_0\|_{L^p_U(\mathbb{R}^N)} \le M_1 \mathrm{e}^{\mu t} t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \|u_0\|_{L^q_U(\mathbb{R}^N)}, \tag{3.3}$$

a and M_1 *depend only on* N, σ *and* $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$ *and* $\mu \in \mathbb{R}$ *can be taken as any number satisfying*

$$-\mu < \inf \left\{ \int_{\mathbb{R}^N} |\nabla \phi|^2 + \int_{\mathbb{R}^N} V(x)|\phi|^2, \quad \phi \in C^\infty_c(\mathbb{R}^N), \ \int_{\mathbb{R}^N} |\phi|^2 = 1 \right\} \tag{3.4}$$

In particular $\mathrm{e}^{(\Delta+V)t}$ *decays in* $L^q_U(\mathbb{R}^N)$ *if and only if it decays in* $L^p(\mathbb{R}^N)$ *for* $1 \le q, p \le \infty$.

ii) *For* $B \subset \mathbb{R}^N$ *bounded,* $u_0 \in L^q_U(\mathbb{R}^N)$, $1 \le q < \infty$,

$$\| \mathrm{e}^{(\Delta+V)t} u_0 - u_0\|_{L^q(B)} \to 0, \quad \text{as } t \to 0.$$

If moreover $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$, $1 \le q \le \infty$, *then*

$$\| \mathrm{e}^{(\Delta+V)t} u_0 - u_0\|_{L^q_U(\mathbb{R}^N)} \to 0, \quad \text{as } t \to 0.$$

As a consequence we get

Theorem 3.4. *If* $V \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > N/2$, *then* $\Delta + V$ *generates an order preserving analytic semigroup in* $L^q_U(\mathbb{R}^N)$, *for* $1 \le q \le \infty$ *which is continuous even at* $t = 0$ *if* $u_0 \in \dot{L}^q_U(\mathbb{R}^N)$. *Moreover, for* $u_0 \in L^q_U(\mathbb{R}^N)$, $1 \le q \le \infty$

$$(0, \infty) \ni t \longmapsto \mathrm{e}^{(\Delta+V)t} u_0 \in \dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$$

is analytic.

4. Nonlinear parabolic problems. Consider the problem

$$u_t = \Delta u + f(x,u), \qquad x \in \mathbb{R}^N, \quad t > 0, \tag{4.1}$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \tag{4.2}$$

In what follows we present the results in [5]. We will assume that

$$f(x,s) = m(x)s + f_0(x,s) + g(x) \quad \text{with} \quad f_0(x,0) = 0, \quad \frac{\partial}{\partial s} f_0(x,0) = 0. \tag{4.3}$$

$$\left| \frac{\partial}{\partial s} f_0(x,s) \right| \le c(1 + |s|^{\rho - 1}) \qquad \text{and} \qquad |f_0(x,s)| \le c(|s| + |s|^{\rho}). \tag{4.4}$$

Theorem 4.1. *Assume* (4.3) *holds with $g \in L^q_U(\mathbb{R}^N)$, $m \in L^p_U(\mathbb{R}^N)$, for some $p > N/2$. Then problem* (4.1)–(4.2) *is well posed in $L^q_U(\mathbb{R}^N)$ provided*

$$1 \le \rho \le \rho_C = 1 + \frac{2q}{N}$$

Solutions of (4.1)– (4.2) *are continuous functions in $L^q_U(\mathbb{R}^N)$ which satisfy the variation of constants formula*

$$u(t,u_0) = T(t)u_0 = S(t)u_0 + \int_0^t S(t-s)\Big(f_0(\cdot, u(s)) + g\Big)\, \mathrm{d}s \tag{4.5}$$

for $0 \le t < T$, where $T = T(u_0)$ is the maximal existence time and $S(t)$ denotes the analytic semigroup in $L^q_U(\mathbb{R}^N)$ generated by $\Delta + m(x)$ with Dirichlet boundary conditions, that is $S(t) = \mathrm{e}^{(\Delta + m(x))t}$.

Below we will assume the following structure assumption on f.

$$f(x,s)s \le C(x)|s|^2 + D(x)|s|, \qquad \text{for all} \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^N \tag{4.6}$$

for some suitable functions $C(x)$ and $D(x) \ge 0$ defined in $\mathbb{R}^N$.

Theorem 4.2. *Assume the conditions for local existence for initial data in $L^q_U(\mathbb{R}^N)$ of* Theorem 4.1. *Assume further that* (4.6) *holds for some $C \in L^\sigma_U(\mathbb{R}^N)$, for some $\sigma > N/2$, and $D \in L^p_U(\mathbb{R}^N)$ with $p > N/2$.*

Then the unique solution of problem (4.1)–(4.2) *with initial data $u_0 \in L^q_U(\mathbb{R}^N)$, is global, remains bounded in $L^\infty(\mathbb{R}^N)$ on compact time intervals away from $t = 0$.*

Proof. Using comparison and maximum principles we get

$$|u(t,x)| \le U(t,x) \qquad x \in \mathbb{R}^N$$

where

$$U_t - \Delta U = C(x)U + D(x), \qquad U(0) = |u_0|.$$

With this, $D \in L^p_U(\mathbb{R}^N)$ and $p > N/2$ implies U is bounded in $L^\infty(\mathbb{R}^N)$ on $[\varepsilon, T]$. Hence, $\|f_0(\cdot,u)\|_{L^q_U(\mathbb{R}^N)} \le C_0(\varepsilon, T), \quad t \in [\varepsilon, T]$ and from the variation of constants formula, the result follows. ∎

The next step is obtaining asymptotic bounds on the solutions

THEOREM 4.3. *Assume $\Delta + C(x)I$ has negative exponential type, i.e. we can take $\mu < 0$ in* (3.4).

Let $0 \le \phi \in W^{2,p}_U(\mathbb{R}^N) \subset C^{\alpha}_U(\mathbb{R}^N)$ be the unique solution of

$$-\Delta\phi = C(x)\phi + D(x) \quad in \quad \mathbb{R}^N.$$

Then $\limsup_{t\to\infty} |u(t,x,u_0)| \le \phi(x)$, uniformly in $x \in \mathbb{R}^N$. If $|u_0(x)| \le \phi(x)$, then $|u(t,x,u_0)| \le \phi(x)$ for all $t > 0$.

Proof. Now

$$|u(t,x)| \le U(t,x) \to \phi(x), \quad \text{as} \quad t \to \infty$$

and we use the variation of constants formula. ∎

With this we can prove

THEOREM 4.4. *With the assumptions above there exists a bounded invariant set $\mathcal{A} \subset BUC(\mathbb{R}^N)$ such that for every bounded set $B \subset L^q_U(\mathbb{R}^N)$,*

$$dist_{C_{loc}(\mathbb{R}^N)}\big(u(t,B),\mathcal{A}\big) \to 0, \quad as \quad t \to \infty.$$

For any compact set $K \subset \mathbb{R}^N$, the set of restrictions to K, $\mathcal{A}_{|K}$, is a compact set in $C(K)$.

EXAMPLE: Assume $m \in L^p_U(\mathbb{R}^N)$, with $p > N/2$ and

$$f(x,s) = m(x)s - s^3, \quad x \in \mathbb{R}^N.$$

Since $\rho = 3$ for local existence we take initial data in $L^q_U(\mathbb{R}^N)$ with $q > N$. On the other hand, global existence follows by taking $C(x) = m(x)$ and $D(x) = 0$.

Assume now that $m(x) = m_0(x) - m_1(x)$, $\quad m_0, m_1 \in L^p_U(\mathbb{R}^N)$ such that

$$e^{(\Delta - m_1(x)I)t} \quad \text{decays exponentially,}$$

e.g. m_1 can be taken as a large positive constant. Hence using Young's inequality we get (4.6) with

$$C = -m_1 \in L^p_U(\mathbb{R}^N), \quad p > N/2, \qquad D(x) \approx |m_0|^{3/2}(x)$$

and the existence of the attractor in the local compact topology follows if $m_0 \in L^r_U(\mathbb{R}^N)$ with $r > \frac{3N}{4}$.

5. Finite dimension of invariant sets. The geometric idea behind the estimate of the dimension of an invariant set $\mathcal{A}$ is to analyze the evolution of a d-dimensional volume under the action of the semigroup. Then one searches for the smallest d for which any d-dimensional volume contracts asymptotically as $t \to \infty$.

Thus we take $u_0 \in \mathcal{A}$ and consider d orthogonal functions in $L^2(\mathbb{R}^N)$ and we denote by $V_d(0)$ the d-dimensional volume delimited by them. Then these vectors and volume evolve by the flow of the equation (4.1) linearized along the trajectory $u(t,u_0)$. Hence the volume $V_d(t)$ is given by the initial volume times the factor $\exp\Big(\int_0^t Tr\big(A_1(s,u_0)\circ Q_d(s)\big)\ \mathrm{d}s\Big)$ where $A_1(t)$ is the linearized operator along $u(t,u_0)$, that is

$$\partial_t U = \Delta U + \frac{\partial f}{\partial u}(x,u(t,u_0))U := A_1(t)U \tag{5.1}$$

and Q_d is a suitable orthogonal projection of rank d. Hence, to obtain the exponential decay of $V_d(t)$ it is enough to show that

$$\limsup_{t\to\infty} \sup_{u_0\in\mathcal{A}} \frac{1}{t}\int_0^t Tr_d(A_1(\tau))\,\mathrm{d}\tau < 0 \tag{5.2}$$

where

$$Tr_d(A_1(t)) := \sup_{E_d} Tr_d(A_1(t), E_d) := \sup_{E_d} \sum_{i=1}^{d} (A_1(t)\varphi_i, \varphi_i)_{L^2(\mathbb{R}^N)} \tag{5.3}$$

is the d-dimensional trace of $A_1(t)$. Here E_d is a d-dimensional subspace of $L^2(\mathbb{R}^N)$, and $\varphi_i \in E_d,\ i = 1, \dots, d$ is an orthonormal basis in $L^2(\mathbb{R}^N)$ of E_d. This implies in fact that $dim_H\mathcal{A} \le dim_F\mathcal{A} < \infty$, see [11, 13, 12, 8].

Using this general technique, we now present the results in [4].

Lemma 5.1. *Assume the Schrödinger operator $\Delta + m(x)I$, with $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > N/2$ has a negative exponential type, that is, we can take $\mu < 0$ in (3.4). Then for every $d \in \mathbb{N}$, we have*

$$Tr_d(\Delta + m(x)I) \le \mu d.$$

Proof. Note that for any orthonormal set $\varphi_i,\ i = 1, \dots, d$ in $L^2(\mathbb{R}^N)$ we have

$$\sum_{i=1}^{d}((\Delta + m(x)I)\varphi_i, \varphi_i)_{L^2(\mathbb{R}^N)} = -\sum_{i=1}^{d}\left(\int_{\mathbb{R}^N}|\nabla\varphi_i|^2 - \int_{\mathbb{R}^N} m(x)\varphi_i^2\right) \le \mu d$$

which according to (5.3) gives the result. ∎

The next result, known as the Lieb-Thirring inequality will also be of great help below, see [13].

Lemma 5.2. *Assume $\{\varphi_1, \dots, \varphi_d\} \subset H^1(\mathbb{R}^N)$ is an orthonormal set in $L^2(\mathbb{R}^N)$ and denote $\rho(x) := \sum_{i=1}^{d}(\varphi_i(x))^2$. Then for any p such that $\max\{1, \frac{N}{2}\} < p \le 1 + \frac{N}{2}$, there exists a constant $K = K(N,p) > 0$ independent of d and of the set $\{\varphi_1, \dots, \varphi_d\}$ such that*

$$K\left(\int_{\mathbb{R}^N}\rho(x)^{\frac{p}{p-1}}\,dx\right)^{\frac{2(p-1)}{N}} \le \sum_{j=1}^{d}\int_{\mathbb{R}^N}|\nabla\varphi_j|^2\,dx.$$

Then we have the following result. Note that unlike references [1, 6, 7, 9], we allow the linear term to be space dependent and sign changing.

Theorem 5.3. *Assume the nonlinear term in (4.1) satisfies (4.6) for some $C \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > 2$ and $0 \le D \in L^p(\mathbb{R}^N)$, $p > \frac{N}{2}$ and*

$$|\frac{\partial^2 f}{\partial s^2}(x,s)| \le C(R), \quad \textit{for} \quad |s| \le R, \quad x \in \mathbb{R}^N.$$

Assume (4.1) has a compact invariant set $\mathcal{A}$ such that

$$\mathcal{A} \subset L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \quad \textit{is bounded.}$$

Assume furthermore that the operator $\Delta + C(x)I$ has a negative exponential type. Then the Hausdorff and fractal dimensions of $\mathcal{A}$ are finite.

Proof. We write the linearization as

$$A_1(t) := \Delta + \frac{\partial f}{\partial u}(x, \varphi(x)) + \frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x)) \tag{5.4}$$

and the key point is to construct $\varphi(x)$ in such a way that $\varphi \in L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ for $p > \frac{N}{2}$ and the operator $\Delta + \frac{\partial f}{\partial u}(x, \varphi(x))$ has a negative exponential type, see [4] for details. Note that $\frac{\partial f}{\partial u}(\cdot, \varphi(\cdot)) \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$ since $|\frac{\partial f_0}{\partial u}(x,u)| \le C(R)|u|$, for $|u| \le R$, and since $\varphi \in L^\infty(\mathbb{R}^N)$ we get $|\frac{\partial f_0}{\partial u}(\cdot, \varphi(\cdot))| \in L^\infty(\mathbb{R}^N) \subset L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$.

Now we write the linearized operator as

$$A_1(t) = (1-\delta)[\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))] + \delta\Delta + \frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x))$$

and we chose $\delta \in (0,1)$ sufficiently small such that the operator $\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))$ has a negative exponential type that we still denote $\mu < 0$. Hence from Lemma 5.1, $Tr_d(\Delta + \frac{1}{1-\delta}\frac{\partial f}{\partial u}(x, \varphi(x))) \le \mu d$ and then for any choice of $\varphi_1, \varphi_2, \dots, \varphi_d \in H^1(\mathbb{R}^N)$, which are orthonormal in $L^2(\mathbb{R}^N)$ and span a subspace E_d, we get

$$\begin{aligned} Tr_d(A_1(t), E_d) \le (1-\delta)\mu d - \delta \sum_{i=1}^d \int_{\mathbb{R}^N} |\nabla \varphi_i|^2 \, \mathrm{d}x \\ + \sum_{i=1}^d \int_{\mathbb{R}^N} \left(\frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x)) \right) \varphi_i^2 \, \mathrm{d}x. \end{aligned}$$

Denoting $\rho(x) := \sum_{i=1}^d \varphi_i(x)^2$ and applying Lemma 5.2, we get

$$\begin{aligned} Tr_d(A_1(t), E_d) \le \frac{(1-\delta)\mu d}{2} - \delta K \int_{\mathbb{R}^N} \rho(x)^{1+\frac{2}{N}} \, \mathrm{d}x \\ + \int_{\mathbb{R}^N} J(t,x)\rho(x) \, \mathrm{d}x. \end{aligned}$$

since $\int_{\mathbb{R}^N} \rho(x) \, \mathrm{d}x = d$, where $J(t,x) := \max\{0, [\frac{\partial f}{\partial u}(x, u(t,x)) - \frac{\partial f}{\partial u}(x, \varphi(x))] + \frac{\mu(1-\delta)}{2}\}$. Then setting $V(t) = \left(\int_{\mathbb{R}^N} |J(t,x)|^{1+\frac{N}{2}} \, \mathrm{d}x \right)^{\frac{2}{N+2}}$ and $y = \left(\int_{\mathbb{R}^N} \rho(x)^{1+\frac{2}{N}} \, \mathrm{d}x \right)^{\frac{N}{N+2}}$ we get

$$Tr_d(A_1(t), E_d) \le \frac{(1-\delta)\mu d}{2} - C y^{\frac{N+2}{N}} + V(t) y.$$

Now, Young's inequality gives, for every $\varepsilon > 0$, $V(t)y \le \varepsilon y^{\frac{N+2}{N}} + C_\varepsilon V(t)^{\frac{N+2}{2}}$ and taking $\varepsilon = \frac{C}{2}$ and the sup in all subspaces E_d we get

$$Tr_d(A_1(t)) \le \frac{(1-\delta)\mu d}{2} + C_1 V(t)^{\frac{N+2}{2}}. \tag{5.5}$$

Hence condition (5.2) is satisfied provided

$$\frac{2C_1}{(1-\delta)|\mu|} \limsup_{t\to\infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t V(\tau)^{\frac{N}{2}+1} \, \mathrm{d}\tau < d. \tag{5.6}$$

Now for any trajectory in $\mathcal{A}$ and $\hat{\delta} > 0$ we split

$$V(t)^{\frac{N}{2}+1} = \int_{\{x\in\mathbb{R}^N,\, |u(t,x)-\varphi(x)|\le\hat{\delta}\}} |J(t,x)|^{\frac{N}{2}+1} + \int_{\{x\in\mathbb{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N}{2}+1}. \tag{5.7}$$

From (5.3), using that $\|u\|_{L^\infty(\mathbb{R}^N)} \le R$ for all $u \in \mathcal{A}$ and $\|\varphi\|_{L^\infty(\mathbb{R}^N)} \le R$, we get

$$|\frac{\partial f}{\partial u}(x,\varphi(x)) - \frac{\partial f}{\partial u}(x,u(t,x))| \le C(R)|\varphi(x) - u(t,x)|, \quad x \in \mathbb{R}^N. \tag{5.8}$$

Hence, we chose $\hat{\delta}$ such that if $|u-\varphi| < \hat{\delta}$, then $C(R)\hat{\delta} < \frac{\mu(\delta-1)}{2}$. Thus we have

$$\int_{\{x\in\mathbb{R}^N,\ |u(t,x)-\varphi(x)|\le\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}}\, \mathrm{d}x = 0.$$

Now we deal with the second term in (5.7). Since $\mathcal{A}$ is bounded in $L^\infty(\mathbb{R}^N)$ and $\varphi \in L^\infty(\mathbb{R}^N)$, we get $|J(t,x)| \le C_{\mathcal{A},\varphi} + \frac{|\mu|(1-\delta)}{2} := K_1$ and thus

$$\int_{\{x\in\mathbb{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}} \le K_1^{\frac{N+2}{2}} |\{x \in \mathbb{R}^N : |u(t,x)-\varphi| > \hat{\delta}\}|. \tag{5.9}$$

Using now that $\mathcal{A}$ is bounded in $L^p(\mathbb{R}^N)$ and $\varphi \in L^p(\mathbb{R}^N)$ we get that for all $u \in \mathcal{A}$

$$\hat{\delta}^p|\{x \in \mathbb{R}^N,\, |u(x)-\varphi(x)| > \hat{\delta}\}| \le \int_{\mathbb{R}^N} |u(x)-\varphi(x)|^p\ \mathrm{d}x \le C = C(\mathcal{A}, \|\varphi\|^p_{L^p(\mathbb{R}^N)}),$$

and then substituting in (5.9) we get

$$\int_{\{x\in\mathbb{R}^N,\, |u(t,x)-\varphi(x)|>\hat{\delta}\}} |J(t,x)|^{\frac{N+2}{2}}\ \mathrm{d}x \le K_1^{\frac{N+2}{2}} \frac{C(\mathcal{A}, \|\varphi\|^p_{L^p(\mathbb{R}^N)})}{\hat{\delta}^p}.$$

From (5.6) we get the result. □

Now we illustrate the scope of the result above with the following example. Consider a prototype problem (4.1) with bistable nonlinear term

$$f(x,s) = m(x)s - n(x)s^3.$$

Note that as soon as $0 \le n \in L^\infty(\mathbb{R}^N)$ then the assumption above are satisfied. Assume there exists a decomposition

$$m(x) = m_0(x) - m_1(x), \quad \text{with} \quad m_0, m_1 \in L^\sigma_U(\mathbb{R}^N), \quad \sigma > N/2,\ \sigma > 2,$$

such that the operator $\Delta - m_1(x)$ has negative exponential type. Hence, using Young's inequality we have that (4.6) is satisfied with

$$C(x) = -m_1(x), \quad D(x) = A\frac{|m_0|^{3/2}(x)}{n^{1/2}(x)}$$

for some constant A.

Moreover, it was proved in [2] that the attractor $\mathcal{A}$ of (4.1) satisfies the remaining assumptions in THEOREM 5.3, that is

$$\mathcal{A} \subset L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \quad \text{is bounded}$$

provided

$$\frac{|m_0|^{3/2}}{n^{1/2}} \in L^r(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$$

for some $r > N/3$, $p > \frac{N}{2}$ and $2 \geq s > \frac{2N}{N+4}$.

Note that a source term $g(x) = f(x, 0)$ can also be considered as long as

$$g \in L^\infty(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N);$$

see [2, Theorem 5.2] for sharper assumptions on g.

The particular case of the theorem above, when f_0 does not depend on x improves conditions in [6, Theorem 3.3].

THEOREM 5.4. *Consider* (4.1) *with* f *as in* (4.3) *and* $f_0 = f_0(s)$ *with* $g \in L^2(\mathbb{R}^N)$, $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, *and*

$$f_0 \in C^2(\mathbb{R}), \quad \text{with} \quad f_0(0) = 0 = f_0'(0). \tag{5.10}$$

Assume a compact invariant set, $\mathcal{A}$, *exists and satisfies*

$$\mathcal{A} \subset L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \quad \text{is bounded.} \tag{5.11}$$

Finally assume the operator $\Delta + m(x)I$ *has a negative exponential type. Then the Hausdorff and fractal dimensions of* $\mathcal{A}$ *are finite.*

Finally, note that the technique above can be adapted for nonlinear terms depending on the gradient, see [4] for further details.

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