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SEMILINEAR PROBLEMS PERTURBED THROUGH THE BOUNDARY CONDITION*

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Abstract. We analyze the semilinear diffusion equation $\Delta u = a(x)u^p$ in a smooth bounded domain Ω subjected to the boundary condition $\partial u/\partial\nu = \lambda u$, where ν is the outward unit normal to $\partial\Omega$, λ is a real parameter. The coefficient $a(x)$ is a nonnegative weight function, which could even vanish in a whole smooth subdomain Ω_0 of Ω. We consider both cases $p > 1$ and $0 < p < 1$, and give a detailed description of the existence, uniqueness or multiplicity and asymptotic behavior of nonnegative solutions. As an additional special feature, the adherence of the portions $\Omega \cap \partial\Omega_0$, $\partial\Omega \cap \partial\Omega_0$ of the boundary of Ω_0 are allowed to meet each other in a smooth manifold.

Key words. Bifurcation, Steklov problem, boundary blow-up, perturbation of domains.

AMS subject classifications. 35J25, 35B40.

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1. Introduction. In this note we are giving a summary of the issues of existence and uniqueness – alternatively multiplicity – of positive solutions to the semilinear problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda u & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a $C^{2,\alpha}$ bounded domain, $a \in C^\alpha(\overline{\Omega})$, $a(x) \geq 0$.

The main features of problem (1.1) are: a) the presence of the bifurcation parameter λ in the boundary condition, b) the possibility that $a \geq 0$ can vanish in a whole subdomain Ω_0 of Ω and c) the two regimes of the problem: "regular" and "degenerate", corresponding to the ranges $p > 1$ and $0 < p < 1$ of the exponent p respectively. In particular, we are paying special attention to the study of the asymptotic profiles of the solutions when the parameter λ tends to critical values λ_c where bifurcation phenomena occur. In the degenerate case, the study of the regions where the solutions u vanish identically ("dead cores") will be an additional issue to deal with.

Problem (1.1) constitutes another example of the group of models where a "dissipative" ("absorption") mechanism is in competition with another one of "production" ("radiation"). The logistic problem under either of the three classical boundary conditions is the paradigmatic example of this setting (see [8], [2], [3], [1] for the discussion of phenomena which are related to the ones covered here).

In the rest of the note we will assume that the coefficient $a(x)$ verifies the following hypothesis:

H) *$a \in C^\alpha(\overline{\Omega})$ is nonnegative and nontrivial. Moreover, it is either strictly positive in Ω or $a \equiv 0$ in a subdomain $\Omega_0 \subset \Omega$ of class $C^{2,\alpha}$.*

For immediate reference we are fixing the following notations: $\Gamma_2 = \Omega \cap \partial\Omega_0$, $\Gamma_1 = \partial\Omega_0 \setminus \Gamma_2$. In addition, $\Omega^+ = \{x \in \Omega : a(x) > 0\}$, $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$. Notice that since $a(x)$ is nontrivial then both Γ_2 and Ω^+ must always be non void meanwhile $\Omega \cap \partial\Omega^+ = \Gamma_2$.

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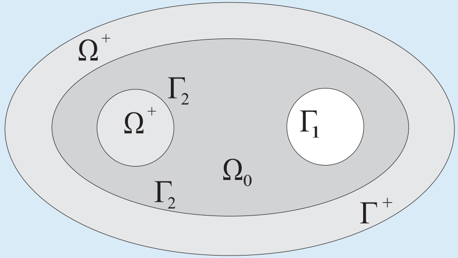

FIG. 1.1. *A possible configuration for an* Ω_0 *satisfying* H$)_s$.

In our previous works [4], [5], an important part of the results was obtained under the simplifying assumption,

H$)_s$ $\overline{\Gamma}_2 = \Gamma_2$,

the bar standing for "adherence". This means that Γ_2 is a closed $N-1$ dimensional manifold and, more importantly, that Γ_1 is separated away from Γ_2 ($\text{dist}(\Gamma_1, \Gamma_2) > 0$). In particular, Ω^+ also defines a $C^{2,\alpha}$ subdomain of Ω, while $\Omega_0, \Omega^+, \Gamma_1, \Gamma_2, \Gamma^+$ consist only of a finite number of connected pieces (FIG.1.1). We are reviewing the above mentioned results in Sections 3 and 4.

In the present work some of the main statements in [4] are significantly extended to cover more natural configurations for Γ_1, Γ_2 in $\partial\Omega_0$ than in $H)_s$. In particular, we allow $\overline{\Gamma}_2 \cap \Gamma_1 \neq \emptyset$ (see details in Sections 3, 4 and Figure 3.2).

REMARK 1. The connectedness condition in Ω_0 can be relaxed without significant changes in the results.

2. Eigenvalue problems. In the analysis of (1.1) several eigenvalue problems which are non-standard appear. In fact, some of them involve the eigenvalue in the boundary condition while other ones exhibit a non usual regime for the coefficients.

A first eigenvalue problem is

$$\begin{cases} \Delta\phi = \lambda\phi & x \in \Omega_0 \\ \dfrac{\partial\phi}{\partial\nu} = \mu\phi & x \in \partial\Omega, \end{cases} \tag{2.1}$$

$\mu > 0$ regarded as a parameter. The existence of a principal eigenvalue $\lambda_1(\mu)$ to (2.1) and its qualitative behavior with respect to μ was first studied in [7] and extended in several aspects in [4]. The corresponding principal eigenfunctions ϕ are a source of sub and supersolutions for problem (1.1) in the regime $p > 1$.

The description of the set of positive solutions for problem (1.1) when $\Omega_0 \neq \emptyset$ requires the introduction of several critical values for λ, characterized by means of some eigenvalue problems. Specifically, we define $\lambda = \sigma_1$ as the principal eigenvalue of the mixed Steklov-type eigenvalue problem:

$$\begin{cases} \Delta\phi = 0 & x \in \Omega_0 \\ \phi = 0 & x \in \Gamma_2 \\ \dfrac{\partial\phi}{\partial\nu} = \lambda\phi & x \in \Gamma_1. \end{cases} \tag{2.2}$$

Since it could happen that $\Gamma_1 = \emptyset$, we set in that case $\sigma_1 := \infty$ (Fig. 2.1).

In the same way we introduce $\lambda = \sigma_1^+$, which is the corresponding principal eigenvalue of the problem:

$$\begin{cases} \Delta\phi = 0 & x \in \Omega^+ \\ \phi = 0 & x \in \Gamma_2 \\ \dfrac{\partial\phi}{\partial\nu} = \lambda\phi & x \in \Gamma^+. \end{cases} \tag{2.3}$$

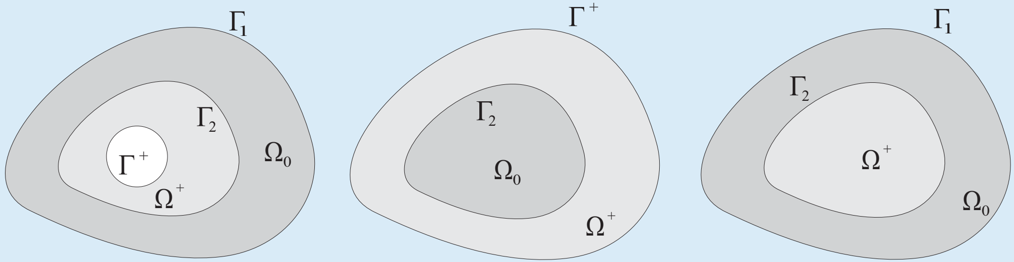

FIG. 2.1. *Several configurations for Ω_0 and Ω^+: $\Gamma_1, \Gamma^+ \neq \emptyset$, $\Gamma_1 = \emptyset, \Gamma^+ \neq \emptyset$ and $\Gamma_1 \neq \emptyset, \Gamma^+ = \emptyset$.*

We also set $\sigma_1^+ := \infty$ whenever $\Gamma^+ = \emptyset$.

We refer to [4] for a detailed study of the properties of the eigenvalue problems (2.2), (2.3) and some other related problems. We remark that σ_1^+ has to be suitably defined in case Ω^+ is not connected. In the first of the configurations in FIG. 2.1, both σ_1 and σ_1^+ are finite. One of them becomes infinite in the rest of the configurations.

3. The regime $p > 1$.

We are next describing the main features of problem (1.1) for the regular case where $p > 1$. In THEOREMS 3.1 to 3.3 it is assumed that $a(x)$ satisfies H) together with the "separation" hypothesis $\mathrm{H})_s$. In the first statement the value $\sigma_1 = \infty$ (see Section 2) is also used when referring to the case $a(x) > 0$ in Ω. Such results are collected in [4]. The section concludes with the study of the case where Γ_1 and $\overline{\Gamma}_2$ meet in a nontrivial way.

THEOREM 3.1. *Assume $a(x)$ satisfies H), $\mathrm{H})_s$ while $p > 1$. Then problem (1.1) admits positive solutions $u \in C^{2,\alpha}(\overline{\Omega})$ if and only if:*

$$0 < \lambda < \sigma_1 \leq \infty,$$

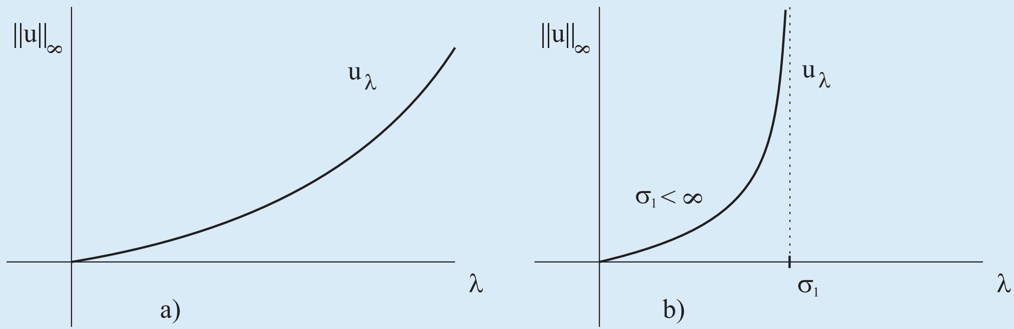

Fig. 3.1. *a)* $a > 0$ *in* Ω *or* $\Omega_0 \neq \emptyset$ *but* $\sigma_1 = +\infty$*, and b)* $\sigma_1 < +\infty$.

the solution $u = u_\lambda(x)$ *being unique in that case. Moreover, the mapping* $\lambda \longmapsto u_\lambda$ *is increasing and real analytic when considered with values in* $C^{2,\alpha}(\overline{\Omega})$*, and* u_λ *is globally attractive (among the positive solutions). Finally,* $u_\lambda \to 0$ *as* $\lambda \to 0+$*, while* $|u_\lambda|_{\infty,\Omega} \to \infty$ *when* $\lambda \to \sigma_1$.

Remark 2. Theorem 3.1 allows the possibility that $a(x)$ vanishes somewhere on $\partial\Omega$ in the case where $a(x)$ is positive in Ω ($\Omega_0 = \emptyset$), also permitting the existence of zeros for $a(x)$ in Γ^+ if both Ω_0 and Γ^+ are nonempty ($a(x) \equiv 0$ on Γ_1!).

It has been established that if $\Omega_0 \neq \emptyset$ and $\sigma_1 < \infty$, positive solutions disappear when λ crosses the value σ_1. The next results explain this interruption by analyzing the singularities developed by the solutions. Even if $\sigma_1 = \infty$, the solution u_λ approaches a finite profile in Ω when λ tends to infinity.

Theorem 3.2. *Assume* $a(x) > 0$ *for every* $x \in \Omega$*, or* $\Omega_0 \neq \emptyset$ *but* $\sigma_1 = \infty$*. Then the solution* u_λ *of* (1.1) *satisfies* $u_\lambda \to u$ *in* $C^{2,\alpha}(\Omega)$ *as* $\lambda \to \infty$ *where* $u(x)$ *is the minimal solution to* *the singular boundary value problem:*

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega \\ u = \infty & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Moreover, we have the following estimate of the asymptotic growth rate:

$$\varliminf_{\lambda\to\infty} \lambda^{-\frac{2}{p-1}} \sup u_\lambda \geq |a|_\infty^{-\frac{1}{p-1}}.$$

Theorem 3.3. *Under the condition $\sigma_1 < \infty$ and hence $\Gamma_1 = \partial\Omega_0 \cap \partial\Omega \neq \emptyset$, the solution u_λ to (1.1) satisfies $u_\lambda \to \infty$ uniformly in $\overline{\Omega}_0$ as $\lambda \to \sigma_1-$. In addition, $u_\lambda \to u$ in $C^{2,\alpha}(\Omega^+)$ where $u(x)$ is the minimal solution to the singular boundary value problem:*

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty & x \in \Gamma_2 \\ \dfrac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+, \end{cases} \tag{3.2}$$

provided that $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega \neq \emptyset$, or $u(x)$ is the minimal solution to the problem:

$$\begin{cases} \Delta u = a(x)u^p & x \in \Omega^+ \\ u = \infty & x \in \partial\Omega^+, \end{cases} \tag{3.3}$$

in case $\Gamma^+ = \emptyset$.

Remark 3. Observe that we prove the existence of nontrivial solutions of (3.2). This is a novelty in view of the boundary condition on Γ^+ not previously treated in the literature

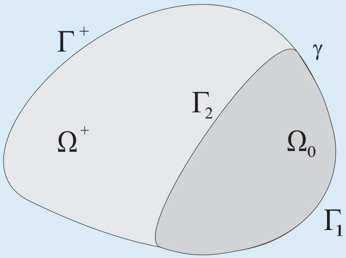

FIG. 3.2. *Configuration for Ω_0 with $\overline{\Gamma}_2$ and Γ_1 contacting in a smooth manifold γ.*

(see specially the corresponding problem in THEOREM 3.4). On the other hand, suitable conditions on the weight $a(x)$ can be given in order to obtain uniqueness of positive solutions to the singular problems (3.1), (3.2) and (3.3).

We are next dealing with the hypothesis H) under a less restrictive assumption than $\mathrm{H})_s$ regarding the separation between Γ_1, $\overline{\Gamma}_2$. Namely, it will be assumed that $\Omega_0 \subset \Omega$ is a C^3 subdomain such that (recall that $\Gamma_2 = \Omega \cap \partial\Omega_0$, $\Gamma_1 = \partial\Omega \cap \partial\Omega_0$),

$\mathrm{H})_m$ $\Gamma_1, \overline{\Gamma}_2$ *are nonempty $N-1$-dimensional manifolds having as common boundary γ, a $(N-2)$-dimensional closed submanifold of $\partial\Omega$* (FIG. 3.2).

The existence, under these conditions on Ω_0, of a unique principal weak eigenvalue $0 < \sigma_1 < \infty$ to (2.2) with a positive associated eigenfunction $\phi_1 \in H^1(\Omega_0) \cap W^{2,s}(\Omega_0)$, $1 < s < 4/3$, can be ensured by variational methods (see [5]).

Our main statement concerning this new framework for problem (1.1) essentially asserts that THEOREMS 3.1 and 3.3 are still valid even when Γ_1 and $\overline{\Gamma}_2$ meet each other.

THEOREM 3.4. *Suppose that* $a \in C^{\alpha}(\overline{\Omega})$ *satisfies* H) *together with condition* $\mathrm{H})_m$. *Then problem* (1.1) *admits a weak positive solution in* $H^1(\Omega)$ *if and only if,*

$$0 < \lambda < \sigma_1.$$

Such solution is unique and defines indeed a classical solution in $C^{2,\alpha}(\overline{\Omega})$. *As a mapping of* $\lambda \in (0, \sigma_1)$, u_λ *is increasing, smooth and bifurcates from zero at* $\lambda = 0$. *Moreover,*

$$u_\lambda \to u,$$

as $\lambda \to \sigma_1-$ *in* $C^{2,\alpha}(\Omega^+) \cap C^{1,\alpha}(\Omega^+ \cup T)$ *for every compact* $T \subset \Gamma^+$ *while,*

$$u_\lambda \to \infty,$$

as $\lambda \to \sigma_1-$ *uniformly on each compact set* $K \subset \Gamma_2$. *In particular:*

$$\lim_{\mathrm{dist}(x,K) \to 0} u(x) = \infty,$$

and thus u *defines a classical solution to problem* (3.2),

$$\begin{cases} \Delta u = a(x) u^p & x \in \Omega^+ \\ u = \infty & x \in \Gamma_2 \\ \dfrac{\partial u}{\partial \nu} = \sigma_1 u & x \in \Gamma^+. \end{cases}$$

REMARK 4. In contrast to the more regular case of the hypothesis $\mathrm{H})_s$, the approach to achieve existence in THEOREM 3.4 is variational. In this regard, existence can be also obtained if Ω_0 is merely Lipschitz (cf. [4, Theorem 2]). An additional regularity argument then shows that any weak $H^1(\Omega)$ solution to (1.1) lies indeed in $L^\infty(\Omega)$ and hence its smoothness improves up to be a classical solution (see further details in [6]). On the other hand, it is a more subtle question to decide which is the possible asymptotic profile of u_λ in the interphase γ as $\lambda \to \sigma_1-$.

4. Degenerate regime. Let us now analyze problem (1.1) in the degenerate range $0 < p < 1$. The results we describe in what follows are contained in the work [5] with the exception of those involving the condition $\mathrm{H})_m$. We are denoting $\alpha_1 = \min\{p, \alpha\}$.

THEOREM 4.1. *Assume that the coefficient $a \in C^{\alpha}(\overline{\Omega})$ satisfies $a(x) > 0$ for every $x \in \Omega$. Then:*

i) [Existence] *Problem (1.1) admits at least one nonnegative solution $u \in C^{2,\alpha_1}(\overline{\Omega})$, $u \not\equiv 0$, for every $\lambda > 0$ and no nonnegative solutions exist if $\lambda \le 0$.*

ii) [Bifurcation from infinity] *For certain $\lambda_0 > 0$ small enough and $0 < \lambda < \lambda_0$ there exists a unique positive solution $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$. The mapping $\lambda \longmapsto u_\lambda$ is real analytic in $(0, \lambda_0)$ with values in $C^{2,\alpha}(\overline{\Omega})$. Moreover, it is decreasing and satisfies:*

$$\lim_{\lambda \to 0+} \lambda^{\frac{1}{1-p}} \, u_\lambda(x) = \left(\frac{1}{|\partial\Omega|} \int_\Omega a \right)^{\frac{1}{1-p}}$$

where the limit is taken in $C^{2,\alpha}(\overline{\Omega})$.

iii) [L^∞ estimate] *There exist constants $\lambda_1 > 0$, $C > 0$ such that every nonnegative solution u corresponding to $\lambda \ge \lambda_1$ satisfies:*

$$0 \le u(x) \le C\lambda^{-\frac{2}{1-p}}.$$

iv) [Dead core formation] *Every nonnegative solution $u_\lambda \not\equiv 0$ corresponding to $\lambda \ge \lambda_2$, for a certain $\lambda_2 > 0$, develops a dead core $\mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$ such that $\mathcal{O}_\lambda \to \Omega$ as $\lambda \to \infty$. More precisely:*

$$\{x : \operatorname{dist}(x, \partial\Omega) \ge d(\lambda)\} \subset \mathcal{O}_\lambda,$$

where $d(\lambda) \to 0+$ as $\lambda \to \infty$. Furthermore, $d(\lambda)$ can be chosen as $d(\lambda) = \dfrac{K}{\lambda}$, for a certain constant $K > 0$, provided that $a(x) > 0$ on $\partial\Omega$.

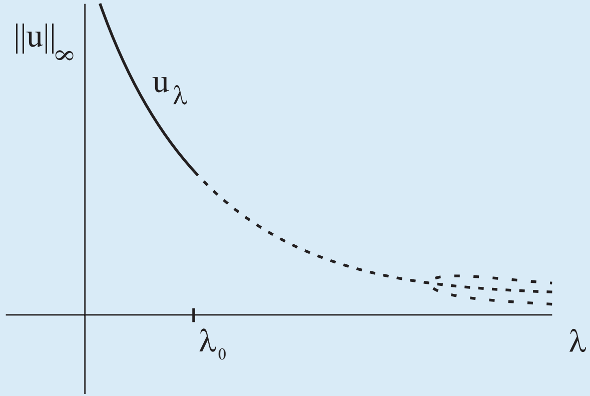

Fig. 4.1. *Bifuraction diagram for $a(x) > 0$: the continuous line means uniqueness.*

An important feature of the degenerate regime is that it exhibits multiple solutions when λ is sufficiently large. This is shown in the next result.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be a $C^{2,\alpha}$ domain whose boundary $\partial\Omega$ splits in k connected components, while $a \in C^{\alpha}(\overline{\Omega})$ is positive in Ω. Then problem (1.1) admits at least $2^k - 1$ nonnegative nontrivial solutions when λ is large enough.*

In view of this result, a question naturally arises: is the non connectedness of $\partial\Omega$ responsible for the multiplicity of solutions? The answer – in the negative – can be found in the next result, where Ω is a ball, hence $\partial\Omega$ is connected.

Theorem 4.3. *Consider problem (1.1) in a ball B of $\mathbb{R}^N$, with the coefficient $a(x)$ radial and positive. Then problem (1.1) admits for every $\lambda > 0$ a radial nonnegative solution $u \neq 0$. Such solution is unique for $0 < \lambda < \lambda_0$. Moreover:*

i) *There exists a* unique *radial nonnegative solution* $u_\lambda \neq 0$ *for large* λ *that satisfies:*

$$\mathit{dist}\,(\mathcal{O}_\lambda, \partial B) \sim \beta\lambda^{-1}, \qquad u_\lambda(1) \sim A\beta^\beta\lambda^{-\beta},$$

as $\lambda \to +\infty$ *where* $\beta = 2/(1-p)$, $A = [\beta(\beta-1)]^{-1/(1-p)}$ *and* $\mathcal{O}_\lambda = \{x \in B : u_\lambda = 0\}$.

ii) *There exists* $\lambda_3 > 0$ *such that problem* (1.1) *admits a solution* $u \neq 0$, *nonnegative and* nonradial *for every* $\lambda \geq \lambda_3$.

Remark 5. Suppose that $a(x)$ satisfies H) with a nonempty domain Ω_0 so that $\sigma_1 = \infty$. Since this is equivalent to $\overline{\Omega}_0 \subset \Omega$ ($\Gamma_1 = \emptyset$, see Section 2) then $\mathrm{H})_s$ is satisfied and it is possible to deduce for problem (1.1) the same conclusions obtained in Theorem 4.1. Namely:

i) For $\lambda > 0$, problem (1.1) admits at least a nonnegative nontrivial solution $u \in C^{2,\alpha_1}(\overline{\Omega})$, and no solutions exist if $\lambda \leq 0$.

ii) There exists a unique positive solution $u \in C^{2,\alpha}(\overline{\Omega})$ for $0 < \lambda < \lambda_0$ which bifurcates from infinity in $\lambda = 0$, with $u_\lambda(x) \sim \left(\frac{1}{|\partial\Omega|}\int_\Omega a\right)^{\frac{1}{1-p}} \lambda^{-\frac{1}{1-p}}$ as $\lambda \to 0+$.

iii) For large λ, nonnegative solutions u verify the estimate $0 \leq u(x) \leq C\lambda^{-\frac{2}{1-p}}$, where the constant C does not depend on u.

iv) Also for large λ, all nonnegative solutions u develop a dead core $\mathcal{O}_\lambda$ which satisfies $\{x : \mathrm{dist}(x, \partial\Omega) \geq d(\lambda)\} \subset \mathcal{O}_\lambda = \{u_\lambda(x) = 0\}$, $d(\lambda) \to 0$ as $\lambda \to \infty$.

However, the situation is completely different to the regular regime $p > 1$ when $\sigma_1 < \infty$ ($\Gamma_1 \neq \emptyset$). In strong contrast with the regular regime $p > 1$, the eigenvalue σ_1^+ plays an important rôle in the degenerate case $0 < p < 1$. The next statement gives precise information on problem (1.1) when $\sigma_1 < \infty$.

Theorem 4.4. *Suppose that* $a(x)$ *satisfies* H) *with* $\sigma_1 < \infty$ *and assume either of the conditions* $\mathrm{H})_s$ *or* $\mathrm{H})_m$. *Then there exists at least one nonnegative nontrivial solution* $u \in C^{2,\alpha_1}(\overline{\Omega})$, $\alpha_1 = \min\{p, \alpha\}$, *with the same properties as the corresponding nonnegative solutions described in* Theorem 4.1 *whenever* $0 < \lambda < \sigma_1$. *Moreover:*

i) *All nonnegative solutions* u *corresponding to* $\lambda \geq \sigma_1$ *satisfy* $u \equiv 0$ *in* Ω_0.

ii) *If* $\sigma_1^+ = \infty$, *there do not exist nonnegative nontrivial solutions for* $\lambda \geq \sigma_1$.

iii) *If* $\sigma_1^+ < \infty$, *nonnegative nontrivial solutions can only occur in the range* $\lambda > \sigma_1^+$. *In particular, such solutions cannot exist for* λ *verifying:*

$$\sigma_1 \leq \lambda \leq \sigma_1^+,$$

assuming $\sigma_1 \leq \sigma_1^+$ *(see* FIG. 4.2*).*

iv) *Assume that condition* H)$_s$ *holds. Then, for* $\sigma_1^+ < \infty$ *certain* $\lambda_1 > \sigma_1^+$ *exists such that problem* (1.1) *admits at least one nonnegative nontrivial solution* u *for every* $\lambda \geq \lambda_1$. *Such solutions satisfy the estimate:*

$$0 \leq u(x) \leq C\ \lambda^{-\frac{2}{1-p}},$$

and develop a dead core which verifies $\mathcal{O}_\lambda \to \Omega$ *as* $\lambda \to \infty$ *in the form described in* REMARK 5-iv).

To conclude with the description of the qualitative properties of nonnegative solutions to problem (1.1) in the degenerate case, we give sufficient conditions providing the existence of a bifurcation from $u = 0$ in $\lambda = \sigma_1$.

THEOREM 4.5. *Assume that* $a(x)$ *satisfies* H) *and either of the hypotheses* H)$_s$ *or* H)$_m$. *If* $\sigma_1 < \infty$ *and one of the following conditions holds: either* $\sigma_1 \leq \sigma_1^+$ *or* $\sigma_1^+ < \sigma_1$ *but* $u = 0$ *is the only nonnegative solution for* $\lambda = \sigma_1$, *then:*

$$u_\lambda \to 0 \qquad as\ \ \lambda \to \sigma_1-,$$

in $C^{2,\alpha_1}(\overline{\Omega})$. *On the other hand, every possible solution develops a dead core* $\mathcal{O}_\lambda \subset \Omega^+$ *such that* $\mathcal{O}_\lambda \to \Omega^+$ *as* $\lambda \to \sigma_1-$.

REMARK 6. Last assertion in THEOREM 4.5 implies that solutions u_λ of (1.1) converging to zero as $\lambda \to \sigma_1-$ develop their dead cores $\mathcal{O}_\lambda$ into Ω^+, being strictly positive in Ω_0. This

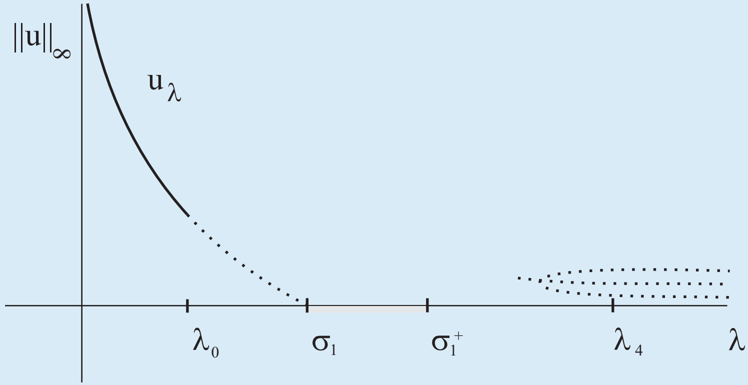

Fig. 4.2. *Bifurcation when $\sigma_1 < \infty$: the continuous line stands for uniqueness.*

should be contrasted with the behavior of nonnegative solutions to the problem (if any) in the range $\lambda \geq \sigma_1$. In fact, the latter must be identically zero in Ω_0 (Theorem 4.4, i)).

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