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REMARKS ON THE INCOMPRESSIBLE NAVIER-STOKES FLOWS FOR LINEARLY GROWING INITIAL DATA*

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Abstract. We deal with the Cauchy problem of the Navier-Stokes equations with linearly growing initial data $U_0 := -Mx + u_0(x)$. Here M is an $n \times n$ matrix with assumptions $\operatorname{tr} M = 0$ and M^2 is symmetric, and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. We establish the local-in-time solvability applied Ornstein-Uhlenbeck semigroup theory. We also show that our solution is analytic in x, if $\| \mathrm{e}^{tM} \| \le 1$ for all $t \ge 0$, nevertheless, the semigroup is not analytic.

Key words. Navier-Stokes equations, mild solution, Ornstein-Uhlenbeck semigroup, regularizing rate, analyticity

AMS subject classifications. 42B20, 42B25, 35J45

1. Introduction. We consider the incompressible and viscous fluid flows in $\mathbb{R}^n$ for initial velocity which grows linearly at space-infinity, which are described the Navier--Stokes equations, i.e.

$$\begin{aligned} U_t - \Delta U + (U,\nabla)U + \nabla P = 0 & \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0 & \quad \text{in} \quad \mathbb{R}^n \times (0,T), \\ U(0) = U_0 \quad \text{with} \quad \nabla \cdot U_0 = 0 & \quad \text{in} \quad \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here $U = U(t) = (U^1(x,t), \dots, U^n(x,t))$ and $P = P(x,t)$ stand for the unknown velocity and the unknown pressure of the fluid; $U_0 = (U_0^1(x), \dots, U_0^n(x))$ is the given initial velocity. There are many contributions of literatures on existence of solutions of (1.1) in the whole space, see e.g. [1, 5, 6, 7, 9, 13, 23]. All these results assume that the initial data decay as $|x| \to \infty$. On the other hand, Okamoto [26] showed that for certain concrete flow problems there exist many exact solutions $\bar{U}$ which have the property that $\bar{U}$ grows linearly as $|x| \to \infty$.

Our purpose is to construct mild solutions to the equations of Navier-Stokes in $L^p_\sigma(\mathbb{R}^n)$, when the initial datum may grow as $-Mx$, where $M = (m_{ij})_{1 \le i,j \le n}$ is a real-valued constant matrix satisfying $\operatorname{tr} M = 0$ and M^2 is symmetric. We hence assume throughout this paper that the initial velocity is of the form

$$U_0(x) = -Mx + u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where $u_0 \in L^p_\sigma(\mathbb{R}^n)^n$ is a function.

In the case $M = 0$, it is well known that there exists a local-in-time smooth solution to (1.1) provided the initial data U_0 belongs to $L^p_\sigma(\mathbb{R}^n)$ for $p \ge n$; see e.g. the articles in the list of References. However, if $M \ne 0$, the situation is more complicated.

We shall explain the reason why we study (1.1) with (1.2) in Physical point of view.

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Let us think about the case M is skew-symmetric, e.g.

$$M = R := \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for $a \in \mathbb{R}$, Notice that $\bar{U} := -Rx$ describes the pure rotation of the fluid. This problem was investigated by Hishida and by Babin, Mahalov and Nicolaenko. Indeed, Hishida constructed in [18, 19, 20] a unique local-in-time mild solution, provided that $u_0 \in H^{1/2}(\mathbb{R}^3)$, and its initial-boundary value problem in the exterior domain is also considered. (We will see the notion of a mild solution below.) Babin, Mahalov and Nicolaenko [3], [4] also proved the existence of a local-in-time mild solution (and global regularity theorem), provided u_0 is in $L^p_\sigma(\mathbb{R}^3)$ or u_0 is a periodic function enjoying the smoothness.

In [29], the second author of this paper proved the existence of a unique local-in-time mild solution, still for $M = R$, provided u_0 belongs to the Besov space $\dot{B}^0_{\infty,1}$

$$\dot{B}^0_{\infty,1} := \{f \in \mathcal{S}'; \sum_{j=-\infty}^{\infty} \|\varphi_j * f\|_\infty < \infty,\ f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}' \text{ sense}\}.$$

Note that $\dot{B}^0_{\infty,1} \subset BUC$ (and this embedding is continuous), where BUC denotes the space of bounded and uniformly continuous functions. The virtue of $\dot{B}^0_{\infty,1}$ and several example of functions are found in [30].

An interesting example of M is

$$M = J := \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix}$$

for $b \in \mathbb{R}$. According to Majda [24], $-Jx$ for $b < 0$ corresponds to the drain along to x_1 and x_2-axises horizontally and to the jet along to x_3-axis of the fluid. He showed that $(\bar{U}, \bar{P})$ is an exact solution of (1.1), where

$$\bar{U} := -Mx, \quad \bar{P} := (\Pi x, x),$$

and $\Pi := \frac{1}{2}[(M^{\mathrm{sym}})^2 + (M^{\mathrm{ant}})^2]$ under the assumptions that $\operatorname{tr} M = 0$ and M^2 is symmetric. We have denoted by M^{sym} and M^{ant} the symmetric and skew-symmetric part of M, respectively, i.e., $M^{\mathrm{sym}} := \frac{1}{2}(M + M^T)$ and $M^{\mathrm{ant}} := \frac{1}{2}(M - M^T)$. Here M^T denotes the transposed matrix of M.

Giga and Kambe [11] also investigated the axisymmetric irrotational flow (mainly, the behavior of its vortex), and studied the stability of the vortex when the velocity field of the fluid U is expressed as $U = -Jx + V$, where $V = (V^1, V^2, 0)$ is a two-dimensional velocity field.

This paper is organized as follows. In Section 2 we state the main results. We shall refer to key lemmas for proving the theorems in Section 3. In Section 4, we show the proof briefly.

2. Main results. In this section we refer to the main results in this paper. Before stating our main theorem, we consider a simple substitute as follows:

$$u := U - \bar{U} = U + Mx, \quad p := \tilde{P} - \bar{P} = P - (\Pi x, x).$$

Note that, if (U, P) is the classical solution of (1.1), then $(u, \tilde{P})$ should satisfy the following equations in classical sense:

$$\begin{aligned} u_t + Au + (u,\nabla)u - 2Mu + \nabla\tilde{P} &= 0 && \text{in } \mathbb{R}^n \times (0,T),\\ \nabla\cdot u &= 0 && \text{in } \mathbb{R}^n \times (0,T), \\ u(0) &= u_0 \quad \text{with } \nabla\cdot u_0 = 0 && \text{in } \mathbb{R}^n. \end{aligned} \tag{2.1}$$

Here we have defined the operator A by

$$Au := -\Delta u - (Mx,\nabla)u + Mu$$

with domain

$$D(A) := \{u \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n); (Mx,\nabla)u \in L^p(\mathbb{R}^n)\}.$$

Thanks to the results of Ornstein-Uhlenbeck semigroup theory by e.g. [25], we know that $-A$ generates a (C_0)-semigroup in $L^p_\sigma(\mathbb{R}^n)$ for $p \in [1,\infty)$. Also, $-A$ generates a semigroup in L^∞_σ. We also have a representation form of semigroup

$$\mathrm{e}^{-tA} f(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}}\,\mathrm{e}^{-tM}\int_{\mathbb{R}^n} f(\mathrm{e}^{tM}x - y)\,\mathrm{e}^{-\frac{1}{4}(Q_t^{-1}y,y)}\,\mathrm{d}y,$$

where $Q_t := \int_0^t \mathrm{e}^{sM}\mathrm{e}^{sM^T}\,\mathrm{d}s$ for all $t > 0$. Note that this semigroup is not analytic; see e.g. [18]. Using this semigroup, we deduce the integral equation by Duhamel's principle:

$$u(t) = \mathrm{e}^{-tA}u_0 - \int_0^t \mathrm{e}^{-(t-s)A}\,\mathbf{P}\nabla\cdot(u(s)\otimes u(s))\,\mathrm{d}s + 2\int_0^t \mathrm{e}^{-(t-s)A}\,\mathbf{P}u(s)\,\mathrm{d}s.$$

Here $\mathbf{P}$ is the Helmholtz projection from L^p to L^p_σ. Since $\nabla\cdot u = 0$, we have used that $(u,\nabla)u = \nabla\cdot(u\otimes u)$, and that A commutes $\mathbf{P}$ (since $\nabla\cdot\mathbf{P}u = 0$). The solution of the integral equation is often called *a mild solution*, we use this terminology. The integral equation is formally equivalent to (2.1). Indeed, once we get the mild solution u, the pair $(u,\tilde{P})$ satisfies (2.1) in classical sense with some $\tilde{P}$; see Remark 2.1(i) below. In what follows we rather discuss the mild solution.

We now state the local-in-time solvability theorem and the uniqueness result for mild solutions in L^p spaces.

Theorem 2.1. *Let $n \geq 2$, $p \in [n,\infty)$. Let M be a real-valued constant $n\times n$-matrix. Assume that $u_0 \in L^p_\sigma(\mathbb{R}^n)$. Then there exist $T_0 > 0$ and a unique mild solution u such that*

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u] \in C([0,T_0); L^q_\sigma(\mathbb{R}^n))$$

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\nabla u] \in C([0,T_0); L^q(\mathbb{R}^n))$$

for all $q \in [p,\infty]$.

Remark 2.1.

(i) In this theorem we may relax the condition of M, although in order to derive (2.1) from (1.1) with (1.2) we need $\operatorname{tr} M = 0$ and M^2 is symmetric. The mild solution u is smooth in x, i.e. $u(t) \in C^\infty(\mathbb{R}^n)$ for all $t \in (0,T_0)$. This comes from the regularizing effect of the semigroup; see (3.6) in Section 3. Hence, $(u, \tilde{P})$ is a classical solution of (2.1) provided we choose $\tilde{P}$ appropriately, for example,

$$\partial_k \tilde{P} = \sum_{i,j=1}^{n} \partial_k R_i R_j u^i u^j - 2 \sum_{i,j=1}^{n} m_{ij} R_l R_j u^i.$$

Uniqueness of classical solutions follows from the argument by [22] and [31].

(ii) When the initial data $u_0 \in L^\infty_\sigma$ or BUC_σ, it is not easy to obtain a unique mild solution in general, because the Helmholtz projection is not a bounded operator in L^∞. However, we can show the existence theorem of the mild solutions $u \in C([0, T_0); \dot{B}^0_{\infty,1})$ provided $u_0 \in \dot{B}^0_{\infty,1}$ with $\nabla \cdot u_0 = 0$.

(iii) In the case $n = p = 2$ we obtain the global-in-time solution. Multiplying (2.1) with u and integrating over $\mathbb{R}^2$, as the standard way, we can derive $\|u(t)\|_2 \le C\|u_0\|_2 \exp\{|M|t\}$ for all $t \ge 0$. Here C is a numerical constant, and $|M| := \max_{i,j} |m_{ij}|$. That is not conservative, however, that sufficiently gives an a priori estimate for extending the mild solution globally-in-time. In 3-dimensional case, we do not know how to get the global solvability as well as the case $M = 0$.

We see that $u \in C^\infty$ in Remark 2.1(i). It is a natural question whether $u \in C^\omega$ or not. We can verify it, if M satisfies an additional condition.

Theorem 2.2. *Assume, furthermore, that*

$$\| \mathrm{e}^{tM} \| \le 1 \quad \text{for all} \qquad t \ge 0. \tag{2.2}$$

Then u is analytic in x.

Besides, it is impossible to get *the analyticity in time*, since the Ornstein-Uhlenbeck semigroup is not analytic. It is clear that (2.2) holds true if M is skew-symmetric. We do not know whether the assumption (2.2) is essential or not. Theorem 2.2 is an application of the regularizing rate estimates of u and its higher order derivatives. We now state them in the case $p = n$ only (for the shake of simplicity).

Proposition 2.3. *Let $n \ge 2$, $u_0 \in L^n_\sigma(\mathbb{R}^n)$ and $r \in (n, \infty)$. Assume that M satisfies* (2.2). *Let u be the local-in-time mild solution of* (2.1) *for some $T > 0$. Assume further that there exist constants M_1 and M_2 such that*

$$\sup_{0<t<T} \|u(t)\|_n \le M_1 < \infty \quad \text{and} \quad \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|u(t)\|_r \le M_2 < \infty.$$

Then there exist constants K_1 and K_2 (depending only on n, r, M, T, M_1 and M_2) such that

$$\|\nabla^m u(t)\|_q \le K_1 (K_2 m)^m t^{-\frac{m}{2} - \frac{n}{2}(\frac{1}{n} - \frac{1}{q})} \tag{2.3}$$

for all $t \in [0, T]$, $m \in \mathbf{N}_0$ and $q \in [n, \infty]$.

Remark 2.2. There exists a constant $C > 0$ such that the size of radius of the Taylor expansion $\rho(t)$ w.r.t. x is estimated as

$$\rho(t) \ge \limsup_{m \to \infty} \left(\frac{\|\nabla^m u(t)\|_\infty}{m!} \right)^{-1/m} \ge C\sqrt{t} \qquad t \in (0, T). \tag{2.4}$$

This estimate follows from Proposition 2.3 with $q = \infty$, the Stirling formula, and Cauchy-Hadamard's criterion. It is clear that (2.4) yields Theorem 2.2.

3. Preliminaries. In this section we prepare the lemmas to prove the theorems. Firstly, we mention the $L^p - L^q$ estimate of e^{-tA}, and its first derivatives.

Lemma 3.1.

(1) *Let $n \geq 1$, and let $1 \leq p \leq q \leq \infty$. Then there exist constants $C > 0$ and $\omega \geq 0$ such that*

$$\| \mathrm{e}^{-tA} f\|_q \leq C \, \mathrm{e}^{\omega t} \, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_p, \tag{3.1}$$

$$\|\nabla \mathrm{e}^{-tA} f\|_p \leq C \, \mathrm{e}^{\omega t} \, t^{-\frac{1}{2}} \|f\|_p \tag{3.2}$$

for all $f \in L^p$ and $t \geq 0$.

(2) *Assume, additionally, that $p < q$. Then*

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \| \mathrm{e}^{-tA} f\|_q \to 0 \quad \text{as} \quad t \to 0, \tag{3.3}$$

$$t^{\frac{1}{2}} \|\nabla \mathrm{e}^{-tA} f\|_p \to 0 \quad \text{as} \quad t \to 0 \tag{3.4}$$

for all $f \in L^p$.

Proof. The first parts of Lemma 3.1 are proved by direct calculations of the kernel of the representation form of the semigroup, combining with Young's inequality. To do so, we use the change the variables $y = Q_t^{1/2} z$. For proving second parts, we first recall that C_0^∞ is densely subset of L^p for $p < \infty$. As same as that in [23], by triangle inequality (3.3) follows from (3.1), obviously. As the same way, the proof of (3.4) is also shown by (3.2).□

Remark that Lemma 3.1 (and Lemma 3.2 below) is shown by [8] for the case $M = \mathrm{Id}$.

To prove Theorem 2.2 (and Proposition 2.3), we need the estimates for higher order derivatives of the Ornstein-Uhlenbeck semigroup, that is to say, we compute $\nabla^m \mathrm{e}^{-tA} f$. The difficulties arise from the fact that ∇ does not commute e^{-tA}. Indeed, we see that

$$\nabla \mathrm{e}^{-tA} f = \mathrm{e}^{tM} \mathrm{e}^{-tA} \nabla f.$$

Nevertheless, thanks to the representation formula, we can get similar estimate to that of the Stokes semigroup.

Lemma 3.2. *Let $n \geq 1$, and let $1 \leq p \leq q \leq \infty$. Then there exist constants $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3 > 0$, ω_1, ω_2, ω_3, $\omega_4 \geq 0$ (depending only on n, p, q, M) such that*

$$\|\nabla^m \mathrm{e}^{-tA} f\|_q \leq \tilde{C}_1 \, \mathrm{e}^{(\omega_1+\omega_2 m)t} \, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|\nabla^m f\|_p \tag{3.5}$$

for all $t > 0$, $m \in \mathbf{N}$ and $f \in W^{m,p}(\mathbb{R}^n)$, and also

$$\|\nabla^m \mathrm{e}^{-tA} f\|_q \leq \tilde{C}_2 (\tilde{C}_3 m)^{m/2} \, \mathrm{e}^{(\omega_3+\omega_4 m)t} \, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|f\|_p \tag{3.6}$$

for all $t > 0$, $m \in \mathbf{N}$ and $f \in L^p(\mathbb{R}^n)$.

Proof. We first consider the case $p = q$. Since $\| \mathrm{e}^{tM} \| = \| \mathrm{e}^{tM^T} \| \leq C \, \mathrm{e}^{\omega_2 t}$ for all $t \geq 0$ with some constants $C > 0$ and ω_2, it follows that

$$\|\nabla^m \mathrm{e}^{-tA} f\|_q \leq \| \mathrm{e}^{tM} \|^m \| \mathrm{e}^{-tA} \nabla^m f\|_q \leq C \, \mathrm{e}^{\omega_2 m t} \, \mathrm{e}^{\omega_1 t} \|\nabla^m f\|_q, \tag{3.7}$$

for some $\omega_1 \geq 0$. This and (3.1) show the assertion (3.5).

To prove (3.6) we compute

$$\|\nabla^m \mathrm{e}^{-tA} f\|_q = \|\nabla \mathrm{e}^{-\frac{t}{2m}A} \, \mathrm{e}^{(m-1)tM} \, \nabla^{m-1} \mathrm{e}^{-(1-\frac{1}{2m})tA} f\|_q \tag{3.8}$$

$$\leq C \Big(\frac{t}{2m}\Big)^{-1/2} \mathrm{e}^{\frac{\omega t}{2m}} \, C \, \mathrm{e}^{\omega(m-1)t} \|\nabla^{m-1} \mathrm{e}^{-(1-\frac{1}{2m})tA} f\|_q. \tag{3.9}$$

We thus see that there exist constants $C > 0$ and $\omega_3, \omega_4 \geq 0$ such that

$$\|\nabla^m \mathrm{e}^{-tA} f\|_q \leq C^m m^{m/2} \mathrm{e}^{\omega_4 mt} \mathrm{e}^{\omega_3 t} t^{-m/2} \| \mathrm{e}^{-\frac{t}{2}A} f\|_q.$$

Finally, we apply (3.1) to obtain (3.6). □

To show Proposition 2.3 with $q = \infty$, we have to prepare the following estimate of set of three operators $\nabla e^{-tA}\mathbf{P}$.

Lemma 3.3. *Let $1 \leq p \leq \infty$. Then there exist constants $C_p > 0$ and $\omega_5 \in \mathbb{R}$ such that*

$$\|\nabla \mathrm{e}^{-tA} \mathbf{P}\|_{\mathcal{L}(L^p)} \leq C_p t^{-1/2} \mathrm{e}^{\omega_5 t}, \qquad t > 0.$$

Remark that $\mathbf{P}$ is not bounded in L^1 and L^∞. We use the Fourier multipliers theorem; refer to [2]. For making short of this paper we omit the proof, since we can find it in [16]. Note also that Lemma 3.3 has already been proved by [10] for the case $A = -\Delta$. They showed it by direct calculation, recalling the $e^{t\Delta} = G_t *$ is a convolution type operator.

Before closing this section we pick up the bilinear estimate of homogeneous Besov spaces to show Remark 2.1(ii).

Lemma 3.4. *There exists a positive constant C such that*

$$\|f \cdot g; \dot{B}^1_{\infty,1}\| \leq C(\|f; \dot{B}^1_{\infty,1}\| \; \|g; \dot{B}^0_{\infty,1}\| + \|f; \dot{B}^0_{\infty,1}\| \; \|g; \dot{B}^1_{\infty,1}\|)$$

for all $f, g \in \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}$.

We can prove this lemma using by the equivalent norm:

$$\|v; \dot{B}^s_{p,q}\| \cong \Big[\int_0^\infty t^{-1-sq} \sup_{|y| \leq t} \|\tau_y v + \tau_{-y} v - 2v\|_p^q \,\mathrm{d}t \Big]^{1/q},$$

which is valid for $1 \leq p, q \leq \infty, 0 < s < 2$, where τ_y is the translation by $y \in \mathbb{R}^n$, that is, $\tau_y f(x) = f(x - y)$. In [17] we found the similar proof so that we may skip the details.

4. Proofs of theorems. We give the proofs of theorems briefly.

Proof of Theorem 2.1. We use the iteration procedure, that is, successive approximation. We only show it for the case $p = n$. Let $n \geq 2$ and $u_0 \in L^n_\sigma(\mathbb{R}^n)$. For $j \geq 1$ and $t > 0$ we define $u_1(t) := \mathrm{e}^{-tA} u_0$ and functions u_{j+1} by

$$u_{j+1}(t) := \mathrm{e}^{-tA} u_0 - \int_0^t \mathrm{e}^{-(t-s)A} \mathbf{P}\nabla \cdot (u_j(s) \otimes u_j(s)) \,\mathrm{d}s + 2 \int_0^t \mathrm{e}^{-(t-s)A} \mathbf{P} M u_j(s) \,\mathrm{d}s.$$

Since e^{-tA} acts on $L^p_\sigma(\mathbb{R}^n)$ it follows from the definition of the Helmholtz projection that the functions u_j are divergence-free for all $t > 0$ and all j.

Let $T \in (0, 1]$, and let $\delta \in (0, 1)$. We settle

$$A_0 := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}} \| \mathrm{e}^{-tA} u_0\|_{n/\delta} \quad \text{and} \quad A_0' := \sup_{0<t\leq T} t^{\frac{1}{2}} \|\nabla \mathrm{e}^{-tA} u_0\|_n,$$

as well as $A_j := A_j$ and $A_j' := A_j'$ where

$$A_j := \sup_{0<t\leq T} t^{\frac{1-\delta}{2}} \|u_j(t)\|_{n/\delta} \quad \text{and} \quad A_j' := \sup_{0<t\leq T} t^{1/2} \|\nabla u_j(t)\|_n, \qquad j \geq 1.$$

We thus obtain the $L^p - L^q$-smoothing of the semigroup and the boundedness of $\mathbf{P}$ from L^p into L^p_σ that

$$\|u_{j+1}(t)\|_{n/\delta} \le t^{-\frac{1-\delta}{2}} A_0 + C \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{\delta}{n})} \|u_j(s) \cdot \nabla u_j(s)\|_r \, \mathrm{d}s + C \int_0^t \|u_j(s)\|_{n/\delta} \, \mathrm{d}s$$

Here $r = \frac{n}{1+\delta}$. We apply the Hölder inequality. Multiplying with $t^{\frac{1-\delta}{2}}$ and taking in t, we have

$$A_{j+1} \le A_0 + C_1 A_j A'_j + C_2 T A_j \tag{4.1}$$

with some constants C_1, C_2 independent of j and T.

Similarly, applying ∇ to the approximation equation and estimating it in the L^n-norm, we obtain

$$A'_{j+1} \le A'_0 + C_3 A_j A'_j + C_4 T A_j \tag{4.2}$$

with some positive constants C_3 and C_4. Lemma 3.1(2) implies that for any $\lambda > 0$, there exists $\tilde{T}_0 > 0$ such that $A_0, A'_0 \le \lambda$ for all $T \le \tilde{T}_0$. Therefore, we obtain bounds for A_j and A'_j for any $T \le \tilde{T}_0$ uniformly in j provided that $\tilde{T}_0$ is small enough.

Using the uniform bounds of A_j and A'_j, it follows that $t^{\frac{1}{2}-\frac{n}{2q}} \|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}} \|\nabla u_j(t)\|_q$ are bounded for $q \in [n, \infty]$, $t \le \tilde{T}_0$ and $j \in \mathbf{N}$. The continuity of the above functions follows from similar calculations.

The estimate on $u_{j+1} - u_j$ is the same as above, essentially. It thus follows that approximations are Cauchy sequences and we conclude that there are unique limit functions

$$t^{\frac{1}{2}-\frac{n}{2q}} u(t) \in C([0, T_0]; L^q_\sigma), \qquad t^{1-\frac{n}{2q}} v(t) \in C([0, T_0]; L^q),$$

of the sequences $(t^{\frac{1}{2}-\frac{n}{2q}} u_j(t))_{j\ge1}$ and $(t^{1-\frac{n}{2q}} \nabla u_j(t))_{j\ge1}$. Finally, note that $v(t) = t^{1/2} \nabla u(t)$ and that u is a mild solution on $[0, T_0]$.

Uniqueness of mild solutions follows as in [12] from Gronwall's inequality. This completes the proof of Theorem 2.1. □

Proof of Proposition 2.3. Suppose that M satisfies (2.2). We start to prove the assertion (2.3) under the additional assumption that the mild solution is smooth:

$$\partial_x^\alpha u \in C((0, T); L^q(\mathbb{R}^n)) \tag{4.3}$$

for all $\alpha \in \mathbf{N}_0^n$. We may assume (4.3), since it can be shown by similar way; see the details in [16].

We use an induction w.r.t. $m \in \mathbf{N}$. That is, we assume that (2.3) holds true for all $m \le k-1$. We now proceed to show it for $m = k$. For simplicity, we suppose that $T \le 1$, $n \ge 3$, $q < \infty$. For $\varepsilon \in (0,1)$ we have

$$\begin{aligned} \|\nabla^k u(t)\|_q \;\le\; & \|\nabla^k \mathrm{e}^{-tA} u_0\|_q + \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k \mathrm{e}^{-(t-s)A} \mathbf{P} \nabla \cdot (u(s) \otimes u(s))\|_q \, \mathrm{d}s \\ & +2 \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t \right) \|\nabla^k \mathrm{e}^{-(t-s)A} \mathbf{P} M u(s)\|_q \, \mathrm{d}s \\ =:\; & B_1 + B_2 + B_3 + B_4 + B_5. \end{aligned}$$

We shall estimate each the above terms $B_1, \dots, B_5$ separately.

The estimates for B_1 are derived from (3.6) as follows:

$$B_1 \le \tilde{C}_2(\tilde{C}_3 k)^{k/2}\,\mathrm{e}^{\omega_3 kt}\,\|u_0\|_n t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} \le C_5(C_6 k)^{k-\delta} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}$$

for $t \in (0,T)$ with constants $C_5 := \tilde{C}_2\|u_0\|_n \le \tilde{C}_2 M_1$ and $C_5 := \tilde{C}_3\,\mathrm{e}^{\omega_3}$. This follows since $k/2 \le k-\delta$ for $k \ge 2$ and $\delta \le 1$. The estimates for B_2, B_4 and B_5 are basically same as above, so we omit to prove.

To derive the estimate B_3, we use the Leibniz rule:

$$\begin{aligned} B_3 \;\le\; & C_7\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}\|\nabla^k u(s)\|_q\|u(s)\|_\infty\,\mathrm{d}s \\ & +C_7\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}\max_{|\beta|=k}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u(s)\|_q\|\partial_x^{\beta-\gamma}u(s)\|_\infty\,\mathrm{d}s \\ =:\; & B_{3a}+B_{3b}. \end{aligned}$$

Here $C_7 = 2\tilde{C}_1 e^{\omega_1}$ is independent of k by (2.2): $\binom{\beta}{\gamma} = \prod_{i=1}^n \frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}$ is a binomial coefficient.

Consider B_{3a}. Firstly, there exists positive constant C such that $\|u(s)\|_\infty \le Cs^{-1/2}$; see the Proposition 3.1 in [15]. Then,

$$B_{3a} \le C_8\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}s^{-1/2}\|\nabla^k u(s)\|_q\,\mathrm{d}s$$

with C_8 is a constant depending only on n,p,q,M,M_1,M_2. Next we deal with B_{3b}. By assumption of induction, we have

$$\begin{aligned} B_{3b} \le\; & C_7\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac{1}{2}}\max_{|\beta|=k}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}K_1(K_2|\gamma|)^{|\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\gamma|}{2}} \\ & \times K_1(K_2|\beta-\gamma|)^{|\beta-\gamma|-\delta}s^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\beta-\gamma|}{2}}\,\mathrm{d}s \\ \le\; & C_7K_1^2K_2^{k-2\delta}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta}\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac{1}{2}}s^{-1-\frac{n}{2q}-\frac{k}{2}}\,\mathrm{d}s. \end{aligned}$$

We now use Kahane's lemma in [21, Lemma 2.1] to get

$$B_{3b} \le C_9K_1^2K_2^{k-2\delta}k^{k-\delta}t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}I(\varepsilon).$$

Here $I(\varepsilon) := \int_{1-\varepsilon}^1(1-\tau)^{-\frac{1}{2}}\tau^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}-\frac{1}{2}}d\tau$, and C_9 is a constant depending only on C_7 and δ (independent of k). Moreover,, and the dependence of C_9 w.r.t. δ is $C_9 \sim \sum_{j=1}^\infty j^{-1/2-\delta/2}$, so we need $\delta > 1/2$.

Define b_ε by

$$b_\varepsilon := \tilde{C}_5(\tilde{C}_6 k/\varepsilon)^{k/2} + C_9K_1^2K_2^{k-2\delta}k^{k-\delta}I(\varepsilon).$$

Here $\tilde{C}_5$ and $\tilde{C}_6$ are constants. Gathering estimates above, we obtain

$$\|\nabla^k u(t)\|_q \le b_\varepsilon t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}} + \tilde{C}_8\int_{(1-\varepsilon)t}^{t}(t-s)^{-1/2}s^{-1/2}\|\nabla^k u(s)\|_q\,\mathrm{d}s.$$

Here $\tilde{C}_8$ is a constant independent of k. By Gronwall's type inequality in [15, Lemma 2.4], there exists a $\varepsilon_k \in (0,1)$ such that

$$\|\nabla^k u(t)\|_q \leq 2b_{\varepsilon_k} t^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{k}{2}}, \qquad t \in (0,T). \tag{4.4}$$

This is possible if we take ε_k small enough. Indeed, there exists k_0 (depending only on n, p, M, M_1, M_2) such that $I(1/k) \leq \frac{1}{2\tilde{C}_8}$ for all $k \geq k_0$.

Finally, we verify $2b_{1/k} \leq K_1(K_2 k)^{k-\delta}$ for suitable choice of K_1 and K_2. Fix a constant $K_0 > 0$ (depending only on n, p, M, M_1, M_2) so that $\|\nabla^k u(t)\|_q \leq K_0$ holds for $k \leq k_0$. For $k \geq 2$, since $I(1/k) \leq 2$, $2b_{1/k} \leq 2\{\tilde{C}_5 \tilde{C}_6^{k-\delta} + 2C_9 K_1^2 K_2^{k-2\delta}\} k^{k-\delta}$. Therefore, we choose

$$K_1 := \max\left(K_0, 4\tilde{C}_5\right) \quad \text{and} \quad K_2 := \max\left(\tilde{C}_6, (4C_9 K_1)^{\delta}\right),$$

then (2.3) holds true for all m.

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