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ON A NONLINEAR NONLOCAL ODE AND ITS APPLICATIONS*

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Abstract. We consider a limit case of a system of two equations arising in magnetic recording for a one-dimensional domain. The system models the tape deflection when it is driven over a magnetic head profile, which is a known function. The system is reduced to a second order nonlinear equation, where the unknown u appears evaluated in a finite set of distinguished points $\{x_i\}_{i=1}^n$ of the domain.

Key words. Nonlocal terms, lubrication theory, Reynolds equation, nonlinear elliptic equations

AMS subject classifications. 34B99, 34B60, 74F17

1. Modelling and main results. The problem consists in a magnetic tape which is driven with constant velocity over a magnetic head and its motion entrains air in the space between the head and the tape. The unknowns of the problem are the position of the tape u and the pressure of the air p in the region between the tape and the head. The position of the tape is modelled by the beam equation and the pressure (when the tape is close to the head) satisfies the compressible Reynolds equation. The head presents trenches to control the position of the tape. If the trench is deep enough, Reynolds equation can not be used to model the problem and the pressure in the trench is considered equal to the atmospheric pressure.

After nondimensionalization one obtains the following system of equations for the position of the tape "u" and the pressure of the air "p" (see [1], [3, Chap. 6] for details)

$$\frac{\partial(ph)}{\partial x} - \varepsilon\frac{\partial}{\partial x}(\alpha h^2\frac{\partial p}{\partial x} + \beta h^3 p\frac{\partial p}{\partial x}) = 0, \qquad x_i < x < x_i + L_i, \tag{1.1}$$

$$-\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^4 u}{\partial x^4} = k(p-1)\left(\sum_{i=1}^{n}\chi_{[x_i,x_i+L_i]}\right), \qquad 0 < x < L, \tag{1.2}$$

$$u(x) = h(x) + \delta(x), \qquad h(x) > 0 \text{ if } x_i \le x \le x_i + L_i, \quad \text{for } 1 \le i \le n \tag{1.3}$$

where

$$0 < x_i < x_i + L_i < x_{i+1} < x_{i+1} + L_i < L, \qquad \text{for } 1 \le i \le n-1, \tag{1.4}$$

$\chi_{[x_i,x_i+L_i]}$ is the characteristic function of the interval $[x_i, x_i + L_i]$ and typically

$$\alpha \sim \frac{1}{10},\ \beta \sim 1,\ x_n + L_n - x_1 \sim 1,\ L \sim 10,\ k \sim 10^4,\ \varepsilon \sim 10^{-2},\ \mu \sim 10^{-3};$$

x_i lies near of the middle of the interval $(0, L)$ for $1 \le i \le n$.

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The boundary conditions are

$$p(x_i) = p(x_i + L_i) = 1, \tag{1.5}$$

$$u = \frac{\partial u}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L. \tag{1.6}$$

We assume throughout the paper that

$$\delta \in C^1[x_i, x_i + L_i] \;\text{ for } 1 \le i \le n; \tag{1.7}$$

$$0 > \delta(L_1) - \delta'(L_1)L_1; \tag{1.8}$$

$$0 > \delta(x_n + L_n) + \delta'(x_n + L_n)(L - x_n - L_n); \tag{1.9}$$

$$\delta(x_{i-1} + L_{i-1}) > \delta(x_i) - \delta'(x_i)(x_i - x_{i-1} - L_{i-1}), \qquad \text{for} \quad 2 \le i \le n; \tag{1.10}$$

$$\delta(x_{i+1}) > \delta(x_i + L_i) - \delta'(x_i + L_i)(x_{i+1} - x_i - L_i), \qquad \text{for} \quad 1 \le i \le n-1. \tag{1.11}$$

The case where δ does not present any trench have been studied in [4] under the concavity assumption $\delta''(x) < 0$. This assumption is very restrictive, not only mathematically, but also physically: Magnetic heads do not generally satisfy the concavity condition. Indeed, in order to reduce the effect of air entrainment (which causes boundary layer for the pressure p near the end of the head) trenches are dug into the head (see [2]). In [5] the problem is studied when the head may present discontinuities and the concavity assumption is removed. Results in [5] are obtained when the side of the trench is small in comparison to the side of the head and compressible Reynolds equation models the air pressure also in the trench zone.

1.1. Limit case: $\varepsilon = 0$, $\mu = 0$. In the special case $\varepsilon = \mu = 0$, the system (1.1)–(1.3) reduces to

$$\frac{\partial (ph)}{\partial x} = 0, \quad x_i < x < x_i + L_i, \quad 1 \le i \le n, \tag{1.12}$$

$$-\frac{\partial^2 u}{\partial x^2} = k(p-1)\left(\sum_{i=1}^{n} \chi_{[x_i, x_i+L_i]}\right), \; 0 < x < L. \tag{1.13}$$

Some of the boundary conditions in (1.5), (1.6) need to the dropped, and we take

$$p(x_i) = 1, \qquad \text{for } 1 \le i \le n, \tag{1.14}$$

$$u(0) = u(L) = 0. \tag{1.15}$$

From (1.12) we see that

$$ph = \text{constant} := C_i \qquad \text{in } x_i < x < x_i + L_i.$$

Since $p(x_i) = 1$, $C_i = h(x_i) = u(x_i) - \delta(x_i)$, so that

$$p(x) = \frac{u(x_i) - \delta(x_i)}{u(x) - \delta(x)} \qquad \text{in } x_i < x < x_i + L_i.$$

Hence (1.13) becomes

$$-\frac{\partial^2 u}{\partial x^2} = k \sum_{i=1}^{n} \left(\frac{u(x_i) - \delta(x_i)}{u(x) - \delta(x)} - 1 \right) \chi_{[x_i, x_i+L_i]}, \qquad 0 < x < L. \tag{1.16}$$

1.2. Main Results. The result of this work concerns on the existence of solution to the problem (1.16), (1.15) and it is presented in the following Theorem.

THEOREM 1.1. *Under assumptions* (1.7)–(1.11) *there exists at least a solution* $u \in W^{2,\infty}(0, L)$ *to* (1.16), (1.15).

The proof of the Theorem is presented in the next section.

2. Existence of solutions: Proof of Theorem 1.1. Let

$$A := \max_{1 \le i \le n} \|\delta'\|_{L^\infty(x_i, x_i+L_i)} + 1.$$

Consider the function $\overline{\delta}$ defined by:

if $0 \le x \le x_1$:

$$\overline{\delta}(x) := \delta(x_1) + \delta'(x_1)(x - x_1);$$

if $x_i \le x \le x_i + L_i$, for $1 \le i \le n$:

$$\overline{\delta}(x) := \delta(x);$$

if $x_i + L_i \le x \le x_{i+1}$, for $1 \le i \le n-1$:

$$\overline{\delta}(x) := \max\{\delta(x_i + L_i) + \delta'(x_i + L_i)(x - x_i),\ \delta(x_{i+1}) + \delta'(x_{i+1})(x - x_{i+1})\};$$

if $x_n + L_n \le x \le L$:

$$\overline{\delta}(x) := \delta(x_n + L_n) + \delta'(x_i + L_i)(x - x_i - L_i).$$

We approximate $\overline{\delta}$ by $\overline{\delta}_\varepsilon$ satisfying the following assumptions:

(i) $\overline{\delta}_\varepsilon(x) = \overline{\delta}(x)$, if $0 \le x \le x_1$ or $x_n + L_n \le x \le L$;

(ii) $\overline{\delta}_\varepsilon(x) = \overline{\delta}(x)$, if $x = x_i$ or $x = x_i + L_i$ for $1 \le i \le n$;

(iii) $\overline{\delta}'_\varepsilon(x_i) = \lim_{\tau \to 0} \overline{\delta}'(x_i + \tau)$ for $\tau > 0$ and $1 \le i \le n$;

(iv) $\overline{\delta}_\varepsilon \in C^2(0, L)$;

(v) $\overline{\delta}_\varepsilon(x) - Ax$ is a decreasing function;

(vi) $\overline{\delta}_\varepsilon \to \overline{\delta}$ in $C^1([x_i, x_i + L_i])$, for $1 \le i \le n$;

(vii) $\overline{\delta}''_\varepsilon \ge 0$ in $(x_i + L_1, x_{i+1})$, for $1 \le i \le n-1$;

(viii) $\overline{\delta}_\varepsilon \to \overline{\delta}$ in $C^0(0, L)$.

Let $I_i \subset \mathbb{R}$ define by $I_i := (\delta(x_i), Ax_i]$, and $I := \Pi_{i=1}^n I_i$. We consider $\lambda := (\lambda_1, \ldots, \lambda_n) \in I$ and the problem

$$\begin{cases} -\dfrac{\partial^2}{\partial x^2} u_\varepsilon(\lambda, x) = k \displaystyle\sum_{i=1}^n \left(\frac{\lambda_i - \delta(x_i)}{u_\varepsilon(\lambda, x) - \overline{\delta}_\varepsilon(x)} - 1 \right) \chi_{[x_i, x_i+L_i]}, & 0 < x < L, \\ u_\varepsilon(\lambda, 0) = u_\varepsilon(\lambda, L) = 0, & \\ u_\varepsilon(\lambda, x) - \overline{\delta}_\varepsilon(x) > 0, \quad \text{if } x_i \le x \le x_i + L_i, & \text{for } 1 \le i \le n. \end{cases} \tag{2.1}$$

LEMMA 2.1. *For each* $\lambda = (\lambda_1, \dots, \lambda_n) \in I$ *and* $\varepsilon > 0$*, there exists a unique solution* $u_\varepsilon(\lambda, x)$ *to* (2.1).

Proof. Set $F_\varepsilon(w,x) = k\sum_{i=1}^{n}\left(\frac{\lambda_i - \delta(x_i)}{w - \overline{\delta}_\varepsilon(x)} - 1\right)\chi_{(x_i, x_i+L_i)}$, for $w > \overline{\delta}_\varepsilon$, then

$$\frac{\partial F_\varepsilon}{\partial w} < 0. \tag{2.2}$$

Consider $\underline{u}_\varepsilon(\lambda, x) = \overline{\delta}_\varepsilon(x) + \gamma$, for $\gamma > 0$ small enough satisfying

$$\overline{\delta}_\varepsilon(0) + \gamma \le 0, \qquad \overline{\delta}_\varepsilon(L) + \gamma \le 0, \quad \|\overline{\delta}_\varepsilon''\|_{L^\infty(0,L)} - \min_{1\le i\le n} k\frac{\lambda_i - \delta(x_i) - \gamma}{\gamma} < 0.$$

Then

$$-\frac{\partial^2}{\partial x^2}\underline{u}_\varepsilon(\lambda, x) - F_\varepsilon(\underline{u}_\varepsilon(\lambda, x), x) \le 0, \qquad \underline{u}_\varepsilon(\lambda, 0) \le 0, \quad \text{and} \quad \underline{u}_\varepsilon(\lambda, L) \le 0,$$

thus $\underline{u}_\varepsilon(\lambda, x)$ is a subsolution to (2.1).

The function $\bar{u} = Ax$ is a supersolution (by (iv)).

We introduce the convex set of functions

$$G = \{\tilde{u} \in C^0[0,L], \underline{u}_\varepsilon(\lambda, x) \le \tilde{u}(x) \le \bar{u}(x)\},$$

and the problem

$$\begin{cases} -\dfrac{\partial^2}{\partial x^2}v_\varepsilon(\lambda, x) = F_\varepsilon(\tilde{u}, x), & 0 < x < L, \\ v_\varepsilon(\lambda, 0) = v_\varepsilon(\lambda, L) = 0. \end{cases} \tag{2.3}$$

By (2.2) and maximum principle we deduce the existence of a unique solution $v_\varepsilon(\lambda, x)$ to (2.3), satisfying

$$\underline{u}_\varepsilon(\lambda, x) \le v_\varepsilon(\lambda, x) \le \bar{u}(x), \; 0 \le x \le L. \tag{2.4}$$

We define the continuous mapping $T : G \subset C^0(0,L) \to C^0(0,L)$ by $T(\tilde{u}) = v_\varepsilon(\lambda, x)$. By (2.4), T maps G into itself. Since $F_\varepsilon(\tilde{u}, x)$ is bounded we have

$$v_\varepsilon(\lambda, x) \in W^{1,\infty}(0,L)$$

$T(G)$ lies in a compact subset of G. Appealing to the Schauder fixed point theorem we conclude that T has a fixed point, which is clearly a solution to (2.1). Finally, uniqueness is a consequence of (2.2). □

We denote by $u_\varepsilon(\lambda, x)$ the solution to (2.1) and define $f : I \to \mathbb{R}^n$ by

$$f(\lambda) := (u_\varepsilon(\lambda, x_1), \dots, u_\varepsilon(\lambda, x_n)). \tag{2.5}$$

LEMMA 2.2. *f is a continuous function.*

Proof. Let $\lambda^0 \in I$. Then

$$\left|\frac{\partial^2}{\partial x^2}(u_\varepsilon(\lambda^0, x) - u_\varepsilon(\lambda, x))\right| \le \sum_{i=1}^{n}\frac{2k|\lambda_i^0 - \lambda_i|}{\min_{x_i<x<x_i+L_i}\{|u_\varepsilon(\lambda^0) - \overline{\delta}_\varepsilon|\}}, \tag{2.6}$$

for λ close to λ_0. Taking limits in (2.6), it results, $u_\varepsilon(\lambda) \to u_\varepsilon(\lambda^0)$ in $C^1(0,L)$, and consequently $u_\varepsilon(\lambda, x_i) \to u_\varepsilon(\lambda^0, x_i)$ as $\lambda \to \lambda_0$ which proves the Lemma. ◻

Lemma 2.3. *$f(I) \subset I$.*

Proof. Since $u_\varepsilon(\lambda, x) \le \overline{u} = Ax$ is a super solution of (2.1) for any $\lambda \in I$, we obtain that $f_i(\lambda) \le Ax_i$. Using the subsolution $\underline{u}_\varepsilon(\lambda, x)$ constructed in the proof of Lemma 2.1, we obtain that $u_\varepsilon(\lambda, x_i) > \underline{u}_\varepsilon(\lambda, x_i) > \delta_\varepsilon(x_i)$ which ends the proof. ◻

Lemma 2.4. *$u_\varepsilon(\lambda, x) \in W^{1,\infty}(0,L)$ for $\lambda \in I$.*

Proof. Integrating (2.1) over $(0,L)$ we obtain

$$-\frac{\partial}{\partial x} u_\varepsilon(\lambda, x)\Big|_{x=0}^{x=L} = k \sum_{i=1}^{n} \int_{x_i}^{x_i+L_i} \left(\frac{\lambda_i - \delta_\varepsilon(x_i)}{u_\varepsilon(\lambda, x) - \overline{\delta}_\varepsilon(x)} - 1 \right) \mathrm{d}x. \tag{2.7}$$

Since $\overline{\delta}_\varepsilon(x) < u_\varepsilon(\lambda, x) < Ax$ it results

$$\frac{\partial}{\partial x} u_\varepsilon(\lambda, x)\Big|_{x=0} < A, \quad \text{and} \quad \frac{\partial}{\partial x} u_\varepsilon(\lambda, x)\Big|_{x=L} > -A \frac{x_n + L_n}{L - x_n - L_n}. \tag{2.8}$$

Then, by (2.7) and (2.8)

$$k \sum_{i=1}^{n} \int_{x_i}^{x_i+L_i} \frac{\lambda_i - \overline{\delta}_\varepsilon(x_i)}{u_\varepsilon(\lambda, x) - \overline{\delta}_\varepsilon(x)} \, \mathrm{d}x < \infty.$$

Integrating over $(0, x)$, since

$$\frac{\lambda_i - \overline{\delta}_\varepsilon(x_i)}{u_\varepsilon(\lambda, x) - \overline{\delta}_\varepsilon(x)} > 0,$$

we prove the lemma. ◻

Lemma 2.5. *f has, at least, a fixed point in I.*

Proof. In order to prove the lemma, we argue by contradiction and assume there is not fixed point of f in I. We extend the function f by the continuous function $\overline{f} : \overline{I} \to \overline{I}$, such that $\overline{f}\big|_I = f$. Since $\overline{I}$ is a compact set and $\overline{f}$ is continuous, by Schauder fixed point theorem, there exists $\lambda^0 \in \overline{I}$ such that $f(\lambda^0) = \lambda^0$. We consider the first $i > 0$ such that $\lambda_i^0 \notin I_i$, i.e. $\lambda_i^0 = \overline{\delta}_\varepsilon(x_i)$. Then

$$\lim_{\lambda \to \lambda^0} f_i(\lambda) = \overline{\delta}_\varepsilon(x_i).$$

By (1.10) it results that

$$\frac{\partial}{\partial x} u_\varepsilon(\lambda, x)\Big|_{x=x_i} - \overline{\delta}'_\varepsilon(x_i) < -\kappa_1 < 0$$

for $|\lambda_i - \delta(x_i)| < \kappa_2$ and κ_1 and κ_2 small enough. Then for $\lambda \in I$ (with $|\lambda - \lambda^0|$ small enough) the solution $u_\varepsilon(\lambda, x_i + \kappa_0) = \overline{\delta}_\varepsilon(x_i + \kappa_0)$ (for some $\kappa_0 > 0$) which contradicts the fact that the subsolution $\underline{u}_\varepsilon(\lambda)$ is strictly bigger than $\overline{\delta}_\varepsilon$ (the proof is similar to the proof of Theorem 2.1 in Friedman-Tello [5] where more details can be found). Then f has a fixed point $\lambda^0 \in I$, such that $u_\varepsilon(\lambda^0, x_i) = \lambda_i^0$ which ends the proof of the lemma. ◻

Notice that $\lambda^0 = \lambda^0(\varepsilon)$ and by using the same approach that in LEMMA 2.5, we may prove that

$$\lim_{\varepsilon \to 0} \lambda^0(\varepsilon) > \delta(x_i). \tag{2.9}$$

We denote by u_ε the solution to (2.1) for $\lambda = \lambda^0$ (the fixed point of f). Then, by LEMMA 2.4 there exists a sequence $\varepsilon_j \to 0$ such that

$$u_{\varepsilon j}(x) \longrightarrow u(x) \text{ uniformly in } \ x \in [0, L], \tag{2.10}$$

and

$$\left\| \frac{\partial u}{\partial x} \right\|_{L^\infty(0,L)} \leq C \ \text{ a.e.} \tag{2.11}$$

By Fatou′s Lemma, (2.9) and (2.7) we deduce that

$$\int_{x_i}^{x_i+L_i} \frac{\mathrm{d}x}{u(x) - \delta(x)} \leq C, \quad \text{for} \quad 1 \leq i \leq n. \tag{2.12}$$

LEMMA 2.6. *There holds:*

$$u(x) - \delta(x) > 0 \ \text{ if } \ x_i \leq x \leq x_i + L_i.$$

Proof. Suppose $u(x_0) - \delta(x_0) = 0$ for some $x_0 \in [x_i, x_i + L_i]$. By (2.9) $x_0 \neq x_i$ and by the Mean Value Theorem,

$$|u(x) - \delta(x)| \leq C|x - x_0|, \quad \text{i.e.} \quad \frac{1}{|u(x) - \delta(x)|} \geq \frac{1}{C|x - x_0|},$$

in an interval with endpoint at x_0. Integrating over the interval we obtain a contradiction to estimate (2.12), which proves the lemma. □

Since $u > \delta$, taking limits when $\varepsilon_j \to 0$, in the equation

$$-\frac{\partial^2 u_{\varepsilon j}}{\partial x^2} = k \sum_{i=1}^{n} \left(\frac{u_{\varepsilon j}(x_i) - \delta(x_i)}{u_{\varepsilon j}(x) - \overline{\delta}_{\varepsilon j}(x)} - 1 \right) \chi_{[x_i, x_i + L_i]}, \qquad 0 < x < L.$$

we get that u is a solution to (1.16).

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