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SUPPORT RE-SPLITTING PHENOMENA CAUSED BY AN INTERACTION BETWEEN DIFFUSION AND ABSORPTION*

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Abstract. Numerical computation for an interaction between diffusion and absorption suggests several interesting phenomena in the dynamical behavior of the support. The most remarkable properties is the occurrence of *support re-splitting phenomena* for the porous media equation with strong absorption. In this paper such phenomena are investigated by use of finite difference scheme, and justified from numerical and analytical points of view.

Key words. porous media equation, interfaces, support splitting, finite extinction, difference scheme

AMS subject classifications. 65M12, 35K65, 35B99

1. Introduction. Nonlinear diffusion equations have played an important role in investigating phenomena in several fields of fluid dynamics, combustion theory, plasma physics, and population dynamics. The interaction between diffusion and absorption is described as one of simple mathematical models. This interaction causes *support splitting phenomena* and *total extinction in finite time*, which mean that the region occupied by the fluid becomes disconnected and the fluid vanishes in finite time. The most remarkable property is the occurrence of

Support re-splitting phenomena. After *support splitting phenomena* appear, the support becomes connected, and thereafter *support splitting phenomena* appear again (see FIG. 1.1).

From only numerical computation it is difficult to justify whether such phenomena are true or not, because the space mesh and the time step are sufficiently small but not zero. So the mathematical analysis is needed.

We shall try to investigate such phenomena in the Cauchy problem for the following one-dimensional homogeneous porous media equation with the interaction between diffusion and absorption:

$$v_t = (v^m)_{xx} - cv^p, \qquad x \in \mathbb{R}^1, \quad t > 0, \tag{1.1}$$

$$v(0,x) = v^0(x), \qquad x \in \mathbb{R}^1, \tag{1.2}$$

where $m(> 1)$, $p(> 0)$, and $c(\geq 0)$ are constants, v denotes the density in the flow of the liquids through an absorbing medium, and $v^0(x) \in C^0(\mathbb{R}^1)$ is nonnegative and has compact support.

The existence and uniqueness of a weak solution and the finite propagation of the support of v are proved by Aronson [1], Oleinik, Kalashnikov and Chzou Yui-Lin [11], Kalashnikov [6, 7], and Herrero and Vázquez [5]. Moreover, $v(t,x)$ is smooth in the open set $\mathcal{P}(v) = \{(t,x) | v(t,x) > 0 \text{ and } t > 0\}$, and the behavior of the support is classified into two cases:

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Case 1. For $c = 0$, or $c > 0$ and $p \geq 1$ the diffusion is active and supp $v(t, \cdot)$ monotonously expands as t increases; that is,

$$\lim_{t\to\infty} \text{supp } v(t,\cdot) = \mathbb{R}^1 \quad \text{for } c = 0, \quad \text{or} \quad c > 0 \text{ and } p \geq m$$

and

$$\lim_{t\to\infty} \text{supp } v(t,\cdot) \subset [M_1, M_2] \quad \text{for } c > 0 \text{ and } m > p \geq 1,$$

where $M_i (i = 1, 2)$ are some constants;

Case 2. For $c > 0$ and $0 < p < 1$ the absorption is active and the solution vanishes identically at some finite time $T^* > 0$; that is, supp $v(\cdot, \cdot)$ becomes compact. Such an absorption is called a strong absorption.

In Case 1 *support splitting phenomena* never appear. In Case 2 there is a possibility of the support to split, when $v^0(x)$ has two local maxima. Rosenau and Kamin [12] suggested this possibility by numerical computation. Chen, Matano and Mimura [3] constructed the initial function for which the support of the solution splits into multiple connected components in a finite time. This motivates us to investigate the detail of the behavior of the support.

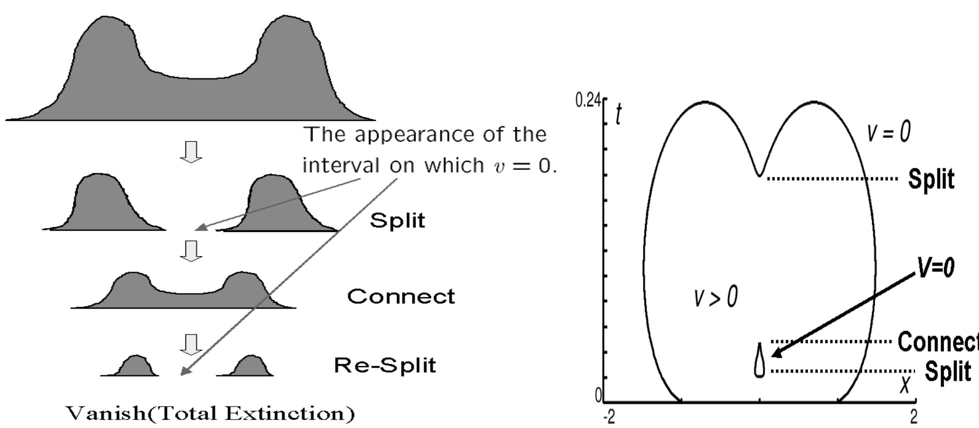

Fig. 1.1. *Support re-splitting phenomena. The graph on the right hand side shows the numerical interfaces for* $v_t = (v^m)_{xx} - cv^p$ $(m = 1.5, p = 0.5$ *and* $c = 5)$.

In this paper we shall justify *support re-splitting phenomena* under the following

Assumption A. $c > 0$, $m + p = 2$ and $0 < p < 1$.

For this end we have to show *support splitting phenomena* in the sense of the appearance of the interval on which $v = 0$ and *support connecting phenomena.* The latter is proved by taking account of the interface equation[9]. It is obvious that the former follows from the following

Conjecture. *A spontaneous appearance of zeros of the solution implies support splitting phenomena in such a sense.*

However, we are unable to prove this conjecture. To avoid the such difficulties we assume that $v^0(x)$ is an even function, which implies that the solution $v(t, x)$ also becomes an even function. We impose some additional conditions on $v^0(x)$ for which zeros of the solution appear at the points $(\tilde{t}, \pm\tilde{x})$, and apply the nonincrease of the number of local maximum points to this solution [3]. Thus the solution identically vanishes on the interval $[-\tilde{x}, \tilde{x}]$ at $t = \tilde{t}$, and *support splitting phenomena in the sense stated above* follow. This is our strategy, where several estimates derived from the finite difference scheme are employed. Unfortunately, in the case where $m + p \neq 2$, $m > 1$ and $0 < p < 1$, we are unable to succeed in constructing the finite difference scheme with convergence. This is the reason why we are concerned with the specific case stated in Assumption A.

2. Finite difference schemes. We put $u = v^{m-1}$ and rewrite (1.1)–(1.2) as follows:

$$u_t = muu_{xx} + a(u_x)^2 - c', \tag{2.1}$$

$$u(0, x) = u^0(x) \equiv (v^0(x))^{m-1}, \tag{2.2}$$

where $a = \dfrac{m}{m-1}$, $c' = (m-1)c$ and the term of absorption is written as the constant $-c'$ by the assumption $m + p = 2$. Our scheme approximates the problem (2.1)–(2.2) instead of (1.1)–(1.2) [4]. Let h be a space mesh width and V_h be the set of the nonnegative and piecewise-linearly interpolated functions $u_h = u_h(x)$ with the mesh $\mathcal{M}_h = \{\ell, Lh, (L+1)h, \cdots, (R-1)h, Rh, r\}$, where the L and R are integers, and ℓ and r denote the left and right interfaces of u_h, respectively. The scheme is described as follows:

Find the sequence $\{u_h^n\}_{n=1,2,\cdots} \subset V_h$ with the mesh

$$\mathcal{M}_h^n = \{\ell_n, L_n h, (L_n + 1)h, \cdots, (R_n - 1)h, R_n h, r_n\}$$

for each $u_h^0 \in V_h$ such that

$$u_h^{n+1} = S_{h,k} u_h^n \quad \text{for } n = 0, 1, 2, \cdots, \tag{2.3}$$

where $u_h^0(x) = u^0(x)$ on $\mathcal{M}_h^0$. Since $S_{h,k}$ is somewhat complicated form, we omit its description [8, 9, 10]. The variable time step $k = k_{n+1} \equiv t_{n+1} - t_n$ $(t_0 = 0)$ is determined by

$$k = \frac{1}{c'} \max(u_L, u_{L+1}) \text{ for the approximation to the left interface, or} \tag{2.4}$$

$$k = \frac{1}{c'} \max(u_R, u_{R-1}) \text{ for the approximation to the right interface.} \tag{2.5}$$

When $S_{h,k} u_h^{n^*} \equiv 0$ holds for some integer $n^* > 0$, we put the numerical extinction time $T_h^* = t_{n^*+1} \equiv t_{n^*} + k_{n^*+1}$, and stop the numerical computation. We define the left (resp. right) numerical interface curves $\ell_h(t)$(resp. $r_h(t)$) by piecewise-linearly interpolating (t_n, ℓ_n)(resp.(t_n, r_n))$(0 \leq n \leq n^*)$. We state several results without proof, which play an important role in constructing the initial function for which *support re-splitting phenomena* appear. For this end we introduce the following

Condition B.

i) $v^0(x) \in C^0(\mathbb{R}^1)$ is a nonnegative function with compact support and $\left((v^0(x))^{m-1}\right)_x \in L^\infty(\mathbb{R}^1) \cap BV(\mathbb{R}^1)$;

ii) $\left((v^0(x))^{m-1}\right)_x$ is absolutely continuous on $\mathbf{I} = \{x | v^0(x) > 0\}$ and $\text{ess.inf}_{\mathbf{I}} \left((v^0(x))^{m-1}\right)_{xx}$ is finite.

We define the constants $C_j(v^0)$ $(j = 0, 1, 2)$ by

$$\begin{cases} C_0(v^0) = \|(v^0)^{m-1}\|_\infty, \quad C_1(v^0) = \left\|\left((v^0)^{m-1}\right)_x\right\|_\infty, \\ C_2(v^0) = -\text{ess.inf}_{\mathbf{I}} \left((v^0(x))^{m-1}\right)_{xx}, \end{cases} \tag{2.6}$$

where $\|\cdot\|_\infty$ denotes $\|\cdot\|_{L^\infty(\mathbb{R}^1)}$.

THEOREM 2.1 (Basic estimates [8], [10]). *Let $u_h^0 \in V_h$. Then u_h^n either becomes extinct or belongs to V_h for each $n \geq 0$, and the following estimates hold for all $n \geq 0$:*

$$T_h^* \leq t_n + \frac{\|u_h^n\|_\infty}{c'}, \tag{2.7}$$

$$0 \leq r_n - \ell_n \leq (r_0 - \ell_0 + 2a\|(u_h^0)_x\|_\infty t_n) \text{ if } u_h^n \not\equiv 0, \tag{2.8}$$

$$0 \leq u_h^n(x) \leq \max(\|u_h^0\|_\infty - c't_n,\, 0) \quad \text{on } \mathbb{R}^1, \tag{2.9}$$

$$\|(u_h^n)_x\|_\infty \leq \|(u_h^0)_x\|_\infty, \tag{2.10}$$

$$TV((u_h^n)_x) \leq TV((u_h^0)_x), \tag{2.11}$$

$$\begin{aligned}\|(u_h^{n+1} - u_h^n)/k_{n+1}\|_{L^1(\mathbb{R}^1)} &\leq (m+a)\|u_h^0\|_\infty TV((u_h^0)_x)\\ &\quad + c'(r_0 - \ell_0 + 2a\|(u_h^0)_x\|_\infty t_n),\end{aligned} \tag{2.12}$$

$$\inf_{i\in\mathbf{Z}} \delta^2 u_i^0 \leq \inf_{i\in\mathbf{Z}} \delta^2 u_i^n, \tag{2.13}$$

where $\delta^2 u$ denotes a usual finite difference approximation to u_{xx}.

THEOREM 2.2 (Convergence of numerical solutions [10]). *Under* CONDITION B *let $\{h\}$ be an arbitrary sequence which tends to zero. Then, there exists the unique weak solution v of* (1.1)–(1.2)*, and*

$$\|v_h - v\|_{L^\infty(\mathcal{H})} \longrightarrow 0 \quad \text{and} \quad |T_h^* - T^*| \longrightarrow 0 \quad \text{as} \quad h \to 0, \tag{2.14}$$

where $\mathcal{H} = [0,\infty) \times \mathbb{R}^1$, $v_h = (u_h)^{1/(m-1)}$, $u_h(t,x) = u_h^n(x)$ on $[t_n, t_{n+1}) \times \mathbb{R}^1$ for all t_n and h, and T^* is the extinction time.

Then, from THEOREMS 2.1 and 2.2 and the fact that $v(t,x)$ is smooth on $\mathcal{P}(v)$ we have

LEMMA 2.3 (Basic estimates). *Assume* CONDITION B. *Then*

$$0 \leq u(t,\cdot) \leq \max(\|u^0\|_\infty - c't,\, 0) \quad \text{on } \mathbb{R}^1, \tag{2.15}$$

$$\|u_x(t,\cdot)\|_\infty \leq \|u_x^0\|_\infty, \tag{2.16}$$

$$\int_{b_1}^{b_2} |u_{xx}(t,x)|\,\mathrm{d}x = TV(u_x(t,\cdot)) \leq TV((u^0)_x) \quad \text{for all } t \text{ and intervals } [b_1,b_2] \subset \mathcal{P}(u), \tag{2.17}$$

$$\text{ess.inf}_{\mathbf{I}}\, u_{xx}^0 \leq u_{xx}(t,x) \quad \text{for } (t,x) \in \mathcal{P}(u). \tag{2.18}$$

THEOREM 2.4 (Convergence of numerical interface curves [9]). *Under* CONDITION B *let there exist a positive constant M such that*

$$((v^0)^{m-1})_x(\ell_0 + 0),\ -((v^0)^{m-1})_x(r_0 - 0) > M. \tag{2.19}$$

Let $M'(< M)$ be an arbitrary positive number. Then,

$$(v^{m-1})_x(t, \ell(t)+0),\ -(v^{m-1})_x(t, r(t)-0) > M' \text{ on } [0,\ T(M', v^0)), \tag{2.20}$$

and $\ell_h(t)$ (resp. $r_h(t)$) converges uniformly to the exact left (resp. right) interface curve $\ell(t)$ (resp. $r(t)$) on $[0,T]$ as h tends to zero for each fixed $T < T(M', v^0)$, where

$$T(M', v^0) = \frac{(M - M')M'}{(2a+m)C_1(v^0)C_2(v^0)M' + 3c'C_2(v^0)}. \tag{2.21}$$

Moreover, the following interface equations

$$\dot{\ell}(t) = -a(v^{m-1})_x(t,\ \ell(t)+0) + \frac{c'}{(v^{m-1})_x(t,\ \ell(t)+0)}, \tag{2.22}$$

$$\dot{r}(t) = -a(v^{m-1})_x(t,\ r(t)-0) + \frac{c'}{(v^{m-1})_x(t,\ r(t)-0)} \tag{2.23}$$

hold for a.e. $t \in\ [0, T(M', v^0))$.

Theorem 2.5 (Support splitting phenomena [10]). *Assume* Condition B. *For $\alpha_1 < \beta_1 < \gamma_1 < \gamma_2 < \beta_2 < \alpha_2$ let $v^0(x)$ satisfy*

$$v^0(x) > 0 \ \text{on}\ (\alpha_1,\ \alpha_2), \quad [\alpha_1,\ \alpha_2] = \operatorname{supp} v^0(x), \tag{2.24}$$

$$\frac{(v^0(\beta_j))^{m-1}}{c' + mC_0C_2} > \frac{\varepsilon^{m-1}}{c'}\ (j=1,2) \ \text{and}\ \ v^0(x) = \varepsilon \ \text{on}\ [\gamma_1, \gamma_2], \tag{2.25}$$

where $C_j = C_j(v^0)$ $(j = 0, 2)$ are given by (2.6). *Then, for sufficiently large $\sigma \equiv \gamma_2 - \gamma_1$, there exist $\tilde{t} > 0$ and $\tilde{x} \in [\gamma_1,\ \gamma_2]$ such that $v(\tilde{t}, \tilde{x}) = 0$ and $v(\tilde{t}, \beta_j) > 0$ $(j = 1, 2)$ hold.*

3. Support re-splitting phenomena. First, we introduce two nonnegative functions $\phi(x;\varepsilon)$ and $\psi(x;\varepsilon, d_1, d_2)$ for arbitrary positive numbers ε, d_1 and d_2, which satisfy the following

Condition C.

i) $\phi(x;\varepsilon)$ satisfies Conditions B with $v^0(x) = \phi(x;\varepsilon)$ and supp $\phi = [0,\ \alpha\ ]$;

ii) $\phi(x;\varepsilon)$ takes the unique local maximum at $x = \beta$, and

$$\phi(x;\varepsilon) = \varepsilon \quad \text{on} \quad [\xi,\ \gamma\ ], \tag{3.1}$$

where $0 < \xi < \gamma < \beta < \alpha$;

iii)

$$\psi(x;\varepsilon, d_1, d_2) = \begin{cases} 0 & \text{if} \quad -\infty < x < d_1, \\ \phi(x - d_1;\varepsilon) & \text{if} \quad d_1 < x < \xi + d_1, \\ \varepsilon & \text{if} \quad \xi + d_1 < x < \gamma + d_1 + d_2, \\ \phi(x - d_1 - d_2;\varepsilon) & \text{if} \quad \gamma + d_1 + d_2 < x. \end{cases} \tag{3.2}$$

Putting

$$v^0(x;\varepsilon, d_1, d_2) = \psi(x;\varepsilon, d_1, d_2) + \psi(-x;\varepsilon, d_1, d_2) \quad \text{on}\ \mathbb{R}^1, \tag{3.3}$$

we choose ε, d_1 and d_2 so that *support connecting and splitting phenomena* appear for the initial function $v^0(x;\varepsilon, d_1, d_2)$.

Next, for $v^0(x;\varepsilon, d_1, d_2)$ we introduce the initial function $v^0_\rho(x;\varepsilon, d_1, d_2)$ $(0 < \rho < \varepsilon)$ satisfying

Condition D.

i) $v^0_\rho(x;\varepsilon,d_1,d_2) = v^0_\rho(-x;\varepsilon,d_1,d_2)$ and $v^0(x;\varepsilon,d_1,d_2) \le v^0_\rho(x;\varepsilon,d_1,d_2)$ hold on $\mathbb{R}^1$;

ii)

$$v^0_\rho(x;\varepsilon,d_1,d_2) = \begin{cases} v^0(x;\varepsilon,d_1,d_2) & \text{if} \quad x \le -d_1-\eta, \text{ or } d_1+\eta \le x \\ \rho & \text{if} \quad -d_1 \le x \le d_1, \end{cases} \tag{3.4}$$

where $0 < \eta < \xi$, and $v^0_\rho(x;\varepsilon,d_1,d_2)$ decreases on $[-d_1-\eta,-d_1]$ and increases on $[d_1,d_1+\eta]$;

iii) $v^0_{\rho'}(x;\varepsilon,d_1,d_2) \le v^0_\rho(x;\varepsilon,d_1,d_2)$ holds for $\rho' \le \rho$;

iv) v^0_ρ satisfies Condition B with $v^0(x) = v^0_\rho(x;\varepsilon,d_1,d_2)$ and

$$\begin{cases} \|u^0_{\rho x}(\cdot;\varepsilon,d_1,d_2)\|_\infty \le \|u^0_x(\cdot;\varepsilon,d_1,d_2)\|_\infty, \\ TV(u^0_{\rho x}(\cdot;\varepsilon,d_1,d_2)) \le \big(TV(u^0_x(\cdot;\varepsilon,d_1,d_2))\big), \\ \text{ess.inf } u^0_{\rho xx}(\cdot;\varepsilon,d_1,d_2) \ge \text{ess.inf } u^0_{xx}(\cdot;\varepsilon,d_1,d_2), \end{cases} \tag{3.5}$$

where $u^0(x;\varepsilon,d_1,d_2)=(v^0(x;\varepsilon,d_1,d_2))^{m-1}$ and $u^0_\rho(x;\varepsilon,d_1,d_2)=(v^0_\rho(x;\varepsilon,d_1,d_2))^{m-1}$ (see Fig. 3.1).

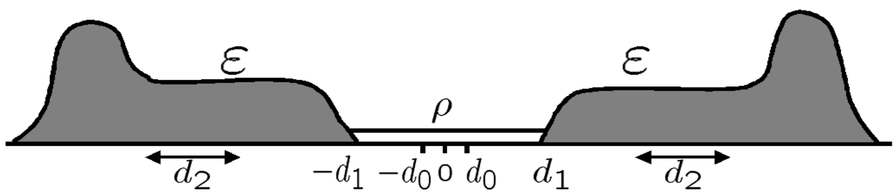

Fig. 3.1. *Initial function* $v^0_\rho(x;\varepsilon,d_1,d_2)$.

Taking the constant ρ sufficiently small, we can show that *support splitting, connecting, and re-splitting phenomena* appear in the behavior of supp $v_\rho(t,x;\varepsilon,d_1,d_2)$, where $v_\rho(t,x;\varepsilon,d_1,d_2)$ is the solution of (1.1) with $v(0,x) = v^0_\rho(x;\varepsilon,d_1,d_2)$. We state our results.

Theorem 3.1 (Support connecting and splitting phenomena [13]). *Let $\phi(x;\varepsilon)$ satisfy* Condition C *and the following inequalities:*

$$\lim_{x\to+0} \big((\phi(x;\varepsilon))^{m-1}\big)_x > \sqrt{\frac{c'}{a}}\ , \tag{3.6}$$

and

$$\frac{(\phi(\beta;\varepsilon))^{m-1}}{c' + mC_0(\phi)C_2(\phi)} > \frac{\varepsilon^{m-1}}{c'}, \tag{3.7}$$

where $C_j(\phi)$ $(j=0,2)$ are given by (2.6) *with $v^0=\phi$. Then, for sufficiently small d_1 and sufficiently large d_2 there exist constants T_1, T_2 $(T_2 > T_1 > 0)$ and $\tilde{x} \in [\xi+d_1,\ \gamma+d_1+d_2]$ such that* supp $v(T_1,\cdot\,;\varepsilon,d_1,d_2)$ *is connected and*

$$\begin{cases} v(T_2,x;\varepsilon,d_1,d_2) = 0 \quad \text{on } [-\tilde{x},\tilde{x}] \text{ and} \\ v(t,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2) > 0 \ \text{for } t \le T_2 \ (j=1,2), \end{cases} \tag{3.8}$$

where $v(t,x;\varepsilon,d_1,d_2)$ is the solution of (1.1) *with $v(0,x) = v^0(x;\varepsilon,d_1,d_2)$ given by* (3.3).

Proof. The support connecting property follows from Theorem 2.4, and the support splitting property is obtained by applying Theorem 2.5 to $v^0(x;\varepsilon,d_1,d_2)$ with $\gamma_1=\xi+d_1$ and $\gamma_2=\gamma+d_1+d_2$. Thus the proof immediately follows. See the details in the proof of [13, Theorem 3.2] by the author. ∎

THEOREM 3.2 (Support re-splitting phenomena). *Let the initial function* $v^0(x;\varepsilon,d_1,d_2)$ *and the constant* T_1 *satisfy the conclusion of* THEOREM 3.1. *Assume that* $v^0_\rho(x;\varepsilon,d_1,d_2)$ *satisfies* CONDITION D. *Then for sufficiently small* $\rho>0$*, there exist constants* T_0*,* T_2 ($T_2>T_1>T_0>0$)*,* $\hat{x}$ *and* $\tilde{x}$ *such that* supp $v_\rho(T_1,\cdot\,;\varepsilon,d_1,d_2)$ *is connected and*

$$\begin{cases} v_\rho(T_0,x;\varepsilon,d_1,d_2)=0 \quad \text{on } [-\hat{x},\hat{x}] \text{ and} \\ v_\rho(T_0,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2)>0 \ (j=1,2), \end{cases} \tag{3.9}$$

$$\begin{cases} v_\rho(T_2,x;\varepsilon,d_1,d_2)=0 \quad \text{on } [-\tilde{x},\tilde{x}] \text{ and} \\ v_\rho(T_2,(-1)^j\beta+(-1)^j(d_1+d_2);\varepsilon,d_1,d_2)>0 \ (j=1,2). \end{cases} \tag{3.10}$$

Thus the appearance of *support re-splitting phenomena* follows from this theorem.

Proof. [*of Theorem 3.2.*] In the following, the constants ε, d_1 and d_2 given by THEOREM 3.1 are fixed. For simplicity we put

$$v_\rho(t,x)=v_\rho(t,x;\varepsilon,d_1,d_2) \quad\text{and}\quad u_\rho(t,x)=u_\rho(t,x;\varepsilon,d_1,d_2)\equiv(v_\rho(t,x;\varepsilon,d_1,d_2))^{m-1}.$$

We note that $v_\rho(t,\pm(d_1+d_2+\beta))>0$ for $t<T_1$ and $\rho>0$. Putting $\mathbf{S}=[0,\ T_1]\times[d_0,d_1]$ for an arbitrary fixed positive constant $d_0<d_1$, we show that $\mathbf{S}$ contains at least one point $(\tilde{t},\hat{x})$ such that $v_{\tilde{\rho}}(\tilde{t},\hat{x})=0$ for some positive constant $\tilde{\rho}$. For this end we assume the contrary; that is, suppose $v_\rho(t,x)>0$ on $\mathbf{S}$ for $\rho>0$. By LEMMA 2.3, CONDITION D and (2.1) we obtain

$$\begin{aligned} \int_{d_0}^{d_1} u_\rho(t,x)\,\mathrm{d}x &= \int_{d_0}^{d_1} u_\rho(0,x)\,\mathrm{d}x \\ &\quad + \int_0^t\int_{d_0}^{d_1}\left\{m u_\rho(t,x)u_{\rho xx}(t,x)+a(u_{\rho x}(t,x))^2-c'\right\}\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.11}$$

$$\begin{aligned} &= (d_1-d_0)\rho^{m-1} \\ &\quad -\int_0^t\left\{(d_1-d_0)c'-(m-2)a\int_{d_0}^{d_1}u_\rho(t,x)u_{\rho xx}(t,x)\,\mathrm{d}x - a\Big[u_\rho(t,x)u_{\rho x}(t,x)\Big]_{d_0}^{d_1}\right\}\mathrm{d}t \\ &\le (d_1-d_0)\rho^{m-1} \\ &\quad -\left\{(d_1-d_0)c'-a\max_{[0,t]\times[d_0,d_1]}u_\rho(t,x)\Big((2-m)TV(u^0_{\rho x})+2\|u^0_{\rho x}\|_\infty\Big)\right\}t \\ &\qquad\qquad \text{for } t\in[0,T_1]. \end{aligned}$$

Let ρ_1 be an arbitrary fixed positive constant such that

$${\rho_1}^{m-1}<\frac{(d_1-d_0)c'}{a\Big((2-m)TV(u^0_x)+2\|u^0_x\|_\infty\Big)}. \tag{3.12}$$

Then, by the continuity of the solution $v_\rho(t,x)$ and the comparison theorem [2] on the initial data there exist positive constants ρ_2 and $\tilde{T}_1 < T_1$ such that

$$\max_{[0,t]\times[d_0,d_1]} u_\rho(t,x) < {\rho_1}^{m-1} \tag{3.13}$$
$$\text{for} \quad t < \tilde{T}_1 \quad \text{and} \quad \rho < \rho_2 < \min\left(\rho_1, \psi(d_1+\eta,\varepsilon,d_1,d_2)\right).$$

We put

$$T(\rho) = \frac{(d_1-d_0)\rho^{m-1}}{(d_1-d_0)c' - a{\rho_1}^{m-1}\Big((2-m)TV(u_x^0) + 2\|u_x^0\|_\infty\Big)}, \tag{3.14}$$

and choose $\tilde{\rho} < \rho_2$ such that $T(\tilde{\rho}) < \tilde{T}_1$. Hence, it follows from (3.11) and CONDITION D that

$$\int_{d_0}^{d_1} u_{\tilde{\rho}}(t,x)\,\mathrm{d}x < 0 \quad \text{for} \quad t \in (T(\tilde{\rho}), \tilde{T}_1], \tag{3.15}$$

which is a contradiction. Thus, $v_{\tilde{\rho}}(T_0,\hat{x}) = 0$ holds for some $(T_0,\hat{x}) \in \mathbf{S}$. Since the solution is an even function and the number of the local maximum points is nonincreasing, it follows that $v_{\tilde{\rho}}(T_0,x;\varepsilon,d_1,d_2) = 0$ holds on $[-\hat{x},\hat{x}]$. It is clear by THEOREM 3.1 and the comparison theorem that supp $v_{\tilde{\rho}}(T_1,\cdot)$ becomes connected. Thus (3.9) follows.

Taking the proof of THEOREM 3.2 [13] and the inequality $u_{\tilde{\rho}}(0,x) = u^0_{\tilde{\rho}}(x) < \varepsilon^{m-1}$ ($x \in [-d_1,d_1]$) into consideration, we find that (3.10) follows, and the proof is complete.□

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