Home Page

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

Proceedings of Equadiff-11
2005, pp. 567–575

CONNECTIONS BETWEEN SPATIAL DECAY OF INITIAL DATA AND TIME ASYMPTOTICS IN A SUPERCRITICAL PARABOLIC EQUATION*

MICHAEL WINKLER†

Abstract. The article collects some recent results concerning the large time behavior of nonnegative solutions to $u_t = \Delta u + u^p$ with supercritical p. Quantitative connections between the spatial asymptotics of the initial data and the grow-up rates (resp. convergence rates) of solutions are established. In particular, a *continuum* of such rates appear whenever p and the space dimension are large enough.

Key words. semilinear diffusion equation, supercritical exponent, grow-up, convergence

AMS subject classifications. 35K55, 35B40

1. Introduction and examples. This paper deals with the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where u_0 is a nonnegative continuous function on $\mathbb{R}^N$, $N \geq 11$ and p is supercritical in the sense that

$$p > p_c := \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)}.$$

*This work was supported by the European Community's Human Potential Programme under contract HPRN-CT-2002-00274, Fronts-Singularities

†Department of Mathematics, RWTH Aachen, 52056 Aachen, Germany (winkler@math1.rwth-aachen.de).

That p_c indeed appears as a critical exponent for (1.1) is suggested by an analysis of the set of radially symmetric stationary solutions: In fact, let $\varphi_\alpha = \varphi_\alpha(r), \alpha > 0$, denote the solution of

$$\begin{cases} \varphi_{\alpha,rr} + \frac{N-1}{r}\varphi_{\alpha,r} + \varphi_\alpha^p = 0, \\ \varphi_\alpha(0) = \alpha, \qquad \varphi_{\alpha,r}(0) = 0, \end{cases} \tag{1.2}$$

where we interpret $s^p = |s|^{p-1}s$ whenever $s < 0$. Then it is well-known that

- the solution of (1.2) remains positive if and only if $p \geq p_S$ holds with the Sobolev exponent

$$p_S = \begin{cases} \dfrac{N+2}{N-2} & \text{for } N \geq 3, \\ \infty & \text{for } N < 3, \end{cases}$$

  but
- if $p_S \leq p < p_c$ any of these positive steady states intersects with other positive steady states (see [9]). However,
- for $p \geq p_c$, it was shown by Gui, Ni and Wang [5] that the set $\{\varphi_\alpha \mid \alpha > 0\}$ is ordered, that is, $\varphi_\alpha(r)$ is strictly increasing in α for each r.

The latter work furthermore reveals that

$$\lim_{\alpha\to 0} \varphi_\alpha(r) = 0 \quad \text{and} \quad \lim_{\alpha\to\infty} \varphi_\alpha(r) = \varphi_\infty(r),$$

where φ_∞ is a singular steady state explicitly written as

$$\varphi_\infty(|x|) = L|x|^{-m}, \qquad |x| > 0$$

with

$$m := \frac{2}{p-1} \quad \text{and} \quad L := \{m\,(N-2-m)\}^{1/(p-1)}.$$

It is also shown in [5] that each positive regular steady state φ_α has the asymptotic behavior

$$\varphi_\alpha(|x|) = \begin{cases} L|x|^{-m} - a|x|^{-m-\lambda_1} + h.o.t. & \text{if } p > p_c \\ L|x|^{-m} - a|x|^{-m-\lambda_1} \log|x| + h.o.t. & \text{if } p = p_c \end{cases} \tag{1.3}$$

as $|x| \to \infty$, where λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N,p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

and $a = a(\alpha, N, p)$ is a positive number that is monotone decreasing in α. We note that the quadratic equation

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0$$

has two positive roots if and only if $p > p_c$; the smaller root is λ_1 and the larger root is given by

$$\lambda_2 = \lambda_2(N,p) := \frac{N-2-2m+\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2}.$$

The goal of this paper is to study the behavior of positive solutions of (1.1) bounded above by the singular steady state. If we assume that u_0 satisfies

$$0 \le u_0(x) \le \varphi_\infty(|x|) \quad \text{for } |x| > 0, \tag{1.4}$$

then the solution (1.1) exists globally in time (see [8]), and by comparison, the solution remains between the trivial steady state and the singular steady state for all $t > 0$.

To get a first idea of what might happen, one can consider initial data u_0 satisfying $u_0(x) = A|x|^{-k} + h.o.t.$ as $|x| \to \infty$, where $k = m$ and $A \le L$, or $k > m$. If either $k > m$ or $k = m$ and $A < L$ then for any $\alpha > 0$, $u_0(x)$ lies below $\varphi_\alpha(|x|)$ for all sufficiently large

$|x|$, which gives rise to the conjecture that φ_α might eventually majorize u as well, so that u should converge to zero as $t \to \infty$. In fact, Poláčik and Yanagida [8] proved that

- if $\limsup_{|x|\to\infty} |x|^m u_0(x) < L$ then $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$.

As to the remaining case $k = m$ and $A = L$, one may expand u_0 one step further and assume $u_0(x) = L|x|^{-m} - b|x|^{-l} + h.o.t.$ as $|x| \to \infty$ for some $b > 0$ and $l > m$. Note that large values of l correspond to small distances of u_0 to $\varphi_\infty(|\cdot|)$. Correspondingly, if we assume some appropriate stability of φ_∞ then it is plausible that $u(\cdot,t)$ will tend to $\varphi_\infty(|\cdot|)$ as $t \to \infty$ and thus be unbounded. Indeed, such a behavior was proved in [8] in the case $l > m + \lambda_1$:

- If $\lim_{|x|\to\infty} |x|^{m+\lambda_1}(\varphi_\infty(|x|) - u_0(x)) = 0$ then $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to \infty$.

If $l = m + \lambda_1$, however, (1.3) shows that a similar statement cannot hold any longer. It is basically due to the ordering property of the φ_α that all these regular steady states are asymptotically stable ([8], [5], [6]) in the sense that for $p > p_c$,

- if $\lim_{|x|\to\infty} |x|^{m+\lambda_1}(u_0(x) - \varphi_\alpha(|x|)) = 0$ then $\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $t \to \infty$.

This particularly means that $u_0(x) = L|x|^{-m} - b|x|^{-m-\lambda_1} + h.o.t.$ as $|x| \to \infty$ implies $u(\cdot,t) \to \varphi_\alpha(|\cdot|)$ as $t \to \infty$, where α has to be chosen such that b coindides with the number a_α in (1.3) (which actually satisfies $a_\alpha = \alpha^{-\frac{\lambda_1}{m}} a_1$).

Once one knows qualitative grow-up and convergence results as listed above, the next step is to determine the quantitative rates at which these occur. It turns out that in both cases a continuum of possible rates appears, and that these can precisely be determined in terms of the spatial decay of the initial data.

Before going into detail, let us remark that to the best of our knowledge nothing is known about the behavior of solutions emanating from initial data satisfying $u_0(x) = L|x|^{-m} -$

$b|x|^{-l} + h.o.t.$ as $|x| \to \infty$ when l is bigger than m but smaller than $m + \lambda_1$. One might guess that such solutions should tend to zero as $t \to \infty$, but we are not aware of a proof of this conjecture.

2. Grow-up rates. The history of quantitative grow-up rates started in [2], where the following upper bound was given in Proposition 3.3.

Theorem 2.1. *Let $p \geq p_c$. Suppose that u_0 satisfies (1.4) and*

$$u_0(x) \leq L|x|^{-m} - b|x|^{-l} \quad \text{for } |x| > R$$

with some constants $l > m + \lambda_1$ and $b, R > 0$. Then there exist positive constants C and T such that the solution of (1.1) satisfies

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > T$.

The upper bound in this theorem is not optimal for large l. In fact, it was shown in [2] that there is a universal upper bound independent of the initial data. A *sharp* universal upper bound was found by Mizoguchi [7].

Theorem 2.2. *Let $p > p_c$. Suppose that u_0 satisfies (1.4). Then there exist positive constants C_1 and T such that the solution of (1.1) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_1 t^{\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}}$$

for all $t > T$. Moreover, there exist u_0 satisfying

$$u_0(x) \geq L|x|^{-m} - b e^{-|x|^2/4} \quad \text{for } |x| > R$$

with some $b, R > 0$ such that the solution of (1.1) satisfies

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq C_2 t^{\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}}$$

with some $C_2 > 0$ *for all* $t > 0$.

Concerning the lower bound, only a partial result was obtained in [2] in the case of $l \in (m + \lambda_1, m + \lambda_2]$.

THEOREM 2.3. *Let* $p > p_c$. *Suppose that* u_0 *satisfies* (1.4) *and*

$$u_0(x) \geq L|x|^{-m} - b|x|^{-l} \quad \text{for } |x| > 0,$$

with some constants $l \in (m + \lambda_1, m + \lambda_2]$ *and* $b > 0$. *Then there exists a positive constant* C *such that the solution of* (1.1) *satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \geq Ct^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > 0$.

This theorem implies that the upper bound obtained in THEOREM 2.1 is optimal for $l \in (m+\lambda_1, m+\lambda_2]$. However, it is clear from the universal upper bound in THEOREM 2.2 that THEOREM 2.3 cannot be extended for all larger l; more precisely, comparing THEOREMS 2.2 and 2.3 suggests a borderline value of l located at $l = m + \lambda_2 + 2$. In fact, very recently this conjecture could be proved [1]:

THEOREM 2.4. *Using the above notation,* THEOREM 2.3 *is valid for any* $l \in (m + \lambda_1, m + \lambda_2 + 2)$.

Altogether we obtain

THEOREM 2.5. *Let* $p > p_c$. *Suppose that* u_0 *satisfies* (1.4) *and*

$$L|x|^{-m} - b_1|x|^{-l} \leq u_0(x) \leq L|x|^{-m} - b|x|^{-l} \quad \text{for } |x| > 1$$

with some $l > m + \lambda_1$ *and* $0 < b_2 < b_1$. *Then*

i) *if* $l < m + \lambda_2 + 2$ *then there exist positive constants* C_1 *and* C_2 *such that the solution* u *of* (1.1) *satisfies*

$$C_1 t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_2 t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}} \qquad \forall\, t > 1;$$

ii) *if* $l \geq m + \lambda_2 + 2$ *then there exist* C_2 *and, given any* $\varepsilon > 0$, $C_{1,\varepsilon}$ *such that*

$$C_{1,\varepsilon} t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}} \leq \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_2 t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}} \qquad \forall\, t > 1.$$

Finally, the biggest possible rate in fact is attained by a class of solutions larger than indicated in Theorem 2.2.

Theorem 2.6. *Let* $p > p_c$. *Suppose that* u_0 *satisfies* (1.4) *and*

$$u_0(x) \geq L|x|^{-m} - b\,\mathrm{e}^{-\nu|x|^2} \qquad \textit{for } |x| \geq 1$$

with some positive constants b *and* ν. *Then there exists a positive constant* C *such that the solution of* (1.1) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \geq C t^{\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}} \qquad \forall\, t > 0.$$

2.1. Formal matched asymptotics. A powerful approach towards precise estimates is provided by the tool of formal matched asymptotics. The first step of this method consists of finding different asymptotic expansions of possible solutions to (1.1), each of these – hopefully – giving a good approximation to u within some subregion of the space-time cylinder. Usually, those expansions contain some parameters which are linked to some unknown quantity related to u (such as $u(0,t)$, for instance). The *matching condition*, requiring that such expansions must coincide at each intersection of these regions, is then expected to determine the values of possible parameters and thereby provide some information about u.

Apart from giving an idea of how u might look like, such a procedure can very often be used as the starting point for a rigorous proof of this behavior.

Let us briefly outline how formal matched asymptotics can be carried out in order to derive the results in THEOREMS 2.1–2.6. For more detailed presentations, consult [2] and [1].

We first observe that any radial solution $u = u(r,t)$, $r = |x|$, of (1.1) satisfies

$$\begin{cases} u_t = u_{rr} + \frac{N-1}{r} u_r + u^p, & r > 0,\ t > 0, \\ u(r,0) = u_0(r), & r > 0. \end{cases} \tag{2.1}$$

Let us seek an 'inner' expansion, describing u for small r (but large t). We therefore rewrite $u(r,t)$ as

$$u(r,t) = \sigma(t)\left\{\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Phi(\xi,t)\right\}, \tag{2.2}$$

where $\sigma(t) := u(0,t)$, $\xi := \sigma^{1/m} r$, and $\psi := \varphi_1(\xi)$ satisfies

$$\begin{cases} \psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \psi^p = 0, & \xi > 0, \\ \psi(0) = 1, \quad \psi'(0) = 0. \end{cases} \tag{2.3}$$

Substituting (2.2) in (2.1), we have

$$\psi_{\xi\xi} + \frac{N-1}{\xi}\psi_\xi + \frac{\sigma_t}{\sigma^p}\left(\Phi_{\xi\xi} + \frac{N-1}{\xi}\Phi_\xi\right) + \left(\psi + \frac{\sigma_t}{\sigma^p}\Phi\right)^p \sim \frac{\sigma_t}{\sigma^p}\left(\psi + \frac{1}{m}\,\xi\psi_\xi\right)$$

under some reasonable assumptions on σ and Φ. In view of (2.3), we may put $\Phi = \Psi(\xi) +$ $h.o.t.$, where Ψ satisfies

$$\Psi_{\xi\xi} + \frac{N-1}{\xi}\Psi_\xi + p\psi^{p-1}\Psi = \psi + \frac{1}{m}\,\xi\psi_\xi. \tag{2.4}$$

Thus we obtain the two-term inner expansion

$$u(r,t) \sim \sigma(t)\Big\{\psi(\xi) + \frac{\sigma_t}{\sigma^p}\Psi(\xi)\Big\}.$$

In the inner region, where $\psi(\xi)$ dominates $(\sigma_t/\sigma^p)\Psi$, this can be reduced, using (1.3), to

$$u \sim \sigma(L\xi^{-m} - a\xi^{-m-\lambda_1}) = Lr^{-m} - a\sigma^{-\lambda_1/m}r^{-m-\lambda_1}. \qquad (2.5)$$

Next, we consider a formal expansion in the 'outer' region which is to be understood as having $r \gg 1$ as $t \to \infty$. Setting

$$u = Lr^{-m} - v$$

and assuming $v \ll r^{-m}$ for $r \gg 1$, we have

$$v_t \sim v_{rr} + \frac{N-1}{r}v_r + \frac{pL^{p-1}}{r^2}v, \qquad r \gg 1.$$

We will find a solution which behaves in a self-similar way for $r \gg 1$:

$$v(r,t) = t^{-l/2}F(\eta), \qquad \eta = t^{-1/2}r,$$

so that the specific scaling for $r \gg 1$ corresponding to the outer region is in fact $r = O(\sqrt{t})$ as $t \to \infty$. Here F satisfies

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{l}{2}F + \frac{pL^{p-1}}{\eta^2}F = 0, \quad \eta > 0. \qquad (2.6)$$

In order that the outer expansion matches with (2.5), F must satisfy

$$\lim_{\eta\to 0} \eta^{m+\lambda_1}F(\eta) = c_1 > 0. \qquad (2.7)$$

On the other hand, F is required to satisfy

$$\lim_{\eta\to\infty} \eta^l F(\eta) = c_2 > 0; \tag{2.8}$$

in view of the linearity of (2.6), c_1 is an arbitrary constant (depending on the initial data), while c_1/c_2 depends only on l, N and pL^{p-1}. It has been shown in [1, Lemma 3.1] that (2.6) has a positive solution satisfying (2.7) and (2.8) if and only if $l \in (m+\lambda_1, m+\lambda_2+2)$. In this case, we obtain the two-term outer expansion

$$u \sim Lr^{-m} - t^{-l/2}F(t^{-1/2}r). \tag{2.9}$$

Now we match the inner expansion (2.5) and the outer expansion (2.9) for $1 \ll r \ll \sqrt{t}$ to obtain

$$\sigma(t) \sim \left(\frac{c_1}{a}\right)^{-\frac{m}{\lambda_1}} t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}.$$

This gives us a very plausible guess on the grow-up rate. Based on this formal argument, one can construct, for instance, suitable subsolutions having a shape similar to (2.5) for $rt^{-\frac{1}{2}}$ small, and similar to (2.9) wherever $rt^{-\frac{1}{2}}$ is large, and thereby prove the lower bounds asserted in the Theorems 2.4–2.6 (cf. [1, Sections 3 and 4]).

3. Stabilization rates. In this section we state a result on the rate of convergence towards regular steady states (cf. [3]).

Theorem 3.1. *Let $p > p_c$ and suppose u_0 satisfies (1.4) as well as*

$$|u_0(x) - \varphi_\alpha(|x|)| \le c(1+|x|)^{-l}, \qquad x \in \mathbb{R}^N,$$

with some $l \in (m+\lambda_1, m+\lambda_2), \alpha \ge 0$ and $c > 0$. Then there exists $C > 0$ such that

$$\|u(\cdot,t) - \varphi_\alpha(|\cdot|)\|_{L^\infty(\mathbb{R}^N)} \le Ct^{-\frac{l-m-\lambda_1}{2}} \qquad \forall\, t > 0.$$

3.1. Formal matched asymptotics. We again confine ourselves to a sketch of the proof, outlining the main ideas of the formal approach and leaving out the technical details; for these, see [3, Sections 3 and 4].
When dealing with convergence to some bounded steady state, it is natural to linearize the equation about this equilibrium. For the formal expansion procedure, we therefore fix $\alpha \geq 0$ and let

$$P_\alpha U := U_{rr} + \frac{N-1}{r} U_r + p\varphi_\alpha^{p-1} U$$

denote the linearized operator at φ_α. Then, if u is radial then $U(r,t) := u(r,t) - \varphi_\alpha(r)$ should, up to lower order terms, satisfy

$$\begin{cases} U_t = P_\alpha U, & r > 0,\ t > 0, \\ U_r(0,t) = 0, & t > 0, \\ U(r,0) = U_0(r), & r \geq 0, \end{cases}$$

where U_0 is a continuous function that decays to zero as $r \to \infty$.

We derive the formal expansion of U near the origin as follows. Set

$$U(r,t) = \sigma(t) f(r,t),$$

where we put $\sigma(t) := U(0,t)$. Substituting this in the above equation, we obtain

$$\begin{cases} P_\alpha f = \dfrac{\sigma_t}{\sigma} f + f_t, & r > 0, \\ f(0,t) = 1, \qquad f_r(0,t) = 0, & t > 0. \end{cases} \tag{3.1}$$

Assuming that

$$|f_t| \ll \left| \frac{\sigma_t}{\sigma} \right| \ll 1,$$

we may put

$$f(r,t) = \psi(r) + \frac{\sigma_t}{\sigma}\Phi(r,t), \tag{3.2}$$

where ψ satisfies

$$\begin{cases} P_\alpha \psi = 0, & r > 0, \\ \psi(0) = 1, & \psi_r(0) = 0. \end{cases} \tag{3.3}$$

The solution of this problem is given by

$$\psi(r) = \frac{\partial}{\partial \alpha}\varphi_\alpha(r).$$

Inserting (3.2) in (3.1) and using (3.3), we see that

$$P_\alpha \Phi = f + \frac{f_t}{\sigma_t/\sigma}.$$

Since the second term on the right hand side is expected to be small, we consider the inhomogeneous equation

$$\begin{cases} P_\alpha \Psi = \psi, & r > 0, \\ \Psi(0) = 0, & \Psi_r(0) = 0. \end{cases} \tag{3.4}$$

Using a solution of this equation, we may write Φ as

$$\Phi(r,t) = \Psi(r) + h.o.t..$$

Thus, for each $r > 0$, a formal expansion near the origin is obtained as

$$U = \sigma\psi + \sigma_t \Psi + h.o.t.. \tag{3.5}$$

Next, let us consider the expansion near $r = \infty$ of a solution of the linearized equation that behaves like r^{-l} for large r. By (1.3), U atisfies

$$U_t = U_{rr} + \frac{N-1}{r} U_r + \frac{pL^{p-1}}{r^2} U + h.o.t., \qquad r \simeq \infty.$$

Setting

$$U(r,t) = r^{-l} + V(r,t),$$

we have

$$V_t = P_\alpha V + g(l) r^{-l-2} + h.o.t.,$$

where we let

$$g(\mu) := \mu^2 - (N-2)\mu + pL^{p-1}.$$

Upon the plausible assumption

$$|P_\alpha V| \ll r^{-l-2}$$

we obtain

$$V(r,t) = g(l)(t+\tau) r^{-l-2} + h.o.t.,$$

where τ is an integral constant. Thus a formal expansion near $r = \infty$ is given by

$$U(r,t) = r^{-l} + g(l)(t+\tau) r^{-l-2} + h.o.t., \qquad r \simeq \infty. \tag{3.6}$$

Now let us math the inner expansion (3.5) and the outer expansion (3.6). Since it is known (see [3, Lemmata 2.3 and 2.4], for example) that

$$\psi(r) = c_\alpha r^{-m-\lambda_1} + h.o.t., \qquad r \simeq \infty,$$

for some $c_\alpha > 0$ and

$$\Psi(r) = C_\alpha r^{-m-\lambda_1+2} + h.o.t., \qquad r \simeq \infty,$$

with some $C_\alpha > 0$, we obtain from (3.5)

$$U(r,t) = c_\alpha \sigma r^{-m-\lambda_1} + C_\alpha \sigma_t r^{-m-\lambda_1+2} + h.o.t., \qquad r \simeq \infty.$$

Equating the first and second order terms of this expansion with (3.6), we find

$$c_\alpha \sigma r^{-m-\lambda_1} = r^{-l} \qquad \text{and} \qquad C_\alpha \sigma_t r^{2-m-\lambda_1} = g(l)(t+\tau) r^{-l-2}.$$

Eliminating r, we obtain

$$C_\alpha \sigma_t = g(l)(t+\tau)(c_\alpha \sigma)^{1+\frac{4}{l-m-\lambda_1}}.$$

Now if $g(l) < 0$ (which precisely corresponds to saying $l \in (m+\lambda_1, m+\lambda_2)$), this shows that

$$\sigma(t) = C(t+\tau)^{-\frac{l-m-\lambda_1}{2}}$$

with some $C > 0$, which is exactly the convergence rate asserted by Theorem 3.1. Conversely, if $g(l) \geq 0$, this does not necessarily lead to the convergence of U to the trivial solution at all. In fact, if $l < m + \lambda_1$, the trivial solution of the linearized equation is unstable (see [5]). The case $l \geq m + \lambda_2$ deserves a further study.

REFERENCES

[1] M. Fila, J. R. King, M. Winkler, E. Yanagida, *Optimal lower bound of the grow-up rate for a supercritical parabolic problem*, submitted.

[2] M. Fila, M. Winkler and E. Yanagida, *Grow-up rate of solutions for a supercritical semilinear diffusion equation*, J. Differ. Equations **205** (2004), 365–389.

[3] M. Fila, M. Winkler, E. Yanagida, *Convergence rate for a parabolic equation with supercritical nonlinearity*, J. Dyn. Differ. Equations **17**(2) (2005), 249–269.

[4] V. A. Galaktionov and J. R. King, *Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents*, J. Differ. Equations **189** (2003), 199–233.

[5] C. Gui, W.-M. Ni and X. Wang, *On the stability and instability of positive steady states of a semilinear heat equation in* $\mathbb{R}^n$, Comm. Pure Appl. Math. **45** (1992), 1153–1181.

[6] C. Gui, W.-M. Ni and X. Wang, *Further study on a nonlinear heat equation*, J. Differ. Equations **169** (2001), 588–613.

[7] N. Mizoguchi, *Growup of solutions for a semilinear heat equation with supercritical nonlinearity*, preprint.

[8] P. Poláčik and E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann. **327** (2003), 745–771.

[9] X. Wang, *On the Cauchy problem for reaction-diffusion equations*, Trans. Amer. Math. Soc. **337** (1993), 549–590.