Proceedings of ALGORITMY 2002
Conference on Scientific Computing, pp. 14–22

NUMERICAL SOLUTION OF DEGENERATE CONVECTION-DIFFUSION PROBLEM USING BROYDEN SCHEME

IVAN CIMRÁK *

Abstract. Nonlinear parabolic convection - diffusion equations with nonlinearity in both convection and diffusion terms lead to many problems in numerical implementation. *Up-Wind* scheme is used to avoid these problems (such as non-physical oscillating of the solution).

The equation is discretized in time by *Rothe's method* and then degenerate elliptic equations occur. After a space discretization on each time level the problem can be transformed into solving of large systems of nonlinear equations. In this paper we use *Broyden method* to solve such systems. We mainly focus on numerical aspects.

Key words. Degenerate parabolic equation, slow diffusion, Broyden method, Barenblatt solution, nonlinear convection.

AMS subject classifications. 35K65, 65M60

Numerical solution of degenerate convection-diffusion problem using Broyden scheme

1. Introduction. The main purpose of this paper is to show several aspects of a numerical computing of degenerate parabolic equations. We will be mainly concerned with the following nonlinear parabolic equation:

$$\partial_t u + M \cdot \nabla(\gamma(u)) - \Delta\beta(u) = 0, \quad \text{in } \Omega \times I, \quad I = (0,T), \tag{1.1}$$

where M is a constant vector, γ is a function describing the nonlinearity in a convective term. The numerical results will be computed with $\gamma(u) = u^p$, $p >= 1$. The function β describes the nonlinearity in a diffusion term. We consider $\beta(u) = u^s$, $s > 1$. The problem will be taken with the Dirichlet boundary condition and the initial condition:

$$u(x,t)|_{\partial\Omega\times I} = 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \text{ in } \Omega. \tag{1.3}$$

The problem with such parameters describes slow diffusion in porous media. In general the solution (if exists) does not need be necessarily smooth, that is why we have to deal with the weak solution of the problem.

1.1. Weak formulation. Notice that the problem (1.1) - (1.3) is a degenerated parabolic equation in the case $\beta'(0) = 0$.

DEFINITION 1. *A function u is called a weak solution to the problem P iff*

i) $u \in L_2(I, H_0^1(\Omega)) \cap L_\infty(I \times \Omega)$;

ii) u satisfies the integral identity

$$\begin{aligned}\int_0^T \int_\Omega (u(x,t) - u_0(x))\;\; v_t(x,t)dx\;dt\;\; + \\ \int_0^T \int_\Omega M \cdot \nabla\left(\gamma(u(x,t))\right)\;\; v(x,t)dx\;dt\;\; + \\ \int_0^T \int_\Omega \nabla\beta(u(x,t))\nabla v(x,t)dx\;dt \;\;&=\;\; 0,\end{aligned}$$

*Department of Mathematical Anlysis, Ghent University, Belgium (`cimo@cage.rug.ac.be`).

for all $v \in L_2(I, H_0^1(\Omega))$ *such that* $v_t \in L_\infty(I \times \Omega)$ *and* $v(.,T) = 0$.

It worth to note that the problem has at most one weak solution (see [11]). In [9] a proof of the existence of a solution as well as convergence of *Rothe's method* used for solving parabolic equations of the form (1.1) for specific M has been shown. In [8] authors suggested a linear approximation scheme for solving the problem (1.1) - (1.3) and they proved the convergence of approximative solutions to an exact solution. For the numerical implementation of this scheme we refer to [10].

2. Time discretization. We discretize the time by backward Euler approximation of the time derivative $\partial_t u \sim \frac{1}{\tau}(u(t_i) - u(t_{i-1}))$. In the literature this approach is called *Rothe's method.* We divide the time interval I into n subintervals $I_i = [t_{i-1}, t_i]$ of the same length $\tau = \frac{T}{n}$. Thus we get n elliptic problems:

$$\frac{1}{\tau}(u_i - u_{i-1}) + M \cdot \nabla(\gamma(u_i)) - \Delta(\beta(u_i)) = 0, \quad \text{in } \Omega, \qquad u_i(x)|_{\partial\Omega} = 0. \tag{2.1}$$

The function u_{i-1} is a solution obtained from the previous time level. In the first time level is u_0 the initial condition.

3. Space discretization. We implement the space discretization in two ways: by finite differences (FDM) and by finite elements (FEM).

3.1. FDM. First we use finite difference method on the mesh with constant grid dh. We approximate the operator $\nabla(\gamma(u))$ by symmetric difference $\mathcal{M}$ which can be in 2D symbolically written as

$$\mathcal{M}(u_{00}) = \frac{1}{2\ dh} \begin{pmatrix} \gamma(u_{+0}) - \gamma(u_{-0}) \\ \gamma(u_{0+}) - \gamma(u_{0-}) \end{pmatrix}. \tag{3.1}$$

Later we use Up-Wind scheme, symbolically, if vector M has positive components:

$$\mathcal{M}(u_{00}) = \frac{1}{2\ dh} \begin{pmatrix} \gamma(u_{+0}) - \gamma(u_{00}) \\ \gamma(u_{0+}) - \gamma(u_{00}) \end{pmatrix}. \tag{3.2}$$

The operator $\Delta(\beta(u))$ is approximated by 5-points rule:

$$\mathcal{L}(u_{00}) = \frac{1}{dh^2}\big(\beta(u_{+0}) + \beta(u_{-0}) + \beta(u_{0+}) + \beta(u_{0-}) - 4\beta(u_{00})\big). \tag{3.3}$$

As the solution is approximated by the values in grid points, we get discrete approximation space

$$V_h = \mathbb{R}^N, \tag{3.4}$$

where N is the number of grid points.

3.2. FEM. Let $\mathcal{T}_h = \{K\}$ be the usual regular nonoverlapping finite element triangulation of the domain $\Omega = \cup_{K \in \mathcal{T}_h} K$. Let $\mathcal{E} = \cup_{i=1}^N n_i$ is the set of vertices of the mesh. We define usual continuous linear finite element space

$$V_h = \{u_h \in H_0^1(\Omega) : u_h|_K \in P_1(K)\}. \tag{3.5}$$

Denote by ϕ_i piecewise linear continuous basis function associated with the vertex n_i of the mesh. The following holds true

$$\phi_i(n_j) = \delta_{ij}, \quad i, j = 1 \ldots N. \tag{3.6}$$

4. Nonlinear system of equations. After performing the space discretization we obtain a nonlinear system of equations. The reason is, that the nonlinear (in a space) problem was not linearized. Unknowns represent the approximated solution in the space V_h. Therefore the problem can be reformulated as seeking the root of the equation:

$$F(v) = 0, \quad F : \mathbb{R}^N \to \mathbb{R}^N. \tag{4.1}$$

There are several numerical iterative methods used in order to find approximation of the exact solution. We will be concerned with the iterative schemes of quasi-Newton type. In [6] author introduces Broyden method for solving such systems. He compares the efficiency of Broyden and well-known Newton-Kantorovich methods. We use numerical implementation of this approach. Hereafter, we will denote all inner iterations of these schemes by an upper index.

Broyden method. This method is based on the so-called Broyden's update formula for quasi-Newton iterations:

$$v^{k+1} = v^k - B_k^{-1} F(v^k). \tag{4.2}$$

In the Newton-Kantorovich we have $B_k = F'(v^k)$. In Broyden method, $B_k, k = 1, 2, ...,$ represent only approximations of the Jacobi matrix $F'(v^k)$. If we denote

$$s_k = v^{k+1} - v^k, \quad y_k = F(v^{k+1}) - F(v^k),$$

then the Broyden's update for a new approximation of the Jacobi matrix B_{k+1} is given by

$$B_{k+1} = B_k + \frac{y_k - B_k s_k}{\|s_k\|^2} {s_k}^T.$$

Thanks to Sherman-Morrison formula (see [1]) we can directly compute B_{k+1}^{-1} if B_k^{-1} is known. That is the main advantage in comparison to solving systems using Newton-Kantorovich method. No systems of linear equations with large matrices must be solved.

Such a formula can be derived by following a nice geometric motivation discussed in a more detail in a book by Allgower and Georg [1]. In this book one can also find a proof of a local super-linear convergence of Broyden's iterates to the root of (4.1). The assumptions needed for the proof of a local super-linear convergence require closeness of the initial iterate v^0 and the root v^*. This requirement can be guaranteed by taking $v^0 = v_{i-1}$ and assuming $0 < \tau \ll 1$.

5. Numerical experiments.

5.1. Implementation of FDM. We solve (1.1) in 2D on the square $(0,1) \times (0,1)$. We choose an initial condition shown in Fig. 5.2. We set the vector of convection $M = (200, 0)$, so the "wind" blows strong from the east to the west. In this case we consider no degeneration of convection, i.e. $\gamma \equiv 1$. While the convective term has been approximated in a space by central difference, we have obtained some oscillations, see Fig. 5.1. As soon as the central difference has been replaced by Up-Wind scheme, oscillations have lost, see Fig. 5.2.

The next case discuss Burgers equation. The setting $\gamma(s) = \frac{s^2}{2}$ yields to the well known Burgers equation:

$$\partial_t u + uM \cdot \nabla(u) - \Delta\beta(u) = 0. \tag{5.1}$$

There are results for the classic setting $\beta(s) = s$ depicted in the Fig. 5.3. The value of convective vector is $M = (15000, 0)$. We consider also the generalization with the setting $\beta(s) = 0.005s^3$. The evolution of the solution can be seen in Fig. 5.4.

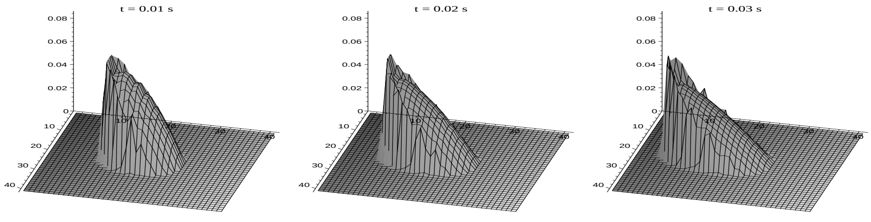

FIG. 5.1. *Evolution with oscillations.*

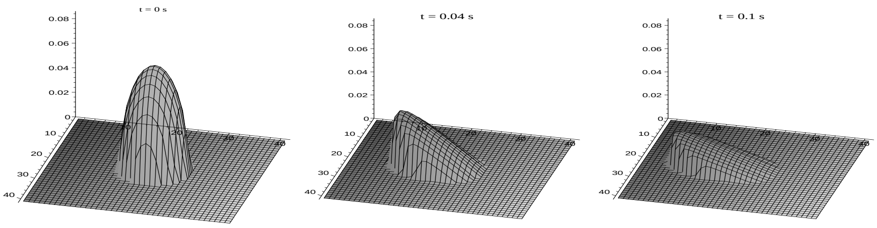

FIG. 5.2. *Evolution without oscillations.*

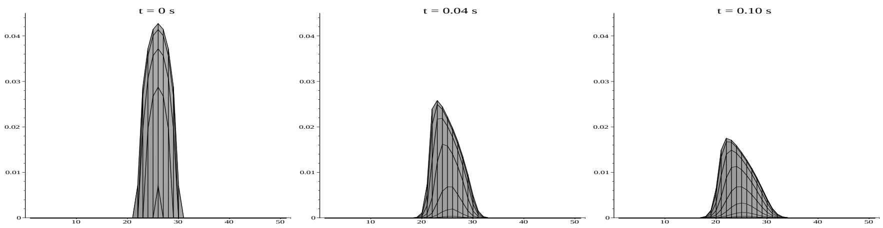

FIG. 5.3. *Burgers solution for $\beta(s) = s$.*

5.2. Implementation of FEM. We use an adaptive hierarchical finite element toolbox ALBERT produced by Alfred Schmidt and Kunibert G. Siebert. This package has been described in details in [12]. The package had to be modified for this problem. For the adaptation of the mesh we use bisection strategy for both refinement and coarsening. We use a local estimator introduced by Verfürth in [13] for marking the elements. Although the theoretical framework in [13] has been dedicated to quasi-linear equations this estimator works for highly nonlinear problems as well. In Fig. 5.6 is depicted the adaptation of the mesh for the problem (1.1) with zero convective term:

$$\partial_t u - \Delta(u^m) = 0. \tag{5.2}$$

We will be mainly concerned with the case of slow diffusion i.e., $m > 1$. In this case the support of initial data $u_0(x)$ (i.e., the closure of the set of x where $u_0(x) > 0$) propagates with finite speed (see [4]). It is desirable to locate the movement of the interface. For the test purposes is very important the exact solution of (5.2) given

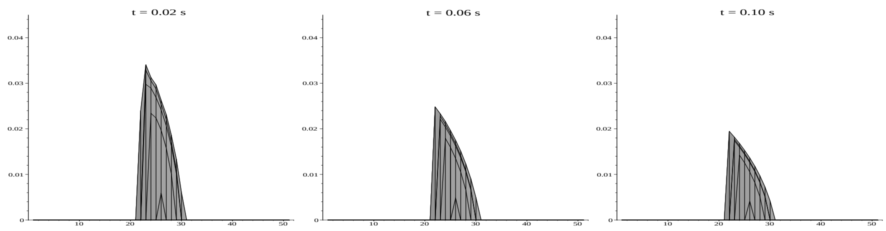

FIG. 5.4. *Generalized Burgers solution for* $\beta(s) = 0.005s^3$.

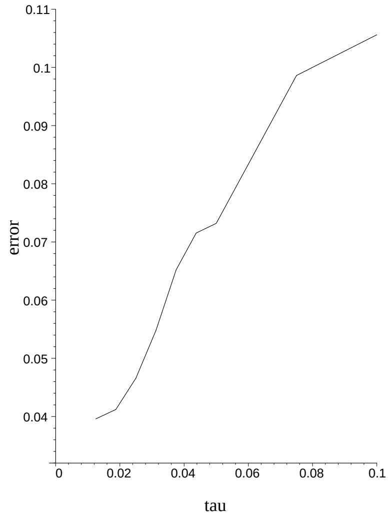

FIG. 5.5. *Semi-discrete error.*

τ	Error
0.100	0.106
0.075	0.099
0.050	0.073
0.044	0.071
0.038	0.065
0.031	0.054
0.025	0.046
0.019	0.041
0.012	0.039

TABLE 5.1
Semi-discrete error.

by Barenblatt in [4]. For comparing the exact u_e and computed u_c solution we have used the following semi-discrete norm:

$$|u_e - u_c| = \frac{1}{N_\tau} \sum_{i=0}^{N_\tau} \left(\int_\Omega (u_e(t_i) - u_c(t_i))^2 \right)^{\frac{1}{2}}, \tag{5.3}$$

where N_τ is the number of time steps in the interval I. The evolution of this error is depicted in Fig. 5.5 and Tab. 5.1.

5.3. Graveleau's exact solution. We consider the differential equation (1.1) with zero convective term:

$$\partial_t u - \Delta(u^m) = 0, \tag{5.4}$$

where $m > 1$ is a constant. This equation describes the evolution of the density u of ideal gas flowing through homogeneous porous media. The initial distribution of the gas is outside of the compact domain and then it diffuses into this domain. In the papers of Aronson [3] and Angenent [2] authors discuss both symmetric and non-symmetric case. There exists an one-parametric family of solutions with respect of radial symmetry. This family was first numerically found by Graveleau in [7] and then was correctly described in [5].

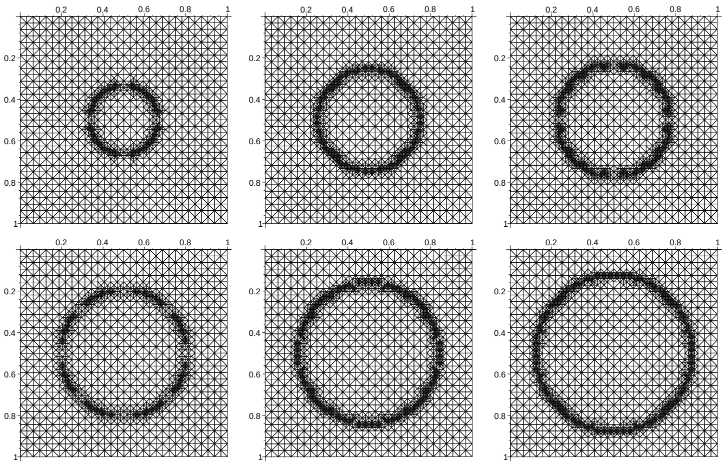

FIG. 5.6. *The adaptation of the mesh in time 0s, 2s, 4s, 8s, 16s, 30s.*

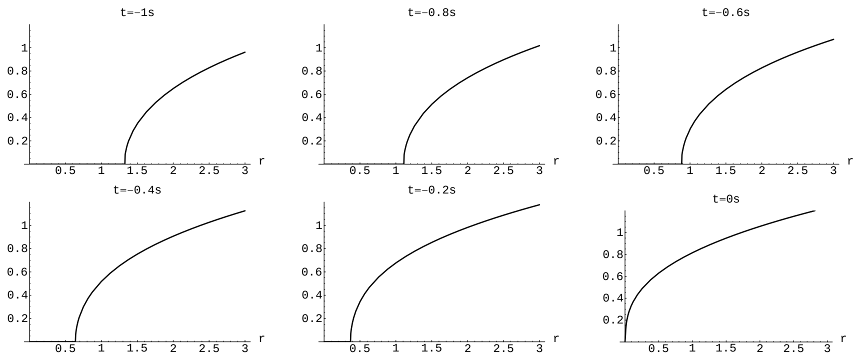

FIG. 5.7. *The evolution of Graveleau's solution.*

Let us denote v a new variable describing the pressure of the gas:

$$v = \frac{m}{m-1} u^{m-1}.$$

In a radial symmetric case $v(r,t)$ corresponds to transformed equation

$$\partial_t v = (m-1)v(\partial_{rr} v + \frac{d-1}{r}\partial_r v) + (\partial_r v)^2, \tag{5.5}$$

where d is a spatial dimension. We seek the solution in $\langle 0,\infty) \times (t_0, T)$ for such $t_0 \in \mathbb{R}$, that

$$v(r, t_0) = v_0(r), \quad r \in \langle 0, \infty), \tag{5.6}$$

where v_0 is given function fulfilling following assumption

$$v_0(r) = \begin{cases} = 0 & , \quad r \in \langle 0, a\rangle \cup \langle b, \infty), \\ > 0 & , \quad r \in (a, b), \end{cases}$$

for some $0 < a < b < \infty$. As the time rises from the value $t = t_0$, the gas flows through the boundary $r = b$ outside and through the boundary $r = a$ inside. Therefore, there exists a non-increasing function $a(t)$ describing the translation of *inner interface* and a non-decreasing function $b(t)$ describing the translation of *outer interface*. The functions $a(t)$ and $b(t)$ are characterized by the assumptions $a(t_0) = a$, $b(t_0) = b$, $a(T) = 0$ and

$$v(r,t) = \begin{cases} = 0 & , \quad r \in \langle 0, a(t)\rangle \cup \langle b(t), \infty), \\ > 0 & , \quad r \in (a(t), b(t)), \end{cases}$$

for $t \in \langle t_0, T\rangle$.

Suppose $T = 0$ and therefore the initial time t_0 is negative. There exists an one-parametric family $\{g_c(r,t)\}$ of solutions equation (5.5) defined for

$$c \in \mathbb{R}^+ \text{ and } (r,t) \in \langle 0, \infty) \times (-\infty, 0\rangle.$$

Every solution covers the interior of the circle with radius a in time $t = 0$ (Graveleau's solution). There exists the numbers $\alpha^*(d,m)$ satisfying

$$\frac{2nd+4}{n(d+2)+4} < \alpha^*(d,m) < \text{Min}\left\{2, \frac{2+nd}{d+1}\right\},$$

where $n = m - 1$. There exists $\gamma(d,m) \in \mathbb{R}^-$ such that for $t < 0$ is fulfilled

$$g_c(r,t) = -r^2 t^{-1} \phi(c\eta), \tag{5.7}$$

while $\phi > 0$ on $(\gamma, 0)$, $\phi = 0$ on $(-\infty, \gamma\rangle$ and

$$\eta = tr^{-\alpha^*}.$$

Function $\phi = \phi(\eta)$ is the solution of degenerate nonlinear ordinary differential equation

$$\begin{aligned} \frac{1}{\eta^2}\phi - \frac{1}{\eta}\phi' &= \frac{1}{\eta^2}(2nd+4)\phi^2 \\ &- \alpha\frac{1}{\eta}\left(n(d+2-\alpha)+4\right)\phi\phi' + n\alpha^2\phi\phi'' + \alpha^2(\phi')^2, \end{aligned} \tag{5.8}$$

with parameter α, solved on the interval $(-\infty, 0)$ with boundary conditions

$$\phi(0) = 0, \quad \phi'(0) = -1.$$

While $\alpha < \alpha^*$ then $\phi > 0$ on $\mathbb{R}^-$ and (5.7) is not satisfactory solution. There exists $\gamma \in \mathbb{R}^-$ for $\alpha = \alpha^*$ such that $\phi > 0$ on $(\gamma, 0)$ and $\phi(\gamma) = 0$. Finally, if we define $\phi = 0$ on $(-\infty, \gamma)$, we obtain $g_c(r,t)$ convenient solution. In general the values α^* must be obtained numerically.

The solutions $g_c(r,t)$ fulfill the following condition:

$$g_c(r, t_0) \begin{cases} = 0 & , \quad r \in \langle 0, a\rangle \\ > 0 & , \quad r > a. \end{cases}$$

We set the values of parameters as follows:

$$m = 3, \quad d = 2, \quad c = 1, \quad t_0 = -1.$$

Using the shooting method we have found the value of $\alpha^*(2,3) = 1.25575$. The degenerate ordinary differential equation was solved by implementation of `NDSolve`

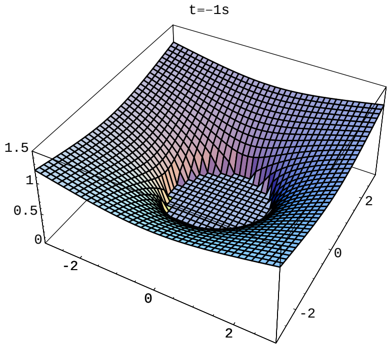

FIG. 5.8. *Initial condition.*

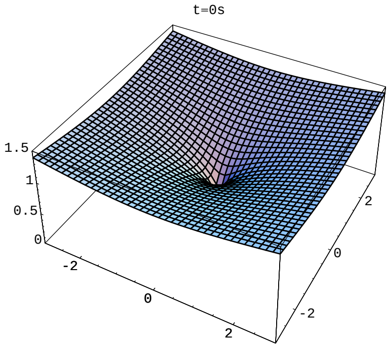

FIG. 5.9. *Final state of Graveleau's solution.*

	Graveleau's solution	
h	$\tau.10^2$	$L_2.10^2$
0.30	0.500	1.38309
0.30	0.250	1.31533
0.30	0.167	1.29867
0.15	0.250	0.54166
0.15	0.167	0.49193

TABLE 5.2
Discrete L_2 norm of the error.

solver in the Mathematica® package. We avoid the degeneration in 0 by setting boundary conditions to

$$\phi(-\varepsilon) = \varepsilon, \quad \phi'(\varepsilon) = -1.$$

The computed solution $g_c(r,t)$ of the equation (5.6) represents the pressure of the gas. Backward transformation gives us the density u as a solution of the equation (5.4) with initial condition

$$u(r,0) = \left(\frac{m-1}{m} g_c(r,0)\right)^{\frac{1}{m-1}}.$$

In Fig. 5.7 is depicted the evolution of the profile of the solution. Tab. 5.2 shows $L_2(\Omega)$ norm of the difference between the exact and computed solution. The initial condition and the final state of Gravaleau's solution is depicted in Fig. 5.8 and 5.9.

REFERENCES

[1] E. L. ALLGOWER AND K. GEORG, *Numerical Continuation Methods*, Springer-Verlag, Berlin, Heidelberg, 1990.

[2] S. B. ANGENENT AND D. G. ARONSON, *Self-similarity in the post focusing regime in porous medium flows*, Euro. Journal of Aplied Mathematics, Vol.7 (1996), 277-285.

[3] S. B. ANGENENT AND D. G. ARONSOM, *The focusing problem for the radially symmetric porous medium equation*, Commun. in Partial Differential Equations, 20 (1995), 1217-1240.

[4] G. I. BARENBLATT, *On a class of exact solutions of the plane one-dimensional problem of unsteady filtration of a gas in a porous medium (Russian)*, Akad. Nauk SSSR. Prikl. Mat. Meh. 17, (1953). 739–742.

[5] D. G. Aronson and J. Graveleau, *A self-similar solution to the focusing problem for the porous medium equation*, Euro. Journal of Aplied Mathematics, 4 (1993), 65-81.

[6] I. Cimrák, *Comparison of Broyden and Newton methods for solving nonlinear parabolic equations*, Journal of Electrical Engineering, Vol. 52 (2001), 48-52.

[7] J. Graveleau, *Quélques solutions auto-semblables pour, l'équation dela chaleur non-linéaire*, Rapport Interne C. E. A., 1972.

[8] W. Jäger, J. Kačur, *Solution of Porous Medium Type Systems by Linear Approximation Schemes*, Numer. Math., 60 (1991), 407-427.

[9] J. Kačur, *Method of Rothe in Evolution Equations*, BSB Teubner Verlagsges, Leipzig, 1985.

[10] K. Mikula, *Numerical Solution of Nonlinear Diffusion with Finite Extinction Phenomenon*, Acta Math. Univ. Comenianae, 64, 2 (1995), 173-184.

[11] F. Otto, *L^1 - contraction and uniqueness for quasilinear elliptic-parabolic equations*, Journal of Differential Equations, 131 (1996), 20-38.

[12] A. Schmidt and K. G. Siebert, *ALBERT—software for scientific computations and applications*, Acta Math. Univ. Comenian. (N.S.) 70 (2000), no. 1, 105–122.

[13] R. Verfürth, *A posteriori error estimates for nonlinear problems. $L^r(0,T;L^\rho(\Omega))$-error estimates for finite element discretizations of parabolic equations*, Math. Comp. 67 (1998), no. 224, 1335–1360.