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APPLICATION OF EXACT SOLUTIONS OF SOME ELLIPTIC EQUATIONS FOR GENERATION OF TWO- AND MULTI-DIMENSIONAL ANALYTICAL GRIDS

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Abstract. Now algebraic approaches to construction of grids are rapidly developing (B-spline, transfinite interpolation etc.) [1, 2]. Their advantage appears in the possibility of fast construction of grids. Traditional approaches based on a numerical solution of differential equations give, as a rule, best grids, but demand much more time for their construction. Especially it is right in calculation of flows in moving boundaries.

In present work, the new method of grid construction based on an analytical solution of some system of elliptic differential equations is suggested. It combines a speed of the algebraic approach with quality of the elliptical approach. In suggested method, the functions which map computational space to physical one (mapping-functions, MF), are obtained in exact form. It allows to receive analytical expressions for metric coefficients and, thereby, completely eliminate errors of their numerical calculation. The method of control of some relevant grid properties is also suggested. The idea of quality control of the obtained grid is based on application of analytical solutions of a boundary value problem for the class of higher order equations. As MF remain analytical, method does not increase grid construction time. The offered grids have some advantage in comparison to conformal ones, where the MF have no sufficient functional arbitrariness. The obtained results are illustrated by examples.

Key words. Analytical grids, mapping-functions, Laplace's equation

1. Introduction. The traditional method of construction of elliptical grids consists in following. Some system of elliptical equations satisfying to an extremum principle is formulated and the isolines of functions obtaining from a solution are taken as new coordinate lines. Extremum principle guarantees a one-to-one mapping between the physical and transformed regions if equations are formulated in physical space. To solve a system, it has to be written in computing space that essentially complicates equations. The simplest implementation of such approach - usage of Laplace's equation for construction of a grid [1]. But even in this case, it is necessary to solve a system of closure of set of non-linear equations in computational space instead of Laplace's equation in physical space. This system converges slowly with increasing of complexity of calculating domain.

In this paper, generation equations are formulated in computational space. Their solution gives a MF which maps points from computing space to physical, i.e. as it is required in the numerical methods. The solution is obtained in the form of final analytical functions. It enables not only to plot necessary grids, but also to predict their properties (including nondegeneracy). As it is known, the boundary value problem for Laplace's equation has analytical solutions for simple domains - rectangle, parallelepiped, circle and full-sphere. They may be extended to the case of greater space dimensions number. These solutions exist in the form of sums of infinite series. The final expressions of these sums exist only for domains in the form of circle and full-sphere and are known as the Poisson integral for a circle and full-sphere respectively [4].

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In this paper, solutions of a boundary value problem for Laplace's equation for square and cube (including multidimensional ones) as final integrals are presented. They allow mapping an arbitrary physical area to a cube or square. One of advantages of such approach is simplicity of its extension on the case of multidimensional space. The appropriate solutions will be presented below.

The idea of quality control of the obtained grid based on application of analytical solutions of a boundary value problem for the class of higher order equations (polyharmonic equation). The Laplace's equation allows only setting the shape of physical area. Increasing of equation order enables to receive a sufficient amount of free parameters permitting to control different properties of grids.

2. Basic principle. Let us consider a solution of a boundary value problem of map of computational space on physical one

$$(x_1, x_2, ..., x_n) \Leftrightarrow (\xi_1, \xi_2, ..., \xi_n) \tag{1}$$

The set of equations for co-ordinate determination is

$$\Delta x_i = 0 \text{ where } \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial \xi_i^2}, \quad n \text{ – dimension of space.} \tag{2}$$

The equations (2) are solved in the space $0 < \xi_i < l$. The boundary conditions (BC) for a case of two dimensions (2D) are

$$\begin{aligned} x_1 &= f_1^{x1}(\xi_2),\ x_2 = f_1^{x2}(\xi_2),\ \text{at } \xi_1 = 0\,; \\ x_1 &= f_2^{x1}(\xi_2),\ x_2 = f_2^{x2}(\xi_2),\ \text{at } \xi_1 = l\,; \\ x_1 &= f_3^{x1}(\xi_1),\ x_2 = f_3^{x2}(\xi_1),\ \text{at } \xi_2 = 0\,; \\ x_1 &= f_4^{x1}(\xi_1),\ x_2 = f_4^{x2}(\xi_1),\ \text{at } \xi_2 = l\,. \end{aligned} \tag{3}$$

Functions f_i are coordinates of the nodes along the physical boundary.

These boundary conditions may be extended to any more than two-dimensional case. So, in three-dimensional case (3D) they will appear not as four BC on edges (as in 2D), but six BC on surfaces.

The solution of boundary value problem (2-3) can be received in the form

$$x_i(\xi_1, ...) = \sum_{k=1}^{Nk} S_k^{xi} \quad \text{where } Nk = 4 \text{ for } 2D; Nk = 6 \text{ for } 3D. \tag{4}$$

a) Two-dimensional case

$$S_k^{xi} = \frac{1}{l} \int_0^l f_k^{xi}(\omega) F_2(\gamma_\omega, \gamma_\xi^k, Q^k)\, d\omega \tag{5}$$

$$F_2(\gamma_\omega, \gamma_\xi, Q) = F_2^+(\gamma_\omega, \gamma_\xi, Q) - F_2^-(\gamma_\omega, \gamma_\xi, Q) \tag{6}$$

$$F_2^\pm(\gamma_\omega, \gamma_\xi, Q) = \frac{1 - Q\cos(\gamma_\omega \pm \gamma_\xi)}{1 - 2Q\cos(\gamma_\omega \pm \gamma_\xi) + Q^2} \tag{7}$$

where

$$\gamma_\omega = \frac{\pi\omega}{l}, \quad Q^k = \frac{\exp(\pi)}{\exp(\beta_k)},$$

$$\gamma_\xi^1 = \gamma_\xi^2 = \frac{\pi\xi_2}{l},, \quad \beta_1 = \pi - \frac{\pi\xi_1}{l}, \ \beta_2 = \frac{\pi\xi_1}{l},$$

$$\gamma_\xi^3 = \gamma_\xi^4 = \frac{\pi\xi_1}{l}, \quad \beta_3 = \pi - \frac{\pi\xi_2}{l}, \ \beta_4 = \frac{\pi\xi_2}{l}.$$

b) Three-dimensional case

$$S_k^{xi} = \frac{1}{2l^2}\int_0^l\int_0^l f_k^{xi}(\omega_1,\omega_2)F_3(\gamma_{\omega1},\gamma_{\omega2},\gamma_{\xi1}^k,\gamma_{\xi2}^k,Q^k)d\omega_1 d\omega_2 \tag{8}$$

$$F_3(\gamma_{\omega1},\gamma_{\omega2},\gamma_{\xi1},\gamma_{\xi2},Q) = F_2(\gamma_{\omega1},\gamma_{\omega2},Q) + F_2(\gamma_{\xi1},\gamma_{\xi2},Q) \tag{9}$$

where

$$\gamma_{\omega1} = \frac{\pi\omega_1}{l}, \ \gamma_{\omega2} = \frac{\pi\omega_2}{l}, \quad Q^k = \frac{\exp(\pi)}{\exp(\beta_k)}$$

$$\gamma_{\xi1}^5 = \gamma_{\xi1}^6 = \gamma_{\xi2}^3 = \gamma_{\xi2}^4 = \frac{\pi\xi_1}{l}, \quad \beta_1 = \pi - \frac{\pi\xi_1}{l}, \ \beta_2 = \frac{\pi\xi_1}{l},$$

$$\gamma_{\xi1}^1 = \gamma_{\xi1}^2 = \gamma_{\xi2}^5 = \gamma_{\xi2}^6 = \frac{\pi\xi_2}{l}, \quad \beta_3 = \pi - \frac{\pi\xi_2}{l}, \ \beta_4 = \frac{\pi\xi_2}{l},$$

$$\gamma_{\xi1}^3 = \gamma_{\xi1}^4 = \gamma_{\xi2}^1 = \gamma_{\xi2}^2 = \frac{\pi\xi_3}{l}, \quad \beta_5 = \pi - \frac{\pi\xi_3}{l}, \ \beta_6 = \frac{\pi\xi_3}{l}.$$

It is evident, that in 3D the integrand is the sum of appropriate functions for 2D. This property is also fulfilled with increasing of space dimension. It gives a simple path of obtained solution generalization on multidimensional spaces. Thus, equation (4) gives distribution of nodes of a grid.

The function in the integral (5) has no singularities because it is the product of two non-singular functions. One of them sets the shape of boundary and, therefore, is non-degenerated by definition. Other functions are the sum of functions of following form:

$$F_2^{\pm}(\gamma_\omega,\gamma_\xi,Q) = \frac{\exp(\beta)\exp(\beta) - \cos(\gamma_\omega \pm \gamma_\xi)\exp(\beta)\exp(\pi)}{\left(\cos(\gamma_\omega \pm \gamma_\xi)\exp(\beta) - \exp(\pi)\right)^2 + \left(\sin(\gamma_\omega \pm \gamma_\xi)\exp(\beta)\right)^2}.$$

Evidently, its denominator is not zero at any point.

Method which allows to receive a solution (4) (i.e. equations (5-7)) in its final form instead of well-known infinite series [4] is explained briefly below.

Most widely used method of a solution of Laplace's equation is the Fourier method of a separation of variables with the subsequent expansion of a required function in an infinite Fourier series. Thus, for 2D Laplace's equation the solution for a rectangle is searched by a following way:

- the required function is represented in the form of sums of four functions, each of which obeys BC on only one edge of domain and equals to zero on other edges [5]. It is possible because Laplace's equation is linear,
- Fourier method is used for a determination of each of the four functions mentioned.

Note that such approach leads to infinite series, the sums of which are unknown.

In the contrary to traditional way, the special substitution of variables is used for each of four mentioned above functions. It permits to receive series in a form which make possible to find the final expression for the sum of these infinite series. According to this approach one can obtain sums in a form $\sum_{n=1}^{\infty} \exp(\beta)^n \cos(\psi n)$ instead of traditional ones. These sums may be reduced then to finite expressions.

3. Method of a grid control. As it was mentioned in the introduction, the idea of adapting of a grid is based on application of solution of higher-order elliptic equations.

Using described above argumentation it is possible to show that a solution of equations

$$\Delta\Delta x_i = 0 \tag{10}$$

with BC (3) may be obtained in a form (4) with one exception: function $\bar{F}_2$ (11) will enter in an integral (5) instead of a function $F_2(\gamma_\omega, \gamma_\xi, Q)$ (6)

$$\bar{F}_2 = F_2(\gamma_\omega, \gamma_\xi, Q) M_k\,, \quad \text{where } M_k = \frac{A_k \exp(\beta_k)^2 + B_k}{A_k \exp(\pi)^2 + B_k}\,, \tag{11}$$

$A_k, B_k = const(\xi_1, ...\xi_i)$. Equations (10) have higher order than the equations (2). Therefore for closure of a boundary value problem it is necessary to put some more BC in addition to BC (3). Then they allow to find A_k and B_k. If it is required to receive a grid with more beforehand-given properties than two free parameters in (11), it is possible to take equation of a kind $\Delta\Delta\cdots\Delta x_i = 0$ instead of (10). It will allow to receive a number of parameters required for grid control.

Developing further this idea, it is possible to receive them as many as necessary for the essence of a problem under consideration.

On figures below, the influence of magnitudes A_k and B_k upon a behavior of a grid is shown. Though A_k and B_k do not depend from ξ_i, they may depend on variable of integration ω. It enables to control grid behavior. As the magnitudes A_k and B_k have arisen because of necessity to satisfy additional BC, their influence is maximum near to boundary and decrease with going off it.

4. Examples. Let us consider examples of application of the obtained solution for construction of grids. (On all figures below, the following labels are used: $x_1 = X, x_2 = Y, x_3 = Z$.)

On Fig.1, the grid constructed with the help of equations (2) in a flat polygon with non-monotone boundaries is shown. It is visible, that the obtained grid lines are smooth, and the influence of angular points decreases with moving away from boundary. It is very difficult to achieve this effect in interpolation methods.

The grid on Fig.2 is an analytical solution of equations (10) with boundary conditions (3). This grid is shown for values A_2= exp(9.0) and B_2= exp(15.0); remaining

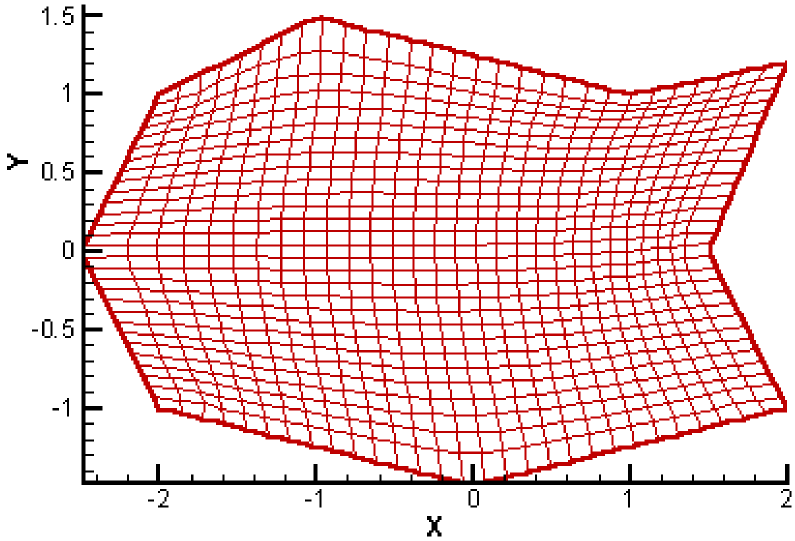

FIG. 1. *Grid as an analytical solution of a boundary value problem (2-3).*

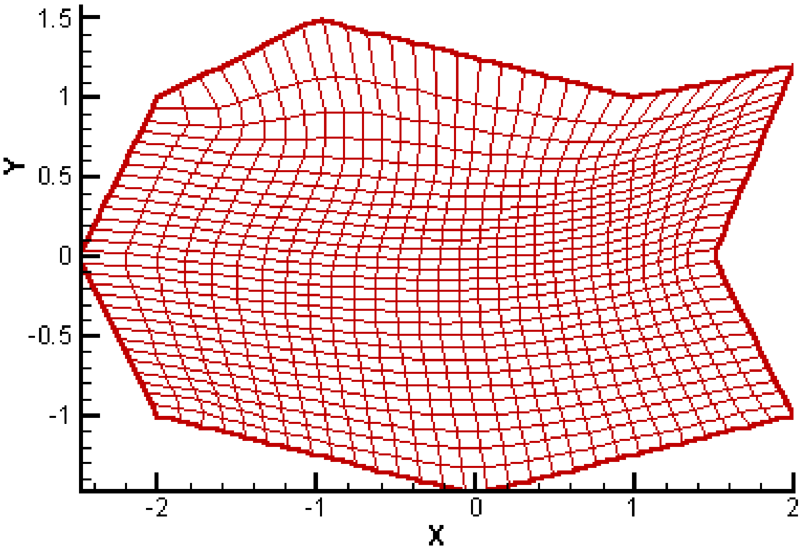

FIG. 2. *Grid as analytical solution of a boundary value problem (10),(3).*

$A_k = 0$, $B_k \neq 0$ (see (4)). In this case grid lines rarefy towards upper bound. By variation of A_k and B_k it is possible to achieve required spacing of the coordinate lines in a desirable place.

Let us consider possibilities of suggested method for construction of spatial grids. On Fig.3, the grid for some spatial domain being an analytical solution of a three-dimensional Laplace's equation (2) is shown.

Evidently the solution of equation (2) for a cube gives Cartesian grid. Fig.4 demonstrates possibilities of an analytical solution of a biharmonic equation for con-

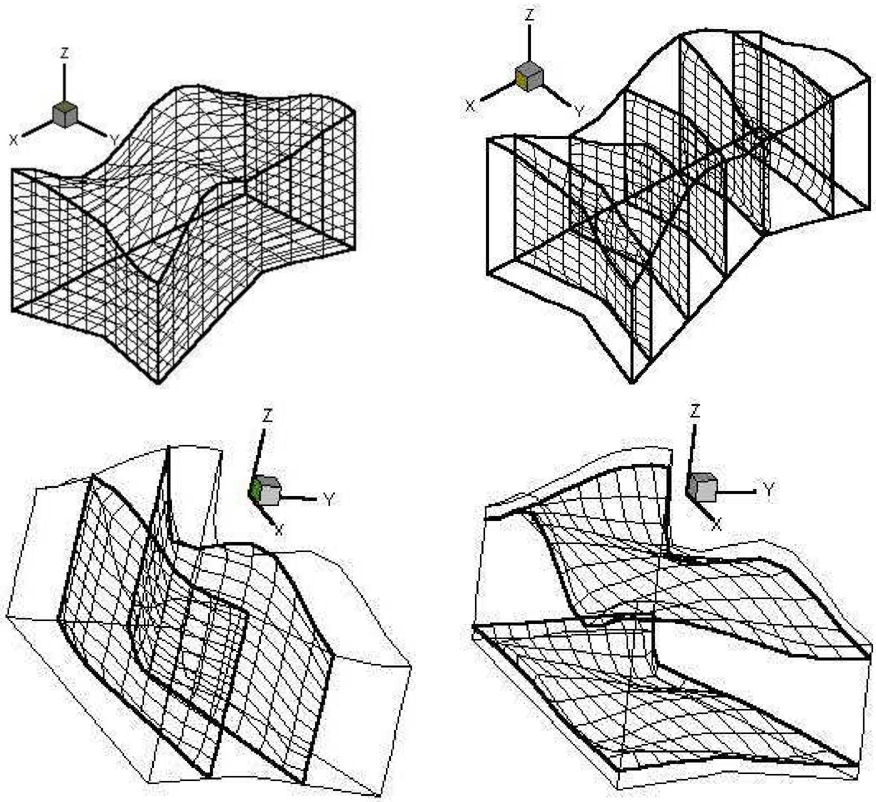

FIG. 3. *Grid as an analytical solution of a boundary value problem (2-3). Three-dimensional case.*

struction of controlled spatial grids. A_k and B_k are taken the same as in previous case. Results on Fig.4 emphasize the nature of A_k and B_k influence upon solution of equation (10) on grid behavior. It manifests in decreasing of their influence with going far from boundary. Let us note that this influence has spatial nature and propagates along any direction from the boundary.

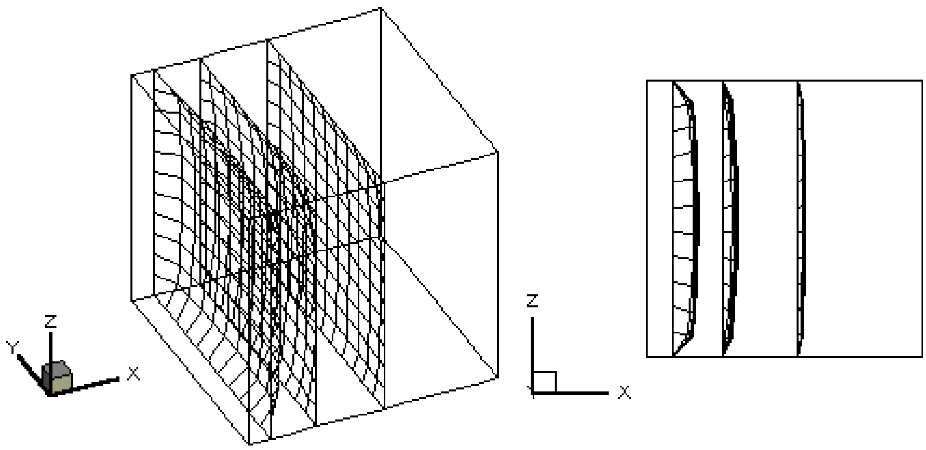

FIG. 4. *Three – dimensional grid adaptation*

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