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PLANAR MULTIFACILITY LOCATION – THE LOCATION-ALLOCATION PROBLEM *

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Abstract.
One of the important logistic tasks is transfer of finished products from producers to customers. If there is a need to supply a large–scale area with large number of customers, it is disadvantageous to deliver products from the only central warehouse or direct from producers. It is suitable to build up local distribution warehouses. The promptitude and quality of customer services improve as the number of warehouses increases. Transport cost created by transfer of products from producers to customers is reduced too. On the other hand, both the fixed costs caused by the warehouse operation and cost of maintenance of greater current, buffer and protective stocks increase all at once.

The contribution is concerned with a common case of a location-allocation problem, which involves both the location of distribution centres or warehouses and an allocation of customers to warehouses. Warehouses receive products from production plants and distribute them to customers such as retail or wholesale outlets.

The well-known Weiszfeld algorithm [1] was adapted for sequentially increasing number of warehouses until adding another warehouse is advantageous from transport cost standpoint. The basic criterion for decision is minimization of freight cost, but increasing expenses of product stocks are considered as well. Distances between warehouses and customers are given by a suitable two-dimensional metric corresponding to the network of routes. Location of customers is given by means of either geographical information systems or GPS system.

Key words. Multifacility location, location-allocation problem, conversion of coordinates

AMS subject classifications. 90B05, 90B06, 90B80, 49M05

1. Introduction. Optimum location of distribution warehouses is an important partial task of optimization of logistic channels. Determination of positions of distribution warehouses is a complex decision-making process [1, 2]. A basic assumption of the process is determination of the number of distribution warehouses and assessment of their approximate positions. Different location models can be used according to characteristics of a distribution space. The models simplify problem of location and remove or evade some difficulties of the location process. The method for estimation of transportation distances between given and located objects in the distribution space influences the choice of an optimization method.

2. The location-allocation problem for distribution warehouses. Simultaneous optimization of the number of warehouses, their allocation to customers and their location is not possible. Optimum location of new distribution warehouses for known customers must be decomposed into three subsequent optimization problems [4]. The first of them is determination of the optimum number of warehouses. It is a complex decision-making process depending on many information pieces, which are often insufficiently quantifiable. The problem is complicated by the fact that explicit expression of the number of warehouses is impossible in the standard location models. The optimization problem is necessary to solve step-by-step as a sequence

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of optimization tasks with increasing number of warehouses. A new warehouse is added by splitting up a chosen old one. The process is repeated if adding further warehouse is advantageous from the standpoint of economy (savings gained by less transportation costs are greater than expenses of further warehouse operation).

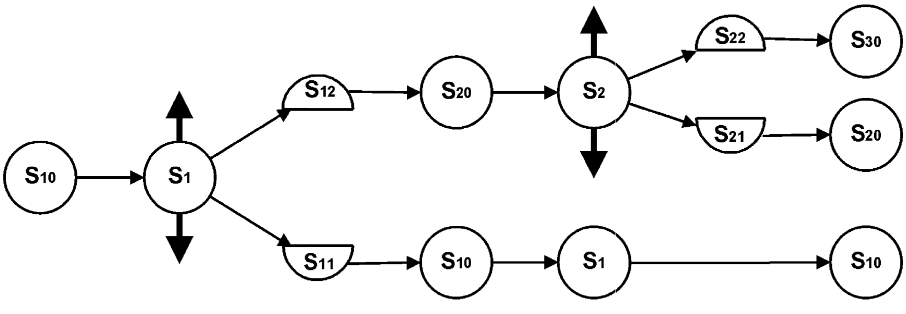

FIG. 1. *The principle of adding a new warehouse by splitting an old one.*

The second one is allocation of customers to new warehouses and determination of distribution areas supplied from every warehouse. It is always a discrete optimization process and its result is mutual optimum allocation of distribution centres (customers) and warehouses. Calculation of the optimum supplied areas is a NP-complete problem. Quantity of calculations increases with the number of warehouses and distribution centres faster than a polynomial.

The third process is continuous or combinatorial optimization of distribution warehouse location in the distribution areas determined in advance. Choice of optimization method depends on the way of determination of distances between objects in the distribution space.

2.1. Formulation of the location-allocation problem for distribution warehouses and centres. The problem of allocation of distribution warehouses to distribution centres and their location in a distribution space can be formulated in following way [4]:

- n existing objects – selling centres P_i are given in a distribution space by their geographical coordinates $[\,\varphi_i, \lambda_i\,]$, $i = 1, 2, \dots, n$, determined by means of geographical information systems,
- amounts of products q_i, $i = 1, 2, \dots, n$, that are planned to be delivered from one of the warehouses within a given time span is given for each distribution centre,
- the distribution space is necessary from reasons of economy (effort at minimization of transportation costs) to divide into several distribution areas T_j,
- each selling centre P_i is allocated by some way into one of the distribution areas T_j, the allocation is not fixed, but it can be changed in the course of calculations according to the situation,
- distance between the distribution centres P_i and the warehouses S_j is calculated by means of a suitable two-dimensional metric $\varrho\,(P_i, S_j)$, so the geographical coordinates $[\,\varphi_i, \lambda_i\,]$, $i = 1, 2, \dots, n$ have to be converted into the rectangular ones $[\,x_i, y_i\,]$, $i = 1, 2, \dots, n$,
- the distances may be corrected by a corrective factor $k_{ij} \geq 1$ in case of need according to local conditions,

- the main goal is to locate r new objects – distribution warehouses S_j in the distribution space and determinate their rectangular coordinates $[\,X_j, Y_j\,]$, $j = 1, 2, \ldots, r$, so that the optimization criterion is minimized [1]:

$$\sum_{j=1}^{r} \sum_{i \in T_j} q_i k_{ij} \varrho\,(P_i, S_j)\,, \tag{1}$$

- the rectangular coordinates $[\,X_j, Y_j\,]$, $j = 1, 2, \ldots, r$ of the new located warehouses are recounted into the geographical ones $[\,\Phi_j, \Lambda_j\,]$, $j = 1, 2, \ldots, r$.

2.2. Conversion of geographical coordinates into planar ones. The Křovák coordinates are the most used planar rectangular coordinates in the Czech Republic. The Křovák mapping is a dual one – the geoid (Bessel, Krasovski, Hayford or WGS-84 ellipsoid) is mapped in a conformal way onto the reference sphere by the Gauss method first and then it is again mapped in a conformal way onto a cone surface in a general position. The course of calculation of the planar coordinates from the geographical ones is as follows (the Bessel ellipsoid is used) [5]:

1. The Gauss conformal mapping of the Bessel ellipsoid onto the reference sphere
$$[\,\varphi\,,\ \lambda\,] \longrightarrow [\,U\,,\ V\,]$$
by means of equations

$$U = 2\ \arctan\left\{\frac{1}{k}\left[\tan\left(\frac{\varphi}{2} + 45^\circ\right)\left(\frac{1 - e\sin\varphi}{1 + e\sin\varphi}\right)^{\frac{e}{2}}\right]^{\alpha}\right\} - 90^\circ, \tag{2}$$

$$V = \alpha\ (\lambda - \lambda_0)\,. \tag{3}$$

The used constants were calculated for the Czech Republic basic parallel $\varphi_0 = 49^\circ\ 30'$:

$$e = 0.08169\ 68308\ 747343, \tag{4}$$

$$\alpha = \sqrt{1 + \frac{e^2 \cos^4 \varphi_0}{1 - e^2}}, \tag{5}$$

$$U_0 = \arcsin \frac{\sin \varphi_0}{\alpha}, \tag{6}$$

$$k = \frac{\left[\tan\left(\frac{\varphi_0}{2} + 45^\circ\right)\left(\frac{1 - e\sin\varphi_0}{1 + e\sin\varphi_0}\right)^{\frac{e}{2}}\right]^{\alpha}}{\tan\left(\frac{U_0}{2} + 45^\circ\right)}, \tag{7}$$

$$\lambda_0 = -17^\circ\ 39'\ 46'', \tag{8}$$

where e is numerical eccentricity of the Bessel ellipsoid and λ_0 is the longitude of the Ferro meridian considering the Greenwich meridian as the basic one.

2. The transformation of the geographical coordinates to the cartographic ones
$$[\,U\,,\ V\,] \longrightarrow [\,S\,,\ D\,]$$
considering the chosen cartographic pole by means of equations

$$S = \arcsin\left[\,\sin U_c \sin U + \cos U_c \cos U \cos\left(V - V_c\right)\,\right], \tag{9}$$

$$D = \arcsin\left[\frac{\sin\left(V - V_c\right)\cos U}{\cos S}\right] \tag{10}$$

for the suitably chosen cartographic pole with coordinates:

$$U_c = 59^\circ\ 42'\ 42.6969'', \tag{11}$$

$$V_c = 42^\circ\ 31'\ 31.41725''. \tag{12}$$

3. The conformal mapping of the reference sphere onto a generally situated cone
$$[\,S\ ,\ D\,] \longrightarrow [\,\varrho\ ,\ \varepsilon\,]$$
by means of equations

$$\varrho = \varrho_0 \left[\frac{\tan\ \left(\frac{S_0}{2} + 45^\circ\right)}{\tan\ \left(\frac{S}{2} + 45^\circ\right)} \right]^n, \tag{13}$$

$$\varepsilon = nD, \tag{14}$$

where the following constants were calculated for the well-represented parallel $S_0 = 78.5^\circ$ and the reference sphere radius $R = 6\ 380\ 703.610\ 5$

$$\varrho_0 = 0.9999 R \cot S_0, \tag{15}$$

$$n = \sin S_0. \tag{16}$$

4. The transformation of the generally situated cone surface described by the polar coordinates onto a plane described by the rectangular ones
$$[\,\varrho\ ,\ \varepsilon\,] \longrightarrow [\,X\ ,\ Y\,]$$

$$X = \varrho \cos \varepsilon, \tag{17}$$

$$Y = \varrho \sin \varepsilon, \tag{18}$$

where X-axis is the image of the basic meridian ($42^\circ\ 30'$ eastward regarding the Ferro meridian) and the origin is the image of the cone apex. The whole Czech republic is placed into the first quadrant (with positive coordinates only) and each point in its territory must satisfy the relation $Y < X$.

2.3. Algorithm of optimum location of distribution warehouses. The algorithm of adaptive sequential optimization of the number of warehouses, their allocation to existing objects and location in the distribution space can be described as follows:

1. the optimum location of warehouses is determined first of all for the limit case of the one warehouse where it is no problem to allocate the distribution centres to distribution warehouses,

2. in every following step another warehouse S_{r+1} is added by splitting a selected warehouse $\mathrm{S}_{r'}$ into two ones, this warehouse is chosen either sequentially or by a suitable heuristic rule,

3. the chosen warehouse is split into two warehouses (the number of the warehouses is increased by one) so that the first one of them is placed at the half distance in the direction to the suitably selected distribution centre and the second one symmetrically in the opposite direction,

4. all the distribution centres allocated up to now to the warehouse $\mathrm{S}_{r'}$ are reallocated to one of the new warehouses according to their transport distance,

5. the optimum location of the warehouses is carried out for the actual allocation of distribution centres,

6. the allocation of the distribution centres to the warehouses is adaptively corrected at the end of the location process,

7. it is advisable to check up the ambiguous allocation items,

8. if the system of the distribution–centre allocation has been corrected, the optimum location of the warehouses has to be performed once more,

9. the whole procedure is repeated until adding another warehouse is economically advantageous considering the total costs.

2.4. Conversion of planar rectangular coordinates into geographical ones. Geographical coordinates φ, λ can be computed from planar rectangular ones X, Y in a very simple way by sequential inversion of equations (17), (18) and (13), (14) and (9), (10) and (3). Only equation (2) cannot be inverted in such a way because the inverse equation in an explicit form is impossible to be found. The value of geographical latitude φ can be calculated from the corresponding value of cartographical latitude U by numerical solution of the following equation for the unknown variable φ

$$\tan\left(\frac{U}{2}+45^\circ\right) = \frac{1}{k}\left[\tan\left(\frac{\varphi}{2}+45^\circ\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{\frac{e}{2}}\right]^{\alpha} \tag{19}$$

or from the following series

$$\varphi = \varphi_0 + 100.14160\,22789\times10^{-2}\,(U-U_0) - 86.87150\,417\times10^{-6}\,(U-U_0)^2 + \\ + 16.70197\times10^{-8}\,(U-U_0)^3 + 117.5089\times10^{-10}\,(U-U_0)^4 \;. \tag{20}$$

3. A real-world example. The above-described algorithm was used for solving the location-allocation problem for a well-known Czech manufacturer of stuffs for

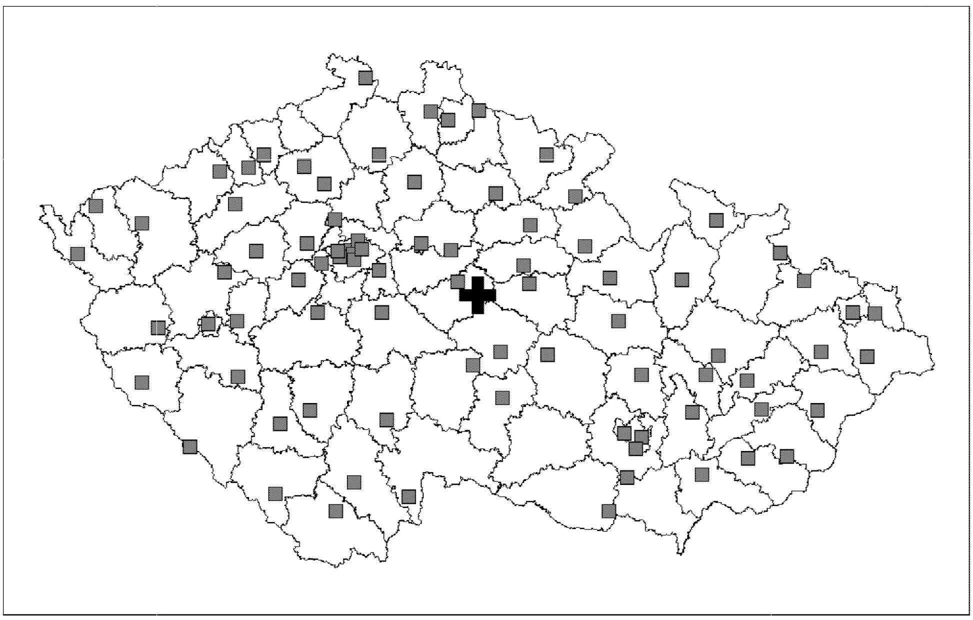

FIG. 2. *Location and allocation of the customers for one warehouse by means of the algorithm.*

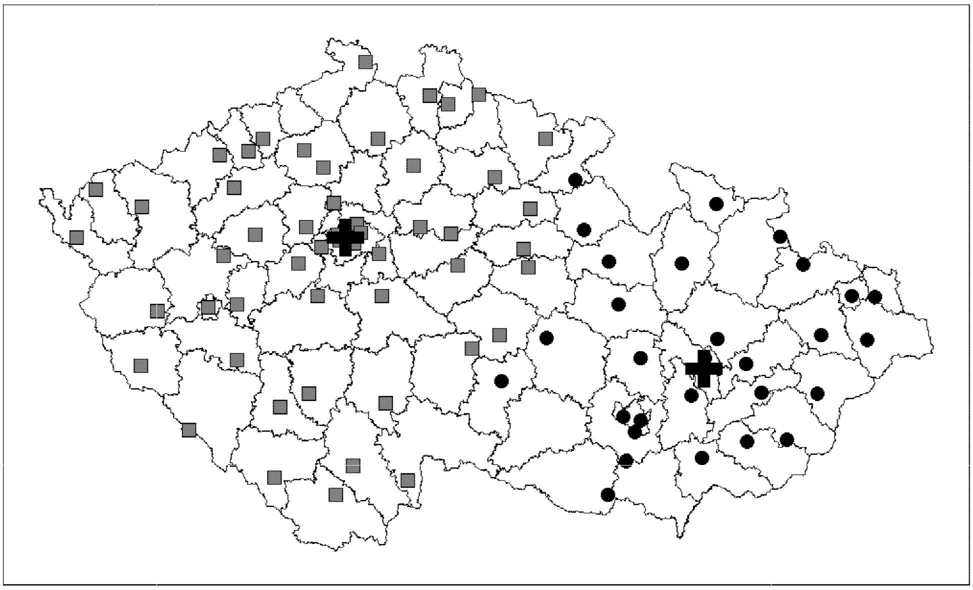

FIG. 3. *Location and allocation of the customers for two warehouses by means of the algorithm.*

cleaning and maintaining cars. There were about a thousand items at his customer list. Locations of all customers were scanned from digital maps by means of MapInfo Professional mapping system as their geographical coordinates. Each customer was described by his location, volume and time interval of regular deliveries. All deliveries were recounted for the same interval of delivery.

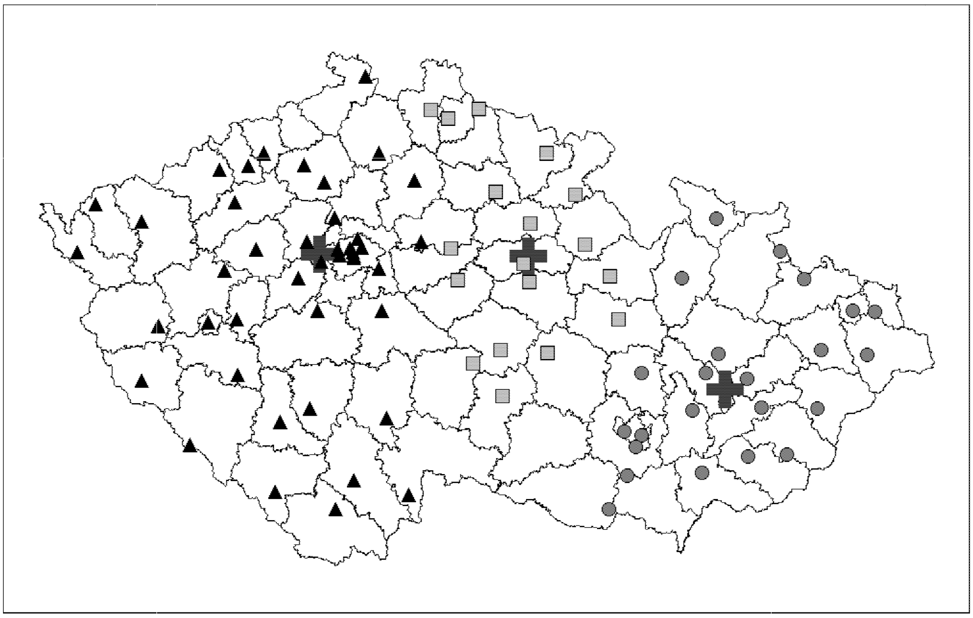

FIG. 4. *Location and allocation of the customers for three warehouses by means of the algorithm.*

All data concerning the customers are confidential and cannot be published in original form. To illustrate the algorithm and its results, all the data were transformed and changed in the following way:

- all customers placed in the same district were replaced by the one aggregated customer,
- its location was calculated as weighted average of customer locations and the volumes of deliveries were used as the weights,
- volumes of deliveries were averaged and transformed by multiplying with the same coefficient.

Table 1

Results of location-allocation algorithm for one warehouse.

Number of warehouses	Location	District	Longitude	Latitude	Criterion	Number of customers	Volume of deliveries
1	Tupadly	Kutná Hora	15.409 746 947	49.879 469 909	217 674.911	84	1853.652

Table 2

Results of location-allocation algorithm for two warehouses.

Number of warehouses	Location	District	Longitude	Latitude	Criterion	Number of customers	Volume of deliveries
1	Praha	Praha	14.412 688 611	50.084 035 627	75 599.930	55	1084.652
2	Žešov	Prostějov	17.122 409 179	49.442 861 635	46 359.951	29	769.000
Total					121 959.881	84	1853.652

Table 3

Results of location-allocation algorithm for three warehouses.

Number of warehouses	Location	District	Longitude	Latitude	Criterion	Number of customers	Volume of deliveries
1	Staré Hradiště	Pardubice	15.788 964 671	50.064 820 299	16 876.554	18	357.075
2	Annín	Přerov	17.281 463 895	49.410 364 719	34 994.550	23	648.750
3	Hájek	Praha-západ	14.211 990 593	50.075 271 585	51 482.186	43	847.827
Total					103 353.290	84	1853.652

Table 4

Results of expert's location procedure for two warehouses.

Number of warehouses	Location	District	Longitude	Latitude	Criterion	Number of customers	Volume of deliveries
1	Praha	Praha	14.309 016 279	50.100 050 872	61 403.393	49	941.402
2	Bousín	Prostějov	16.919 271 237	49.469 585 257	63 205.482	35	912.250
Total					124 608.876	84	1853.652

After the transformation of the data, a set of 84 aggregated customers was used as the input data for the algorithm. The algorithm determined in sequence the location of one, two and three warehouses and simultaneous allocation of customers (Fig. 2.3, Fig. 2.4 and Fig. 2.4). The transport operation was improved gradually from 100% for one to 56.0% for two and finally to 47.5% for three warehouses. Calculations were stopped after creating three warehouses because the economical parameters started to worsen by adding another warehouse. Results of the algorithm (see Tab. 1, Tab. 2 and Tab. 3) were compared with those created empirically by an expert (Fig. 4 and Tab. 4). The algorithm gave better values of used criterion (total transport operation) than the expert did [3]. On the other hand, the total transport operation was divided among warehouses better for the expert's results.

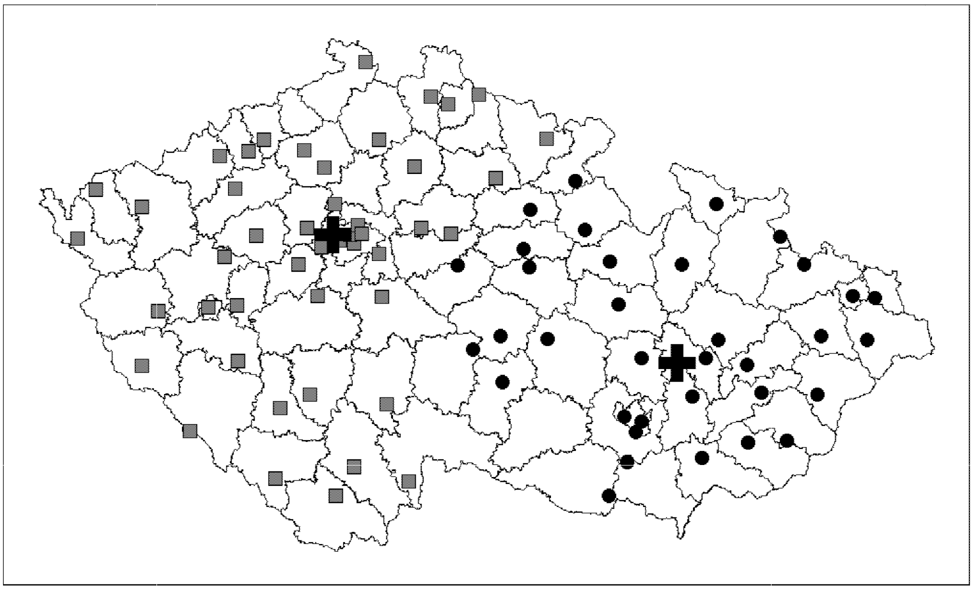

FIG. 5. *Location and allocation of customers for two warehouses by the expert.*

Calculations were performed both by a program developed with the help of Borland Pascal 7.0 programming language and in MS Excel spreadsheet system using its built-in functions and optimizer Solver. Some calculations were necessary to carry out in two steps because the Excel functions for location and selection in spreadsheets usually work over ordered lists only. Both results were the same with high accuracy.

4. Conclusions. The location-allocation problem for multifacility location was solved as a sequence of optimization problems for increasing number of warehouses. Initial estimates for new warehouses and their coordinates were generated by suitable heuristic rules. Locations of the existing object (customers, distribution centres, retail outlets) were scanned from digital maps as geographical coordinates. Algorithms for planar location were used for optimal location of new warehouses. Planar coordinates were transformed into geographical to enable their location in digital maps. All used algorithms were implemented in sheets of MS Excel using its standard means. The described procedure is easy to use for an inexpert user with no programming knowledge and experience who is able to work in the MS Excel.

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