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ORDER OF CONVERGENCE ESTIMATES IN TIME AND SPACE FOR AN IMPLICIT EULER, MIXED FINITE ELEMENT DISCRETIZATION OF RICHARDS' EQUATION BY EQUIVALENCE OF MIXED AND CONFORMAL APPROACH

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Abstract. We analyse a discretization method for a class of degenerate parabolic problems that includes the Richards' equation. This analysis applies to the pressure-based formulation and considers both variably and fully saturated regimes. To overcome the difficulties arising from the lack in regularity, we first apply the Kirchhoff transformation and then formulate a continous mixed variational formulation for a time-integrated version of the equation. Based on this we discretize using in time a scheme equivalent with backward Euler and in space the lowest order Raviart-Thomas elements. Simultaneously, a continous and a semidiscrete (continous in time) conformal variational formulations are stated and the equivalence between the corresponding mixed and conformal schemes is proved. This allows the use of techniques specific for conformal elements to get error estimates for the mixed finite element approach. Numerical results are presented to confirm our theoretical analysis, in particular showing the convergence of the scheme. The advantage of our approach is that the convergence was obtained without any extra regularity assumptions.

Key words. error estimates, Euler implicit scheme, mixed finite elements, regularization, degenerate parabolic problems, porous media, Richards' equation.

AMS subject classifications. 65M12, 65M15, 65M60, 76S05, 35K65, 35K5.

1. Introduction. An appropriate model for the ground water movement, taking into acccount the unsaturated subregions near the surface, is the Richards' equation, a nonlinear degenerate parabolic partial differential equation. In this paper the equation will be considered in its pressure formulation

$$\partial_t \Theta(\psi) - \nabla \cdot K(\psi)\nabla(\psi + z) = 0 \tag{1}$$

where ψ is the pressure head, Θ the saturation, K the conductivity and z the height against the gravitational direction. The Richards' equation describes the flow of a wetting fluid (water) in a porous medium in the presence of a non-wetting fluid (air) supposed to be at constant pressure, 0. It includes partially to fully water saturated regimes but can not be applied to completely dry soils. It results from the requirement of mass conservation (in form of volume conservation assuming the incompressibility of water):

$$\partial_t \Theta(\psi) + \nabla \cdot \mathbf{q} = 0 \tag{2}$$

and Darcy's law

$$\mathbf{q} = -K(\psi)\nabla(\psi + z), \tag{3}$$

with $\mathbf{q}$ denoting the flux. There are two coefficients functions: the soil-water retention $\Theta(\psi)$, relating the saturation and the pressure and the unsaturated hydraulic conduc-

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tivity $K(\psi)$, relating the conductivity and the pressure. Different functional dependencies (retention curves) between ψ, K and Θ are proposed in the literature. These are provided essentially by soil particularities and allow reducing all the unknowns in the above equation to a single one. For negative pressure values, corresponding to unsaturated subregions, the nonlinearities are monotone non-decreasing, therefore (1) is a (strongly) nonlinear parabolic equation there, but positive pressure values lead to a constant value of maximum saturation and represent the region below the ground water table, where the pressure obeys an elliptic equation. The transition from unsaturated to dry is here not considered, therefore the equation never becomes hyperbolic. As a consequence we deal with a nonlinear elliptic-parabolic equation whose solution is tipically lacking in regularity.
More regular unknowns could be obtained by applying the Kirchhoff transformation

$$\begin{aligned} \mathcal{K} : \mathbb{R} &\longrightarrow \mathbb{R} \\ \psi &\longmapsto \int_0^{\psi} K(\Theta(s))\, ds. \end{aligned} \tag{4}$$

Since $K(\Theta(s))$ is positive, this transformation can be inverted and equation (1) can be rewritten in terms of a new variable, $u := \mathcal{K}(\psi)$. Defining now

$$\begin{array}{lcl} b(u) & := & \Theta \circ \mathcal{K}^{-1}(u) \\ k(b(u)) & := & K \circ \Theta \circ \mathcal{K}^{-1}(u), \end{array} \tag{5}$$

and letting e_z denote the vertical unit vector, equation (1) becomes

$$\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\, e_z) = 0 \quad \text{in} \quad (0, \mathrm{T}) \times \Omega. \tag{6}$$

Also after the transformation the equation still remains degenerate and we expect only $\partial_t b(u) \in L^2(0,T; H^{-1}(\Omega))$ which does not allow for a mixed variational formulation being the basis for a mixed finite element discretization. To overcome this we follow an ideea of Nochetto [15], used also from Arbogast [2] and Woodward [22], to formulate a continuous mixed formulation for a time-integrated version of the conservation equation (2). Based on this we discretize using in time a scheme equivalent with backward Euler, together with a regularization step, and in space the lowest order Raviart-Thomas elements for the flux variable and piecewise constant elements for the pressure head. Specifically, with $N > 0$ integer, set $\tau = T/N$ and let $\mathcal{T}_h$ being a decomposition of Ω into closed d-simplices; h stands for the mesh-size. Then the numerical scheme under consideration reads

$$\begin{aligned} b_\epsilon(p_h^n) + \tau \nabla \mathbf{q_h^n} &= b_\epsilon(p_h^{n-1}), \\ \mathbf{q_h^n} + \nabla p_h^n + k(b(p_h^n))\mathbf{e_z} &= 0, \end{aligned}$$

for $n = \overline{1,N}$; p_h^0 approximates u^0 in the finite dimensional approximation space. Here b_ϵ is a regular approximation of b depending on the small parameter $\epsilon > 0$. By p_h^n we denote the piecewise constant approximation of u and $\mathbf{q_h^n}$ is the Raviart-Thomas (RT_0) approximation of the flux -$(\nabla u + k(b(u))\mathbf{e_z})$, based on $\mathcal{T}_h$, both at $t = n\tau$.

Convergence is shown by obtaining first error estimates for the time discrete scheme, by following the ideas in [15]. Next, using the procedure described in [2], error estimates for the fully discrete scheme are obtained. In this setting, the equivalence between the two different formulations becomes essential since in this way results obtained for the conformal method can be transferred to the mixed one and viceversa. The results are given here without proofs, which can be found in [19].

2. Equivalent variational formulations. In what follows Ω is a domain in $\mathbb{R}^d$ (with $d = 1, 2$ or 3). Let $J = (0, T]$ be a finite time interval. We are interested in solving equation (6) endowed with initial and boundary conditions,

$$
\begin{array}{rcll}
\partial_t b(u) - \nabla \cdot (\nabla u + k(b(u))\mathbf{e_z}) & = & 0 & \text{in} \quad \mathrm{J} \times \Omega, \\
u & = & u^0 & \text{in} \quad 0 \times \Omega, \\
u & = & 0 & \text{on} \quad \mathrm{J} \times \Gamma.
\end{array} \tag{7}
$$

Throughout this paper we make use of the following assumptions:
(A1) Ω is bounded with Lipschitz continuous boundary.
(A2) $b \in C^1$ is non-decreasing and Lipschitz continuous.
(A3) $k(b(z))$ is continuous and bounded in z and satisfies, for all $z_1, z_2 \in \mathbb{R}$,

$$
| \, k(b(z_2)) - k(b(z_1)) \, |^2 \leq C_k (b(z_2) - b(z_1))(z_2 - z_1).
$$

(A4) $b(u_0)$ is essentially bounded (by 0 and 1) in Ω and $u_0 \in L^2(\Omega)$.

Because fully air saturated regime has been not included in our analysis the Assumption (A2) is generally satisfied. Assumption (A3) is a relaxation of the Lipschitz continuity of k with respect to the saturation (e.g as is assumed in [22]).

Here and below $(\cdot, \cdot)$ stands for the inner product on $L^2(\Omega)$ or the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $\|\cdot\|$ for the norm in $L^2(\Omega)$, $\|\cdot\|_1$ and $\|\cdot\|_{-1}$ for the norms in $H^1(\Omega)$, respectively $H^{-1}(\Omega)$. We use analogous notations for the inner product and the corresponding norm on $L^2(0, T; \mathcal{H})$, with $\mathcal{H}$ being either $L^2(\Omega)$, $H^1(\Omega)$, or $H^{-1}(\Omega)$. In addition, we often write u or $u(t)$ instead of $u(t, x)$ and use C to denote a generic positive constant, not depending on the discretization or regularization parameters.

Existence, uniqueness and essential bounds for a weak solution of problem (7) is studied in several papers (see, for example, [1], [16], and the references therein). Following [2] or [22] we integrate (7) in time and obtain, for every $t \in J$,

$$
b(u(t)) + \nabla \cdot \int_0^t \mathbf{q} \, (s) \, ds = b(u^0) \tag{8}
$$

in L^2 sense. From [2], the flux $\mathbf{q} := -\left(\nabla u + k(b(u))\mathbf{e_z}\right)$ satisfies

$$
\int_0^t \mathbf{q} \, d\tau \ \in \ H^1(J; (L^2(\Omega))^d) \cap L^2(J; (H^1(\Omega))^d) =: X. \tag{9}
$$

We proceed by stating the variational formulations. Essential for the convergence proof will be the equivalence between the conformal formulation and the mixed one.

2.1. *The continuous case.* Integrated in time, problem (7) becomes
Problem 1. Find $u \in L^2(J, H_0^1(\Omega))$ such that $b(u) \in L^\infty(J \times \Omega)$, and for all $t \in J$ and $\phi \in H_0^1(\Omega)$ it holds

$$
(b(u(t)) - b(u^0), \phi) + \int_0^t (\nabla u(s) + k(b(u(s)))\mathbf{e_z}, \nabla \phi) ds = 0. \tag{10}
$$

Here $b(u)$ models the water content, hence it is natural to assume it bounded almost everywhere in $J \times \Omega$. Moreover, $u \in L^2(0, T; H_0^1(\Omega))$ yields $b(u) \in L^2(0, T; H_0^1(\Omega))$ due to the Lipschitz continuity of b. Since $b(u) \in H^1(0, T; H^{-1}(\Omega))$ we have $b(u) \in C(0, T; L^2(\Omega))$ (see [14], chapter I), allowing a simplified mixed variational formulation.

A mixed formulation for Problem (7) reads

Problem 2. Find $(p, \tilde{\mathbf{q}}) \in L^2(J \times \Omega)) \times X$ such that $b(p) \in L^\infty(J \times \Omega)$ and for all $t \in J$ the equations

$$(b(p(t)) - b(p^0), w) + (\nabla \tilde{\mathbf{q}}(t), w) = 0, \tag{11}$$

$$(\tilde{\mathbf{q}}(t), \mathbf{v}) - \int_0^t (p(s), \nabla \mathbf{v}) ds + \int_0^t (k(b(p(s)))\mathbf{e_z}, \mathbf{v}) ds = 0, \tag{12}$$

hold for all $w \in L^2(\Omega)$ and $\mathbf{v} \in H(div, \Omega)$, with $p^0 = u^0 \in L^2(\Omega)$.

The two problems are equivalent, as stated below.

PROPOSITION 2.1. *$u \in L^2(J, H_0^1(\Omega))$ solves Problem 1 iff $(p, \tilde{\mathbf{q}}) \in L^2(J \times \Omega)) \times X$ defined as*

$$(p, \tilde{\mathbf{q}}) = (u, -\int_0^t (\nabla u(s) + k(b(u(s)))\mathbf{e_z}) ds) \tag{13}$$

solves Problem 2. Moreover, in this case we have $p \in L^2(J, H_0^1(\Omega))$.

2.2. <u>*The semidiscrete case*</u>**.** As mentioned in the introduction, difficulties due to degeneracy can be overcomed by perturbing the original equation to a regular parabolic one. Such a technique has been successfully applied in the analysis of degenerate problems, and also allows developing effective numerical schemes (see, for example, [15], [8], or [18]). Here we approximate b by b_ϵ, where $\epsilon > 0$ is a small perturbation parameter. A possible choice reads

$$b_\epsilon(u) = b(u) + \epsilon u. \tag{14}$$

b_ϵ has the same properties as b but its derivative is bounded from below by ϵ.

With $N > 1$ being an integer giving the time step $\tau = T/N$ and $t_n = n\tau$, the regularized semidiscrete conformal problem reads

Problem 3. Let $n = \overline{1, N}$ and u^{n-1} be given. Find $u^n \in H_0^1(\Omega)$ such that, for all $\phi \in H_0^1(\Omega)$,

$$(b_\epsilon(u^n) - b_\epsilon(u^{n-1}), \phi) + \tau(\nabla u^n + k(b(u^n))\mathbf{e_z}, \nabla \phi) = 0. \tag{15}$$

Its mixed time discrete counterpart becomes

Problem 4. Let $n = \overline{1, N}$ and p^{n-1} given. Find $(p^n, q^n) \in L^2(\Omega) \times H(div, \Omega)$ such that

$$(b_\epsilon(p^n) - b_\epsilon(p^{n-1}), w) + \tau(\nabla \mathbf{q}^n, w) = 0, \tag{16}$$

$$(\mathbf{q^n}, \mathbf{v}) - (p^n, \nabla \mathbf{v}) + (k(b(p^n))\mathbf{e_z}, \mathbf{v}) = 0, \tag{17}$$

for all $w \in L^2(\Omega)$, respectively $\mathbf{v} \in H(div, \Omega)$, with $p^0 = u^0 \in L^2(\Omega)$.

As in the continuous case, the two problems above are equivalent.

PROPOSITION 2.2. *Let $n = \overline{1, N}$ be fixed and assume $u^{n-1} = p^{n-1}$. Then $u^n \in H_0^1(\Omega)$ solves Problem 3 iff $(p^n, \mathbf{q^n}) \in L^2(\Omega) \times H(div, \Omega)$ defined as*

$$(p^n, \mathbf{q^n}) = (u^n, -(\nabla u^n + k(b(u^n))\mathbf{e_z})) \tag{18}$$

solve Problem 4. Moreover, we have $p^n \in H_0^1(\Omega)$.

3. Error Estimates. Due to the equivalences proven above, stability and error estimates for the time discrete mixed formulation can be obtained by analyzing, using technigues from [15], the Euler implicit scheme applied to Problem 3.

3.1. *Error estimates for the semidiscrete scheme.* We use the notations

$$\begin{array}{rcl} \overline{u}^n & = & \frac{1}{\tau}\int_{t_{n-1}}^{t_n} u(t)dt, \\ p_\Delta(t) & = & p^n, \quad \text{for} \quad t \in (t_{n-1}, t_n], \\ e_b(u) & = & b(u) - b_\epsilon(p_\Delta), \end{array} \tag{19}$$

where $n = \overline{1, N}$ and $\overline{u}^0 = u^0$.

For the semidiscrete mixed discretization scheme we obtain the following

THEOREM 3.1. *Assuming (A1) - (A4), if u is the weak solution of Problem 1 and $(p^n, \mathbf{q}^n)$ solve Problem 4 ($n = \overline{1, N}$), we get*

$$\begin{array}{rl} & \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (b_\epsilon(u(t)) - b_\epsilon(p^n), u(t) - p^n)dt \\ + & \|\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (u(t) - p^n)dt\|_1^2 + \|\tilde{\mathbf{q}}(T) - \tau\sum_{n=1}^N \mathbf{q}^\mathbf{n}\|^2 \\ \leq & C(\tau + \epsilon). \end{array} \tag{20}$$

Remark 3.1. *Since b_ϵ is a perturbation of order ϵ for b we can replace the scalar product in (20) by $\int_0^T (b(u(t)) - b(p_\Delta(t)), u(t) - p_\Delta(t))dt$. This immediately implies an error estimate for the saturation,*

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|b(u(t)) - b(p^n)\|^2 dt \leq C(\tau + \epsilon).$$

3.2. *Estimates for the fully discrete scheme.* For the spatial discretization we let $\mathcal{T}_h$ be a regular decomposition of $\Omega \subset \mathbb{R}^d$ into closed d-simplices; h stands for the mesh-size. To avoid technicalities, Ω is assumed polygonal, satisfying $\overline{\Omega} = \cup_{T\in\mathcal{T}_h} T$.

The discrete subspaces $W_h \times V_h \subset L^2(\Omega) \times H(div, \Omega)$ are defined as

$$\begin{array}{rcl} W_h & := & \{p \in L^2(\Omega) |\ p \text{ is constant on each element } T \in \mathcal{T}_h\}, \\ V_h & := & \{\mathbf{q} \in H(div, \Omega) | \mathbf{q}_{|T} = \mathbf{a} + b\mathbf{x} \ \text{ for all } T \in \mathcal{T}_h\}. \end{array} \tag{1}$$

So W_h denotes the space of piecewise constant functions, while V_h is the RT_0 space (see [5]). Further we make use of the usual L^2 projector

$$P_h : L^2(\Omega) \to W_h, \quad ((P_h w - w), w_h) = 0 \qquad \forall w_h \in W_h. \tag{2}$$

Taking $\widetilde{V} = (H^1(\Omega))^d$ a projector Π_h can be defined as (see [5], p.131)

$$\Pi_h : \widetilde{V} \to V_h, \quad (\nabla \cdot (\Pi_h \mathbf{v} - \mathbf{v}), w_h) = 0 \tag{3}$$

for all $w_h \in W_h$. With $r \geq 0$, for the operators defined above we have

$$\begin{array}{rcl} \|w - P_h w\| & \leq & Ch^r \|w\|_r, \\ \|\mathbf{v} - \Pi_h \mathbf{v}\| & \leq & Ch^r \|\mathbf{v}\|_r, \end{array} \tag{4}$$

for any $w \in H^r(\Omega)$ and $\mathbf{v} \in (H^r(\Omega))^d$.

In order to can apply the projector Π_h to the flux variable we have to assume some more regularity:

(A5) $\mathbf{q} \in L^\infty(0, T; (H^1(\Omega))^d)$.

Remark 3.2. *Obviously (A5) is fulfilled in one spatial dimension, since in this case $H(div, \Omega)$ and $H^1(\Omega)$ coincide.*

Before proceeding with the fully discrete approximation scheme we rewrite Problem 4 (continuous in space) as

Problem 5. Let $n = \overline{1,N}$. Find $(p^n, \mathbf{q^n}) \in L^2(\Omega) \times H(div, \Omega)$ such that

$$(b_\epsilon(p^n), w) - (b_\epsilon(p^0), w) + \tau(\sum_{j=1}^{n} \nabla \mathbf{q}^j, w) = 0, \tag{5}$$

$$(\mathbf{q^n}, \mathbf{v}) - (p^n, \nabla \mathbf{v}) + (k(b(p^n))\mathbf{e_z}, \mathbf{v}) = 0, \tag{6}$$

for all $w \in L^2(\Omega)$ and $v \in H(div, \Omega)$, with $p^0 = u^0$.

The fully discrete mixed finite element approximation reads

Problem 6. Let $n = \overline{1,N}$. Find $(p_h^n, q_h^n) \in W_h \times V_h$ such that

$$(b_\epsilon(p_h^n), w_h) + \tau(\sum_{j=1}^{n} \nabla \mathbf{q}_h^j, w_h) = (b_\epsilon(p_h^0), w_h), \tag{7}$$

$$(\mathbf{q_h^n}, \mathbf{v_h}) - (p_h^n, \nabla \mathbf{v_h}) + (k(b(p_h^n))\mathbf{e_z}, \mathbf{v_h}) = 0, \tag{8}$$

for all $w_h \in W_h$ and $\mathbf{v_h} \in V_h$.

Applying techniques developed in [2] we estimate the errors induced by the spatial discretization.

THEOREM 3.2. *Assuming (A1)-(A5), if $(p^n, \mathbf{q^n}) \in L^2(\Omega) \times H(div, \Omega)$, $(p_h^n, \mathbf{q_h^n}) \in W_h \times V_h$ solve, for $n = \overline{1,N}$, Problems 5 and 6, we obtain*

$$\begin{aligned} &\textstyle\sum_{n=1}^{N}(b_\epsilon(p^n) - b_\epsilon(p_h^n), p^n - p_h^n) + \tau \sum_{n=1}^{N} \|\Pi_h \mathbf{q^n} - \mathbf{q_h^n}\|^2 \quad + \\ &\quad + \tau \| \textstyle\sum_{n=1}^{N}(\mathbf{q^n} - \mathbf{q_h^n})\|^2 + \tau \| \sum_{n=1}^{N}(p^n - p_h^n)\|^2 \quad \leq C\frac{h^2}{\tau}. \end{aligned} \tag{9}$$

Combining the estimates in Theorems 3.1 and 3.2 we get, for the fully discrete scheme

THEOREM 3.3. *Assuming (A1)-(A5) there holds*

$$\begin{aligned} &\| \textstyle\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (u(t) - p_h^n)dt\|^2 + \| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\mathbf{q}(t) - \mathbf{q_h^n})dt\|^2 \leq \\ &\quad \leq C(\tau + \epsilon + h^2). \end{aligned} \tag{10}$$

4. Numerical Results. To confirm our theoretical results we present a numerical test. We consider a problem allowing for a travel wave solution, as proposed in [9], which refers to the Richards' equation in its form after the Kirchhoff transformation (6), without gravitation term and with

$$b(u) = \begin{cases} \frac{\pi^2}{2} - \frac{u^2}{2} & \text{for} \quad u \leq 0 \\ \frac{\pi^2}{2} & \text{for} \quad u > 0. \end{cases}$$

For this problem an exact solution is known

$$u_{\text{ex}}(t, x, y) = \begin{cases} \frac{-2(e^s - 1)}{e^s + 1} & \text{for} \quad s \geq 0 \\ -s & \text{for} \quad s < 0 \end{cases}$$

where $s = x - y - t$. The equation has been solved in the unit square Ω, with Dirichlet boundary condition given by $u = u_{\text{ex}}$ on $\partial\Omega$ and initial value u_{ex} at $t = 0$. Computations are carried out for final time $T = 1.0$.

TABLE 1
Numerical results.

N	τ	h	$\tau + h^2$	error	convergence order
1	0.04	0.25	1.025000e-01	6.344201e-06	—
2	0.02	0.176	5.125000e-02	3.620119e-06	0.81
3	0.01	0.125	2.562500e-02	2.057356e-06	0.82
4	0.005	0.088	1.281250e-02	9.574634e-07	1.10
5	0.0025	0.0625	6.406250e-03	5.362175e-07	0.84
6	0.00125	0.044	3.203250e-03	2.431734e-07	1.14
7	0.000625	0.03125	1.601562e-03	1.355397e-07	0.84

For mixed finite element discretizations the emerging algebraic system of equations is difficult to solve due to being the solution of a saddle point problem. A common implementation procedure is to enlarge the system by adding Lagrange multipliers on edges (hybridization of the method). Briefly, within one time step the resulting algorithm reads: first the flux variable is eliminated on each element, then the continuity equation is locally solved for pressure by a variably damped Newton's method. The global system is set for the Lagrange multipliers and solved using again a Newton procedure. Linear iterations are solved by multigrid methods (see [21] for implementation details). The algorithm is implemented in *UG* (version 3.8, see also [3]) and calculations are done on a SUN workstation.

We have started performing computations on a uniform triangular mesh with $h = 0.25$ and a time step $\tau = 0.04$. Then τ and h^2 are successively halved, up to $\tau = 0.000625$ and $h = 0.03125$. Knowing the exact solution, the square of the total error (as written in (10)) is given by

$$E_{tot}^2 = \| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (u_{\mathrm{ex}}(t) - p_h^n)\, dt \|^2 + \| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\mathbf{q_{ex}}(t) - \mathbf{q_h^n})\, dt \|^2,$$

where $\mathbf{q_{ex}} = -\nabla\, u_{\mathrm{ex}}$ is the exact flux. The order of convergence (for the squared error) is estimated by dividing the errors above, computed for two sets of parameters (refined according to the procedure mentioned above). Dividing the natural logarithm of the result by the natural logarithm of the refinement ratio yields an approximation of the convergence order. Results are displayed in Table 1. As predicted by Theorem 3.3, the convergence order approaches 1. Hence we can conclude that numerical results are in concordance with our theoretical analysis, in particular proving the convergence of the scheme.

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