Proceedings of EQUADIFF 2017
pp. 163–172

SEMI-ANALYTICAL APPROACH TO INITIAL PROBLEMS FOR SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT DELAY.

HELENA ŠAMAJOVÁ *

Abstract. This paper deals with the differential transform method for solving of an initial value problem for a system of two nonlinear functional partial differential equations of parabolic type. We consider non-delayed as well as delayed types of coupling and the different variety of initial functions are thought over. The convergence of solutions and the error estimation to the presented procedure is studied. Two numerical examples for non-delayed and delayed systems are included.

Key words. nonlinear partial differential equation, parabolic type equation, delayed equation, system of partial differential equation, initial problem

AMS subject classifications. 35K55, 35K51 35K61

1. Introduction. We consider a system of two nonlinear functional partial differential equations of parabolic type with constant delays

$$\begin{aligned} \frac{\partial y_1(x,t)}{\partial t} &= \frac{\partial^2 y_1(x,t)}{\partial x^2} + K_1(y_2(x,t-\tau_1) - y_1(x,t)) + \eta_1 y_1^3(x,t) \\ \frac{\partial y_2(x,t)}{\partial t} &= \frac{\partial^2 y_2(x,t)}{\partial x^2} + K_2(y_1(x,t-\tau_2) - y_2(x,t)) + \eta_2 y_2^3(x,t) \end{aligned} \tag{1.1}$$

with the given initial function $\tilde{\psi}_i(x,t)$, constant delays τ_i , constants η_i , and K_i where $i = 1, 2$.

We may rewrite the system (1.1) into the vector form

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \kappa_1 u(x,t) + \kappa_2 \hat{u}(x,t) + \eta \tilde{u}(x,t) \tag{1.2}$$

where we consider square matrix

$$\kappa_1 = \begin{pmatrix} -K_1 & 0 \\ 0 & -K_2 \end{pmatrix}; \qquad \kappa_2 = \begin{pmatrix} 0 & K_1 \\ K_2 & 0 \end{pmatrix}; \qquad \eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \tag{1.3}$$

and the vector form of functions

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix};\ \hat{u}(x,t) = \begin{pmatrix} u_1(x,t-\tau_1) \\ u_2(x,t-\tau_2) \end{pmatrix};\ \tilde{u}(x,t) = \begin{pmatrix} u_1^3(x,t) \\ u_2^3(x,t) \end{pmatrix}. \tag{1.4}$$

We consider the system where the time of response may be 0 or different from 0. A real time of response causes that solutions do not affect each other in the same time.

*Dept. of Applied Mathematics, Faculty of Mechanical Engineering, University of Žilina, Univerzitná 1, Slovakia (helena.samajova@fstroj.uniza.sk).

Some types of nonlinear parabolic equation with a constant delay are exactly solved in [5] by functional constraints method. This method brings exact solutions that are supposed to be in the generalized separable form

$$u(x,t) = \sum_{n=1}^{N} \varphi_n(x)\psi_n(t)$$

where $N \in \mathbb{N}$. Functions $\varphi_n(x)$ and $\psi_n(x)$ are established by additional functional constrains given by difference or functional equation. The results in the cited paper are extended to a class of nonlinear partial differential-difference equations with linear differential operators which are defined as separated differential operators with respect to the independent variables x, t and to some partial functional differential equations with time delay. The presented way of solution in [5] requires an assumption that initial functions to an initial problem of a delayed equation are obliged to satisfy the considered equation.

An approach established in this paper enables us to use different types of initial functions that need not indispensable to fulfill the system (1.1).

2. Main Properties of 2D Differential Transform Method (DTM). In the next it is proposed a procedure which allows us to combine DTM and method of steps to obtain semi-analytical solutions for given system of two equations (1.1). This method is used for example in [2, 6, 7] and the references given therein.

The two dimensional Differential transformation method (DTM) for a function $g(x,t)$ is defined by

$$G(m,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n} g(x,t)}{\partial x^m \partial t^n}\right]_{x=x_0,t=t_0}.$$

An inverse transform of $G(m,n)$ leads to

$$g(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} G(m,n)(x-x_0)^m(t-t_0)^n$$

and if $x=0, t=0$ then

$$g(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} G(m,n)x^m t^n.$$

The main properties of the DTM are given in the overview:

Let functions $G, G_i(n), \quad i = 1,2,3$ are differential transforms of the functions $g, g_i(n), \quad i = 1,2,3$, constants $r, s \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{R}$

1. $g(x,t) = \alpha g_1(x,t) + \beta g_2(x,t) \qquad G(m,n) = \alpha G_1(m,n) + \beta G_2(m,n)$
2. $g(x,t) = x^r t^s \qquad G(m,n) = \delta(m-r, n-s) = \delta(m-r)\delta(n-s)$
3. $g(x,t) = \mathrm{e}^{\alpha x + \beta t} \qquad G(m,n) = \frac{\alpha^m \beta^n}{m!n!}$
4. $g(x,t) = \sin(\alpha x)t^s \qquad G(m,n) = \dfrac{\alpha^m}{m!}\sin(\dfrac{m\pi}{2})\delta(n-s)$

5. $g(x,t) = \cos(\alpha x)t^s \qquad G(m,n) = \dfrac{\alpha^m}{m!}\cos(\dfrac{m\pi}{2})\delta(n-s)$

6. $g(x,t) = g_1(x,t)g_2(x,t)g_3(x,t)$
   $G(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{k=0}^{n}\sum_{l=0}^{n-k} G_1(i,n-k-l)G_2(j,k)G_3(m-i-j,l)$

7. $g(x,t) = \dfrac{\partial g_1(x,t)}{\partial x}\dfrac{\partial g_2(x,t)}{\partial t}$
   $G(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(m-i+1)(n-j+1)G_1(m-i+1,j)G_2(i,n-j+1).$

   For delayed functions in the next we suppose $N \to \infty$

8. $g(x,t) = g_1(x,t+\tau) \qquad G(m,n) = \sum_{h=n}^{N}\binom{h}{n}\tau^{h-n}G_1(m,h)$

where $\delta(n)$ is the Kronecker delta symbol and $N \in \mathbb{N}$.
The main steps of the DTM, as a tool for solving different classes of nonlinear problems, are the following. First, we apply the differential transform to the presented problem, and then the functions $G(m,n)$ are given by the recurrence relations. In the second, the iterative solution of this relations and using the inverse differential transform, lead to the solution of the problem as polynomials of two independent variables.
Applying this rules for system (1.1) one obtains following recurrence relations for $\tau = 0$

$$\begin{aligned}
Y_1(m,n+1) &= \frac{1}{n+1}\Big[(m+2)Y_1(m+2,n) + K_1\left[Y_2(m,n) - Y_1(m,n)\right] \\
&+\eta_1 \sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\sum_{s_1=0}^{n}\sum_{s_2=0}^{n-s_1} Y_1(r_1,n-s_1-s_2)Y_1(r_2,s_1)Y_1(m-r_1-r_2,s_2)\Big] \\
Y_2(m,n+1) &= \frac{1}{n+1}\Big[(m+2)Y_2(m+2,n) + K_2\left[Y_1(m,n) - Y_2(m,n)\right] \\
&+\eta_2 \sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\sum_{s_1=0}^{n}\sum_{s_2=0}^{n-s_1} Y_2(r_1,n-s_1-s_2)Y_2(r_2,s_1)Y_2(m-r_1-r_2,s_2)\Big].
\end{aligned} \tag{2.1}$$

2.1. Initial problem for systems of delayed functions. If we suppose delayed system with $\tau_i > 0,\ i = 1,2$ the system is considered with the known initial functions ψ_i, $i = 1,2$

$$\psi_i(x,t) = \begin{cases} 0, & t < -\tau; \\ \tilde{\psi}_i(x,t), & t \in \langle -\tau, 0\rangle; \\ 0, & t > 0. \end{cases} \tag{2.2}$$

We consider different types of functions on the intervals $(-\tau_i, 0)$, as an initial functions for unknown solutions $y_i(x,t)$, $i = 1,2$ to the system (1.1).

A different types of initial functions produce appertaining initial conditions for the system (1.1) and some of them are presented in the table below.

- Calculations are valid on minimum length of the intervals $(0,\tau_i)$, $i = 1,2$
- $\tilde{\psi}_1(x,t)$, $\tilde{\psi}_2(x,t)$ are considered as constant, polynomial, exponential, sin, cos functions
- Recurrent relations are used for evaluations of coefficients $Y_1(m,n),\ Y_2(m,n)$

TABLE 2.1
Types of initial functions.

Initial functions $\Psi_i(x,t)$	**Initial condition** $\Psi_i(x,0)$
$\Psi_i(x,t) = x^r t^s$	$\Psi_i(x,0) = 0$
$\Psi_i(x,t) = x^r \epsilon^{st} \quad r \neq 0$	$\Psi_i(x,0) = x^r$
$\Psi_i(x,t) = t^s \epsilon^{rx} \quad s \neq 0$	$\Psi_i(x,0) = 0$
$\Psi_i(x,t) = x^r \cos st$	$\Psi_i(x,0) = x^r$
$\Psi_i(x,t) = x^r \sin st$	$\Psi_i(x,0) = 0$

- An individual evaluation for the initial functions and initial conditions is required
- Functions $y_1(x,t-\tau_2)$, $y_2(x,t-\tau_1)$ are replaced by the initial functions $\tilde{\psi}_1(x,t)$, $\tilde{\psi}_2(x,t)$ on the intervals $(-\tau_2,0)$, $(-\tau_1,0)$ respectively
- The multi-step differential transform method (MsDTM) given in [1, 3] may be used to extend the domain for the obtained solutions.

In the Table 2.1 we give some examples of types of the initial functions and the initial conditions connected to the initial functions.

The DT method applied to the system (1.1) with $\tau_i > 0$ gives

$$
\begin{aligned}
Y_1(m,n+1) &= \frac{1}{n+1}\Bigg[(m+2)Y_1(m+2,n) + K_1\left(\sum_{h=n}^{N}\binom{h}{n}\tau_1^{h-n}\Psi_2(m,h) - Y_1(m,n)\right) \\
&\quad + \eta_1 \sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\sum_{s_1=0}^{n}\sum_{s_2=0}^{n-s_1} Y_1(r_1,n-s_1-s_2)Y_1(r_2,s_1)Y_1(m-r_1-r_2,s_2)\Bigg] \\
Y_2(m,n+1) &= \frac{1}{n+1}\Bigg[(m+2)Y_2(m+2,n) + K_2\left(\sum_{h=n}^{N}\binom{h}{n}\tau_2^{h-n}\Psi_1(m,h) - Y_2(m,n)\right) \\
&\quad + \eta_2 \sum_{r_1=0}^{m}\sum_{r_2=0}^{m-r_1}\sum_{s_1=0}^{n}\sum_{s_2=0}^{n-s_1} Y_2(r_1,n-s_1-s_2)Y_2(r_2,s_1)Y_2(m-r_1-r_2,s_2)\Bigg].
\end{aligned}
\tag{2.3}
$$

3. Convergence of the 2D Differential Transform Method. In this section, the convergence of the 2-dimensional DTM when applied to a system of partial differential equations is studied. Moreover there is given the sufficient condition for a convergence of the vector function.

This condition of the convergence leads to an estimation of the maximum absolute error of the approximate solutions.

Let consider functions $f_1(x,t): \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $f_2(x,t): \mathbb{R}\times\mathbb{R}\to\mathbb{R}$

$$f_1(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_1(m,n)(x-x_0)^m(t-t_0)^n;$$

$$f_2(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_2(m,n)(x-x_0)^m(t-t_0)^n;$$

and

$$\vec{f}(x,t) = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}.$$

For the vector function we define the vector norm L^∞

$$\|\vec{f}\|_\infty = \max_i |f_i|, \qquad i \in \{1,2\}.$$

The theorem stated below is a special case of the Banach fixed point theorem [4]. In the next this theorem is adapted for 2D DTM.

Theorem 3.1.

Let there exist two series for functions

$$f_1(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_1(m,n)(x-x_0)^m(t-t_0)^n$$

$$f_2(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_2(m,n)(x-x_0)^m(t-t_0)^n.$$

Then the vector series $\vec{f}(x,t)$ converges if there exists $0<\alpha<1$ such that

$$\|\vec{f}_{k+1}(x,t)\| \le \alpha \|\vec{f}_k(x,t)\|$$

for any $k \ge k_0$, for some $k_0 \in \mathbb{N}$.

The estimation of the error of the vector series is a part of the proof of Theorem 3.1.

Proof. We denote $(C(A), \|.\|)$ the Banach space of all continuous vector functions on a domain A with the norm $\|f(x,t)\| = \max_{(x,t)\in A} \|f(x,t)\|$ where $A = [x_0-\varepsilon, x_0+\varepsilon] \times [t_0-\tau, t_0+\tau]$.

Denote individual terms $\varphi^1_{(m,n)}(x,t)$, $\varphi^2_{(m,n)}(x,t)$, $\Phi_{(m,n)}(x,t)$ as

$$\varphi^i_{(m,n)}(x,t) = F_i(m,n)(x-x_0)^m(t-t_0)^n \qquad i=1,2\ ,$$

$$\Phi_{(m,n)}(x,t) = \begin{pmatrix} F_1(m,n)(x-x_0)^m(t-t_0)^n \\ F_2(m,n)(x-x_0)^m(t-t_0)^n \end{pmatrix} = \begin{pmatrix} \varphi^1_{(m,n)}(x,t) \\ \varphi^2_{(m,n)}(x,t) \end{pmatrix}.$$

We define the sequence of vector partial sums $\{S_n\}_{n=0}^\infty$ as follows

$$S_n = \Phi_{(0,0)}(x,t)+\Phi_{(1,0)}(x,t)+\Phi_{(0,1)}(x,t)+\Phi_{(2,0)}(x,t)+\Phi_{(1,1)}(x,t)+\Phi_{(0,2)}(x,t)+\ldots+$$

$$\Phi_{(n,0)}(x,t)+\Phi_{(n-1,1)}(x,t)+\ldots+\Phi_{(1,n-1)}(x,t)+\Phi_{(0,n)}(x,t) =$$

$$\sum_{j=0}^{n}\sum_{i=0}^{j}\Phi_{(i,j-i)}(x,t).$$

In the next we will show that $\{S_n\}_{n=0}^\infty$ is a Cauchy sequence in the Banach space. For this purpose

$$\|S_{n+1}-S_n\| = \left\|\sum_{i=0}^{n+1}\Phi_{(i,n+1-i)}(x,t)\right\| \le \alpha \left\|\sum_{i=0}^{n}\Phi_{(i,n-i)}(x,t)\right\| \le \ldots \le$$

$$\leq \alpha^{n-k_0+1} \left\| \sum_{i=0}^{k_0} \Phi_{i,k_0-i}(x,t) \right\| =$$

$$= \alpha^{n-k_0+1} \max_{(x,t)\in A} \left\{ \sum_{i=0}^{k_0} \left| \varphi^1_{(i,k_0-i)}(x,t) \right|, \sum_{i=0}^{k_0} \left| \varphi^2_{(i,k_0-i)}(x,t) \right| \right\}.$$

For any $i, j \in \mathbb{N},\ i > j > k_0$ we have

$$\|S_i - S_j\| = \left\| \sum_{l=j}^{i-1} (S_{l+1} - S_l) \right\| \leq \sum_{l=j}^{i-1} \|(S_{l+1} - S_l)\|$$

$$\leq \sum_{l=j}^{i-1} \alpha^{l-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \|\Phi_{s,k_0-s}(x,t)\|$$

$$= \frac{1-\alpha^{i-j}}{1-\alpha} \alpha^{j-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \|\Phi_{s,k_0-s}(x,t)\|$$

and whereas $0 < \alpha < 1$, we obtain

$$\lim_{i,j\to\infty} \|(S_i - S_j\| = 0.$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C(A), \|.\|)$ and the vector series

$$\begin{pmatrix} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varphi^1_{(m,n)}(x,t) \\ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varphi^2_{(m,n)}(x,t) \end{pmatrix}$$

converges. The proof is complete. $\Box$

Under the condition that there exists $\alpha \in (0,1)$ such that

$$\sum_{s=0}^{k+1} \left\| \Phi_{(s,k+1-s)}(x,t) \right\| \leq \alpha \sum_{s=0}^{k} \left\| \Phi_{(s,k-s)}(x,t) \right\|$$

for any $k \geq k_0$ where $k_0 \in \mathbb{N},$ power series solution converges to the exact solution.

We define constants α_k for any $k \geq k_0$

$$\alpha_{k+1} = \begin{cases} \dfrac{\sum_{s=0}^{k+1} \left\| \Phi_{(s,k+1-s)}(x,t) \right\|}{\sum_{s=0}^{k} \left\| \Phi_{(s,k-s)}(x,t) \right\|} & \text{for} \quad \sum_{s=0}^{k} \left\| \Phi_{(s,k-s)}(x,t) \right\| \neq 0; \\ 0 & \text{for} \quad \sum_{s=0}^{k} \left\| \Phi_{(s,k-s)}(x,t) \right\| = 0. \end{cases}$$

If $\forall k > k_0 :\ 0 \leq \alpha_k < 1$, then an approximate solution in the form of finite series converges to the exact solution $\vec{u}(x,t)$.

THEOREM 3.2. *Let the approximate solution be in the form*

$$\vec{f}(x,t) = \begin{pmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_1(m,n)(x-x_0)^m(t-t_0)^n \\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} F_2(m,n)(x-x_0)^m(t-t_0)^n \end{pmatrix}$$

and converges to the solution

$$\vec{u}(x,t) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}.$$

If the finite series

$$\begin{pmatrix} \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} F_1(m,n)(x-x_0)^m(t-t_0)^n \\ \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} F_2(m,n)(x-x_0)^m(t-t_0)^n \end{pmatrix}$$

is considered as an approximation to the solution, then the estimation of the absolute error is given as

$$\left\| \vec{u}(x,t) - \begin{pmatrix} \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} \varphi^1_{(m,n)}(x,t) \\ \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} \varphi^2_{(m,n)}(x,t) \end{pmatrix} \right\| \le$$

$$\le \frac{1}{1-\alpha}\alpha^{j-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \|\Phi_{(s,k_0-s)}(x,t)\| \tag{3.1}$$

where $j = \min\{\mu,\nu\}$, $\mu,\nu \in \mathbb{N}$.

Proof. From the Theorem 3.1 we obtained

$$\|S_i - S_j\| \le \frac{1-\alpha^{i-j}}{1-\alpha}\alpha^{j-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \left\|\Phi_{(s,k_0-s)}(x,t)\right\|$$

Since the term $(1-\alpha^{i-j}) < 1$ under the condition that there exists an $\alpha \in (0,1)$ and $k_0 \le j \le i$, the inequality above can be simplify to

$$\|S_i - S_j\| \le \frac{1}{1-\alpha}\alpha^{j-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \left\|\Phi_{(s,k_0-s)}(x,t)\right\|.$$

If we consider that $i \to \infty$ then $S_i \to \vec{u}(x,t)$ - two dimensional power series vector solution converges to the vector solution and the estimation of an absolute error is determined by (3.1). □

In accordance with Theorem 3.2 the estimation of the absolute error is given by the inequality below

$$\left\| \vec{u}(x,t) - \begin{pmatrix} \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} F_1(m,n)(x-x_0)^m(t-t_0)^n \\ \sum_{m=0}^{\mu}\sum_{n=0}^{\nu} F_2(m,n)(x-x_0)^m(t-t_0)^n \end{pmatrix} \right\| \le$$

$$\frac{1}{1-\beta}\beta^{j-k_0+1} \max_{(x,t)\in A} \sum_{s=0}^{k_0} \left\|\Phi_{(s,k_0-s)}(x,t)\right\|,$$

where $\beta = \max\{\alpha_k,\ k = k_0+1, k_0+2, \dots, j+1\}$.

As an example of non-delayed and delayed coupling there are given pairs of figures of solutions $y_1(x,t)$ and $y_2(x,t)$. For different types of initial functions the Figures (3.1) and (3.3) represent non-delayed coupling, the Figures (3.2) and (3.4) delayed coupling. For calculation the system Mathematica was used.

For parameters $K_1 = 0.5$, $K_2 = 1.1$, $\eta_1 = 0.5$, $\eta_2 = 0.3$, $N = 6$ and initial functions $\tilde{\psi}_1 = 0$ and $\tilde{\psi}_2 = \cos x$ the solutions to the system (1.1) for a non-delayed case are on Fig. 3.1

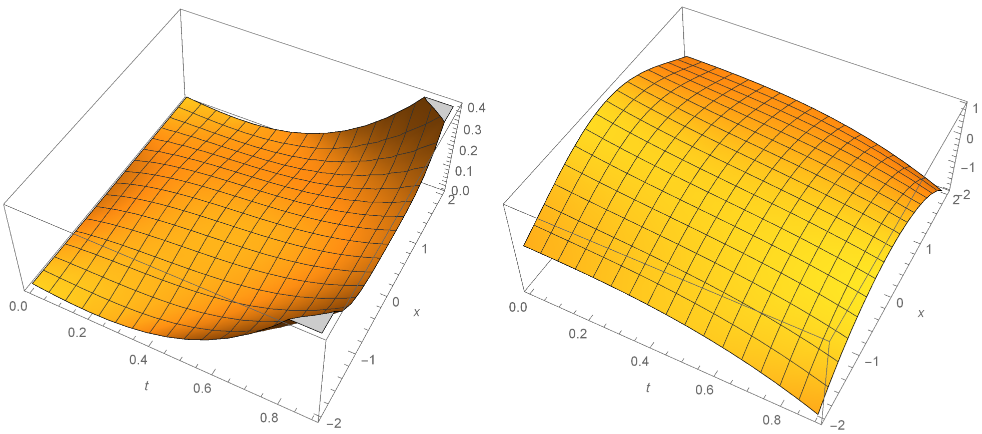

FIG. 3.1. *Solutions from left:* $y_1(x,t)$, $y_2(x,t)$, $\tau = 0$.

where

$$y_1(x,t) = 0.5t - 0.9t^2 + 0.6967t^3 - 0.25tx^2 + 0.2833t^2x^2 + 0.0208tx^4$$
$$y_2(x,t) = 1 - 2.1t + 2.1467t^2 - 1.5438t^3 - 0.5x^2 + 0.7167tx^2 - 0.64t^2x^2 + 0.0417x^4 - 0.0542tx^4 - 0.0014x^6.$$

For a delayed case with $\tau_i = 0.8$ the solutions are in Fig. 3.2

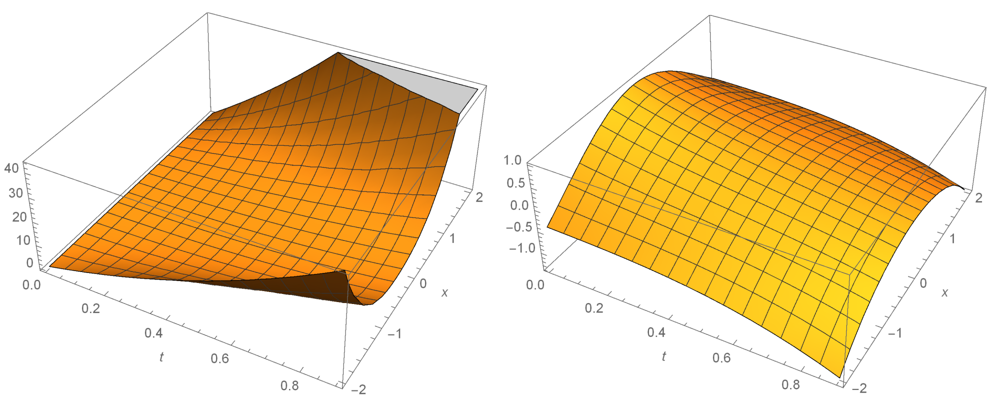

FIG. 3.2. *Solutions from left:* $y_1(x,t)$, $y_2(x,t)$, $\tau_1 = \tau_2 = 0.8$.

where

$$y_1(x,t) = 2.5t + 3.75t^2 + 4.7917t^3 + 2.5tx + 5t^2x + 2.5tx^2 + 6.25t^2x^2 + 2.5tx^3 + 2.5tx^4$$
$$y_2(x,t) = 1 - 2.1t + 1.8717t^2 - 1.0213t^3 - 0.5x^2 + 0.7167tx^2 - 0.5025t^2x^2 + 0.0417x^4 - 0.0542tx^4 - 0.0014x^6.$$

For parameters $K = 0.2,\ \eta_1 = 1.5,\ \eta_2 = 0.3$ and initial functions $\tilde{\psi}_1 = \cos x$ and $\tilde{\psi}_2 = \sin x$, a non-delayed case is on Fig. 3.3

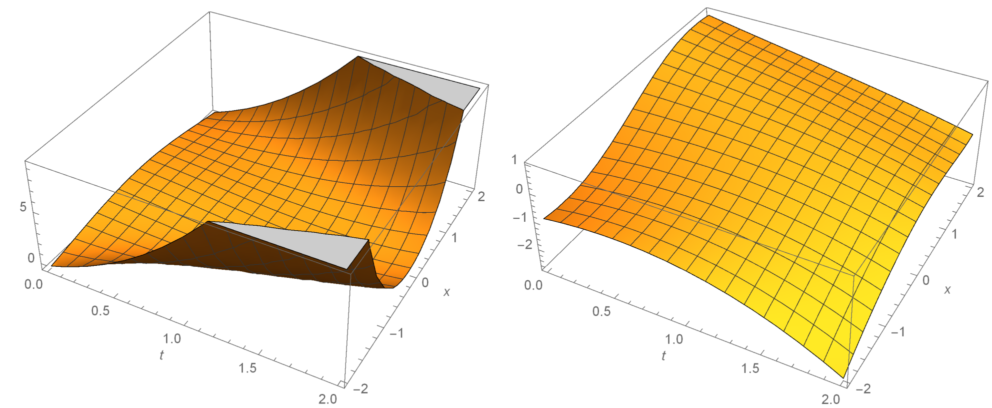

Fig. 3.3. *Solutions from left:* $y_1(x,t),\ y_2(x,t),\quad \tau = 0.$

where

$$\begin{aligned} y_1(x,t) =& 1 - 0.7t - 0.9183t^2 + 0.5294t^3 + 0.2tx + 0.01t^2x - 0.5x^2 \\ & -0.4833tx^2 + 1.0425t^2x^2 - 0.0333tx^3 + 0.0417x^4 + 0.4208tx^4 \\ & -0.0014x^6 \\ y_2(x,t) =& 0.2t - 0.19t^2 - 0.063t^3 + x - 0.7tx + 0.2025t^2x - 0.1tx^2 \\ & -0.0217t^2x^2 - 0.0083x^3 + 0.075tx^3 + 0.0083tx^4 + 0.0083x^5. \end{aligned}$$

Solutions for a delayed case with $\tau_i = 0.8$ are on Fig.3.4

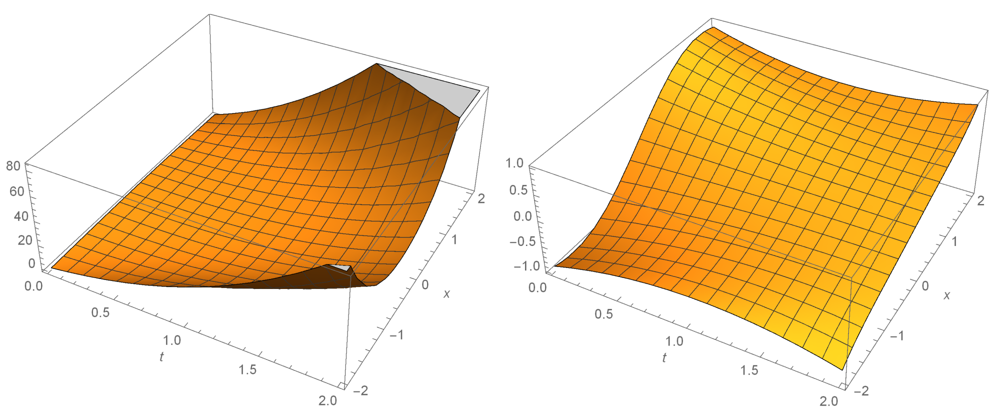

Fig. 3.4. *Solutions from left:* $y_1(x,t),\ y_2(x,t),\quad \tau_1 = \tau_2 = 0.8.$

where

$$\begin{aligned} y_1(x,t) =& 1 + 0.3t + 1.2117t^2 + 3.2051t^3 + tx + 2.65t^2x - 0.5x^2 \\ & +0.5167tx^2 + 3.4525t^2x^2 + tx^3 + 0.0417x^4 + 1.4208tx^4 \\ & -0.0014x^6 \\ y_2(x,t) =& x - 0.7tx + 0.1825t^2x - 0.1667x^3 + 0.075tx^3 + 0.0083x^5. \end{aligned}$$

REFERENCES

[1] Benhammouda, B., Vazquez-Leal, H. *A new multi-step technique with differential transform method for analytical solution of some nonlinear variable delay differential equations* SpringerPlus, (2016), 5, 1723. DOI 10.1186/s40064-016-3386-8

[2] Khan Y., Svoboda Z., Šmarda Z., *Solving certain classes of Lane-Emden type equations using the differential transformation method*, Advances in difference equations, 174, (2012).

[3] Odibat, Z. M., Bertelle C., Aziz-Alaouic M.A., Duchampd H. E. G. *A multi-step differential transform method and application to non-chaotic or chaotic systems* Computers and Mathematics with Applications, 59, (2010), pp 1462-1472.

[4] Odibat, Z. M., Kumar S., Shawagfeh N., Alsaedi A., Hayat T. *A study on the convergence conditions of generalized differential transform method* Mathematical Methods in the Applied Sciences, 40, (2017), pp 40-48.

[5] Polyanin A. D., Zhurov A. I. *Functional constraints method for constructing exact solutions to delay reactiondiffusion equations and more complex nonlinear equations* Commun. Nonlinear Sci. Numer. Simulat., 19, (2014), pp 417-430.

[6] Rebenda J., Šmarda Z., *A differential transformation approach for solving functional differential equations with multiple delays*, Commun. Nonlinear Sci. Numer. Simulat., 48, (2017), pp 246-257.

[7] Rebenda J., Šmarda Z., Khan Y., *A New Semi-analytical Approach for Numerical Solving of Cauchy Problem for Differential Equations with Delay*, FILOMAT, 31, (2017), pp 4725-4733.