

L^p-THEORY OF THE NAVIER-STOKES FLOW IN THE EXTERIOR OF A MOVING OR ROTATING OBSTACLE

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ABSTRACT. In this paper we describe two recent approaches for the L^p -theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle.

1. INTRODUCTION

Consider a compact set $O \subset \mathbb{R}^n$, the obstacle, with boundary $\Gamma := \partial O$ of class $C^{1,1}$. Set $\Omega := \mathbb{R}^n \setminus O$. For t > 0 and a real $n \times n$ -matrix M we set

$$\Omega(t) := \{ y(t) = e^{tM} x, x \in \Omega \} \text{ and } \Gamma(t) := \{ y(t) = e^{tM} x, x \in \Gamma \}.$$

Then the motion past the moving obstacle O is governed by the equations of Navier-Stokes given by

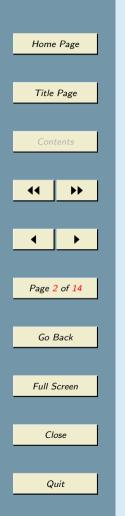
$$\partial_t w - \Delta w + w \cdot \nabla w + \nabla q = 0, \qquad \text{in } \Omega(t) \times \mathbb{R}_+, \\ \nabla \cdot w = 0, \qquad \text{in } \Omega(t) \times \mathbb{R}_+, \\ w(y,t) = My, \qquad \text{on } \Gamma(t) \times \mathbb{R}_+, \\ w(y,0) = w_0(y), \qquad \text{in } \Omega.$$

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(2)

Here w = w(y,t) and q(y,t) denote the velocity and the pressure of the fluid, respectively. The boundary condition on $\Gamma(t)$ is the usual no-slip boundary condition. Quite a few articles recently dealt with the equation above, see [2], [3], [4], [5], [6], [8], [10], [11], [15], [16].

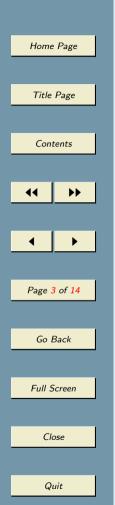
In this paper, we describe two approaches to the above equations for the L^p -setting where $1 . The basic idea for both approaches is to transfer the problem given on a domain <math>\Omega(t)$ depending on t to a fixed domain. The first transformation described in the following Section 2 yields additional terms in the equations which are of Ornstein-Uhlenbeck type. We shortly describe the techniques used in [15] and [12] in order to construct a local mild solution of (1).

In contrast to the first transformation, the second one, inspired by [17] and [6], allows to invoke maximal L^p -estimates for the classical Stokes operator in exterior domains and like this we obtain a unique strong solution to (1). This approach is described in section 3.

2. MILD SOLUTIONS

In this section we construct mild solutions to the Navier-Stokes problem (1). To do this we first transform the equations (1) to a fixed domain. Let Ω , $\Omega(t)$ and $\Gamma(t)$ be as in the introduction and suppose that M is unitary. Then by the change of variables $x = e^{-tM}y$ and by setting $v(x,t) = e^{-tM}w(e^{tM}x,t)$ and $p(x,t) = q(e^{tM}x,t)$ we obtain the following set of equations defined on the fixed domain Ω :

$$\begin{array}{rcl} \partial_t v - \Delta v + v \cdot \nabla v - Mx \cdot \nabla v + Mv + \nabla p &=& 0, & \text{ in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot v &=& 0, & \text{ in } \Omega \times \mathbb{R}_+, \\ v(x,t) &=& Mx, & \text{ on } \Gamma \times \mathbb{R}_+ \\ v(x,0) &=& w_0(x), & \text{ in } \Omega. \end{array}$$



(3)

Note that the coefficient of the convection term $Mx \cdot \nabla u$ is unbounded, which implies that this term cannot be treated as a perturbation of the Stokes operator.

This problem was first considered by Hishida in $L^2_{\sigma}(\Omega)$ for $\Omega \subset \mathbb{R}^3$ and $Mx = \omega \times x$ with $\omega = (0,0,1)^T$ in [15] and [16]. The L^p -theory was developed by Heck and the authors in [12] even for general M.

We will construct mild solutions for $w_0 \in L^p_{\sigma}(\Omega)$, $p \ge n$, to the problem (2) with Kato's iteration (see [18]).

The starting point is the linear problem

$$\partial_t u - \Delta u - Mx \cdot \nabla u + Mu + b \cdot \nabla u + u \cdot \nabla b + \nabla p = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot u = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \\ u = 0, \qquad \text{on } \Gamma \times \mathbb{R}_+, \\ u(x,0) = w_0(x), \qquad \text{in } \Omega, \end{cases}$$

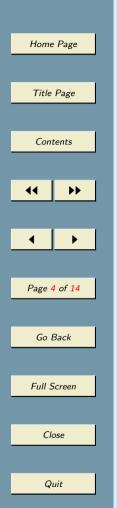
where $b \in C_c^{\infty}(\overline{\Omega})$. The additional term $b \cdot \nabla u + u \cdot \nabla b$ simplifies the treatment of the Navier-Stokes problem (see (11) below). We will first show that the solution of (3) is governed by a C_0 -semigroup on $L_{\sigma}^p(\Omega)$. More precisely, let $L_{\Omega,b}$ be defined by

$$L_{\Omega,b}u := P_{\Omega}\mathcal{L}_{b}u$$

$$D(L_{\Omega,b}) := \{u \in W^{2,p}(\Omega) \cap W_{0}^{1,p}(\Omega) \cap L_{\sigma}^{p}(\Omega) : Mx \cdot \nabla u \in L^{p}(\Omega)\},\$$

where $\mathcal{L}_b u := \Delta u + Mx \cdot \nabla u - Mu + b \cdot \nabla u + u \cdot \nabla b$. Then the following theorem is proved in [12].

Theorem 2.1. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$ -boundary. Assume that tr M = 0 and $b \in C_c^{\infty}(\overline{\Omega})$. Then the operator $L_{\Omega,b}$ generates a C_0 -semigroup $T_{\Omega,b}$ on $L_{\sigma}^p(\Omega)$.



(4)

Sketch of the proof. The proof is devided into several steps. First it is shown that $L_{\Omega,b}$ is the generator of an C_0 -semigroup $T_{\Omega,b}$ on $L^2_{\sigma}(\Omega)$. Then a-priori L^p -estimates for $T_{\Omega,b}$ are proved. Once we have shown this we can easily define a consistent family of semigroups $T_{\Omega,b}$ on $L^p_{\sigma}(\Omega)$ for $1 . In the last step the generator of <math>T_{\Omega,b}$ on $L^p_{\sigma}(\Omega)$ is identified to be $L_{\Omega,b}$.

We start by showing that $L_{\Omega,b}$ is the generator of a C_0 -semigroup on $L^2_{\sigma}(\Omega)$. Choose R > 0such that $\operatorname{supp} b \cup \Omega^c \subset B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$. We then set

$$D = \Omega \cap B_{R+5}(0),$$

$$K_1 = \{x \in \Omega : R < |x| < R+3\},$$

$$K_2 = \{x \in \Omega : R+2 < |x| < R+5\}.$$

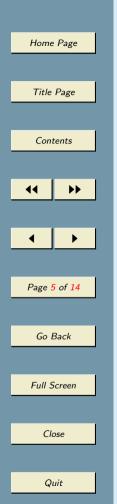
Denote by B_i for $i \in \{1, 2\}$ Bogovskii's operator (see [1], [9, Chapter III.3], [13]) associated to the domain K_i and choose cut-off functions $\varphi, \eta \in C^{\infty}(\mathbb{R}^n)$ such that $0 \leq \varphi, \eta \leq 1$ and

$$\varphi(x) = \begin{cases} 0, & |x| \le R+1, \\ 1, & |x| \ge R+2, \end{cases} \text{ and } \eta(x) = \begin{cases} 1, & |x| \le R+3, \\ 0, & |x| \ge R+4. \end{cases}$$

For $f \in L^p_{\sigma}(\Omega)$ we denote by f^R the extension of f by 0 to all of \mathbb{R}^n . Then, since $C^{\infty}_{c,\sigma}(\Omega)$ is dense in $L^p_{\sigma}(\Omega)$, $f^R \in L^p_{\sigma}(\mathbb{R}^n)$. Furthermore, we set $f^D = \eta f - B_2((\nabla \eta)f)$. Since $\int_{K_2} (\nabla \eta)f = 0$ it follows from [9, Chapter III.3] that $f^D \in L^p_{\sigma}(D)$.

By the perturbation theorem for analytic semigroups there exists $\omega_1 \geq 0$ such that for $\lambda > \omega_1$ there exist functions u_{λ}^D and p_{λ}^D satisfying the equations

$$\begin{aligned} &(\lambda - \mathcal{L}_b)u_{\lambda}^D + \nabla p_{\lambda}^D &= f^D, & \text{ in } D \times \mathbb{R}_+, \\ & \nabla \cdot u_{\lambda}^D &= 0, & \text{ in } D \times \mathbb{R}_+, \\ & u_{\lambda}^D &= 0, & \text{ on } \partial D \times \mathbb{R}_+. \end{aligned}$$



Moreover, by [14, Lemma 3.3 and Prop. 3.4], there exists $\omega_2 \ge 0$ such that for $\lambda > \omega_2$ there exists a function u_{λ}^R satisfying

(5)

$$\begin{aligned} &(\lambda - \mathcal{L}_0) u_{\lambda}^R &= f^R, & \text{ in } \mathbb{R}^n \times \mathbb{R}_+, \\ &\nabla \cdot u_{\lambda}^R &= 0, & \text{ in } \mathbb{R}^n \times \mathbb{R}_+. \end{aligned}$$

For $\lambda > \max\{\omega_1, \omega_2\}$ we now define the operator $U_{\lambda} : L^p_{\sigma}(\Omega) \to L^p_{\sigma}(\Omega)$ by

(6)
$$U_{\lambda}f = \varphi u_{\lambda}^{R} + (1-\varphi)u_{\lambda}^{D} + B_{1}(\nabla\varphi(u_{\lambda}^{R}-u_{\lambda}^{D})),$$

where u^R_λ and u^D_λ are the functions given above, depending of course on f. By definition, we have

(7)
$$U_{\lambda}f \in \{v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_{\sigma}(\Omega) : Mx \cdot \nabla v \in L^p_{\sigma}(\Omega)\}$$

Setting $P_{\lambda}f = (1 - \varphi)p_{\lambda}^{D}$, we verify that $(U_{\lambda}f, P_{\lambda}f)$ satisfies

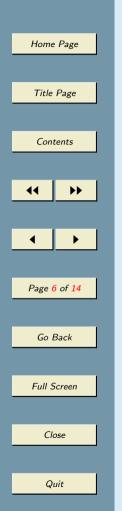
$$\begin{aligned} (\lambda - \mathcal{L}_b)U_{\lambda}f + \nabla P_{\lambda}f &= f + T_{\lambda}f, & \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot U_{\lambda}f &= 0, & \text{in } \Omega \times \mathbb{R}_+, \\ U_{\lambda}f &= 0, & \text{on } \partial\Omega \times \mathbb{R}_+ \end{aligned}$$

where T_{λ} is given by

$$T_{\lambda}f = -2(\nabla\varphi)\nabla(u_{\lambda}^{R} - u_{\lambda}^{D}) - (\Delta\varphi + Mx \cdot (\nabla\varphi))(u_{\lambda}^{R} - u_{\lambda}^{D}) + (\nabla\varphi)p_{\lambda}^{D} + (\lambda - \Delta - Mx \cdot \nabla + M)B_{1}((\nabla\varphi)(u_{\lambda}^{R} - u_{\lambda}^{D})).$$

It follows from [12, Lemma 4.4] that for $\alpha \in (0, \frac{1}{2p'})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, there exists a strongly continuous function $H: (0, \infty) \to \mathcal{L}(L^p_{\sigma}(\Omega))$ satisfying

(8)
$$||H(t)||_{\mathcal{L}(L^p_{\sigma}(\Omega))} \le Ct^{\alpha-1} \mathrm{e}^{\tilde{\omega}t}, \quad t > 0$$



(9

for some $\tilde{\omega} \geq 0$ and C > 0 such that $\lambda \mapsto P_{\Omega}T_{\lambda}$ is the Laplace Transform of H. We thus easily calculate

$$\|P_{\Omega}T_{\lambda}\|_{\mathcal{L}(L^{p}_{\sigma}(\Omega))} \leq C\lambda^{-\alpha}, \quad \lambda > \omega.$$

Therefore, $R_{\lambda} := U_{\lambda} \sum_{j=0}^{\infty} (P_{\Omega}T_{\lambda})^j$ exists for λ large enough and $(\lambda - L_b)R_{\lambda}f = f$ for $f \in L^2_{\sigma}(\Omega)$. Since $L_{\Omega,b}$ is dissipative in $L^2_{\sigma}(\Omega)$, $L_{\Omega,b}$ generates a C_0 -semigroup $T_{\Omega,b}$ on $L^2_{\sigma}(\Omega)$. Moreover, we have the representation

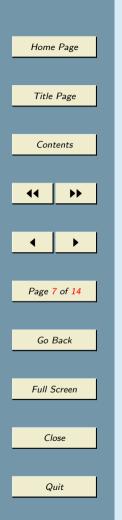
$$T_{\Omega,b}(t)f = \sum_{n=0}^{\infty} T_n(t)f, \quad f \in L^2_{\sigma}(\Omega),$$

where $T_n(t) := \int_0^t T_{n-1}(t-s)H(s) \, \mathrm{d}s$ for $n \in \mathbb{N}$ and

$$T_0(t) = \varphi T_R(t) f^R + (1 - \varphi) T_{D,b}(t) f^D + B_1((\nabla \varphi) (T_R(t) f^R - T_{D,b}(t) f^D)), \quad t \ge 0.$$

Here T_R denotes the semigroup on $L^p_{\sigma}(\mathbb{R}^n)$ generated by $L_{\mathbb{R}^n,0}$ and $T_{D,b}$ denotes the semigroup on $L^p_{\sigma}(D)$ generated by $L_{D,b}$. Note that $\lambda \mapsto U_{\lambda}$ is the Laplace Transform of T_0 . Since the right hand side of the representation (9) is well defined and exponentially bounded in $L^p_{\sigma}(\Omega)$ by [12, Lemma 4.6], we can define a family of consistent semigroups $T_{\Omega,b}$ on $L^p(\Omega)$ for $1 . Finally, the generator of <math>T_{\Omega,b}$ on $L^p(\Omega)$ is $L_{\Omega,b}$ which can be proved by using duality arguments (cf. [12, Theorem 4.1]). \Box

- **Remark 2.2.** (a) The semigroup $T_{\Omega,b}$ is not expected to be analytic since, by [16, Proposition 3.7], the semigroup $T_{\mathbb{R}^3}$ in \mathbb{R}^3 is not analytic.
- (b) As the cut-off function φ is used for the localization argument similarly to [15] the purpose of η is to ensure that $f_D \in L^p_{\sigma}(\Omega)$. This is essential to establish a decay property in λ for the pressure P^D_{λ} (cf. [12, Lemma 3.5]) and T_{λ} .
- (c) The crucial point for a-priori L^p -estimates for $T_{\Omega,b}$ on $L^2_{\sigma}(\Omega)$ is the existence of H satisfying (8).



Since L^p - L^q smoothing estimates for T_R and $T_{D,b}$ follow from [14, Lemma 3.3 and Prop. 3.4] and [12, Prop. 3.2], the representation of the semigroup $T_{\Omega,b}$ given by (9) and estimates for sums of convolutions of this type (cf. [12, Lemma 4.6]) yield the following proposition.

Proposition 2.3. Let $1 and let <math>\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$ boundary. Assume that tr M = 0 and $b \in C_c^{\infty}(\overline{\Omega})$. Then there exist constants $C > 0, \omega \ge 0$ such that for $f \in L^p_{\sigma}(\Omega)$

(a)
$$||T_{\Omega,b}(t)f||_{L^q_{\sigma}(\Omega)} \le Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{\omega t} ||f||_{L^p_{\sigma}(\Omega)}, \quad t > 0,$$

(b)
$$\|\nabla T_{\Omega,b}(t)f\|_{L^p(\Omega)} \le Ct^{-\frac{1}{2}} e^{\omega t} \|f\|_{L^p_{\sigma}(\Omega)}, \qquad t > 0$$

Moreover, for $f \in L^p_{\sigma}(\Omega)$

(1)

(1)

$$\|t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}T_{\Omega,b}(t)f\|_{L^{q}_{\sigma}(\Omega)}+\|t^{\frac{1}{2}}\nabla T_{\Omega,b}(t)f\|_{L^{p}(\Omega)}\to 0, \quad for \quad t\to 0$$

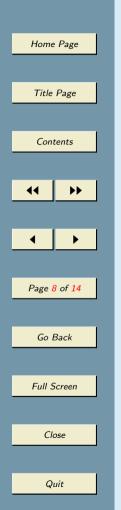
In order to construct a mild solution to (2) choose $\zeta \in C_c^{\infty}(\mathbb{R}^n)$ with $0 \leq \zeta \leq 1$ and $\zeta = 1$ near Γ . Further let $K \subset \mathbb{R}^n$ be a domain such that $\operatorname{supp} \nabla \zeta \subset K$. We then define $b : \mathbb{R}^n \to \mathbb{R}^n$ by

0)
$$b(x) := \zeta M x - B_K((\nabla \zeta) M x),$$

where B_K is Bogovskii's operator associated to the domain K. Then div b = 0 and b(x) = Mxon Γ . Setting u := v - b, it follows that u satisfies

1)

$$\begin{array}{rcl}
\partial_t u - \mathcal{L}_b u + \nabla p &= F & \text{in } \Omega \times (0,T), \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (0,T), \\
u &= 0 & \text{on } \Gamma \times (0,T), \\
u(x,0) &= u_0(x) - b(x), & \text{in } \Omega,
\end{array}$$



with $\nabla \cdot (u_0 - b) = 0$ in Ω and $F = -\Delta b - Mx \cdot \nabla b + Mb + b \cdot \nabla b$, provided u satisfies (2). Applying the Helmholtz projection P_{Ω} to (11), we may rewrite (11) as an evolution equation in $L^p_{\sigma}(\Omega)$:

(12)
$$\begin{aligned} u' - L_{\Omega,b}u + P_{\Omega}(u \cdot \nabla u) &= P_{\Omega}F, \quad 0 < t < T, \\ u(0) &= u_0 - b. \end{aligned}$$

Note that we need the compatibility condition $u_0(x) \cdot n = Mx \cdot n$ on $\partial\Omega$ to obtain $u_0 - b \in L^p_{\sigma}(\Omega)$. In the following, given $0 < T < \infty$, we call a function $u \in C([0,T); L^p_{\sigma}(\Omega))$ a mild solution of (12) if u satisfies the integral equation for 0 < t < T

$$u(t) = T_{\Omega,b}(t)(u_0 - b) - \int_0^t T_{\Omega,b}(t - s)P_\Omega(u \cdot \nabla u)(s) \,\mathrm{d}s + \int_0^t T_{\Omega,b}(t - s)P_\Omega F(s) \,\mathrm{d}s.$$

Then the main result of [12] is the following theorem.

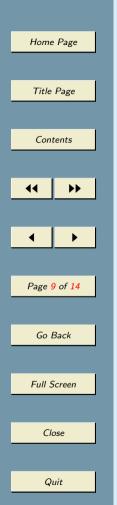
Theorem 2.4. Let $n \geq 2$, $n \leq p \leq q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$ -boundary. Assume that tr M = 0 and $b \in C_c^{\infty}(\overline{\Omega})$ and $u_0 - b \in L_{\sigma}^p(\Omega)$. Then there exist $T_0 > 0$ and a unique mild solution u of (12) such that

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} u(t) \in C\left(\left[0, T_0\right]; L^q_{\sigma}(\Omega)\right),$$

 $t \mapsto t^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{2}} \nabla u(t) \in C\left([0, T_0]; L^q(\Omega)\right).$

3. Strong solutions

In this section we construct strong solutions to problem (1) for $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and tr M = 0. The main difference to the method presented in the previous section is another change of



variables. Indeed, we construct a change of variables which coincides with a simple rotation in a neighborhood of the rotating body but it equals to the identity operator far away from the rotating body. More precisely, let $X(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ denote the time dependent vector field satisfying

$$\begin{array}{lll} \frac{\partial X}{\partial t}(y,t) &=& -b(X(y,t)), \quad y \in \mathbb{R}^n, \ t > 0, \\ X(y,0) &=& y, \qquad \qquad y \in \mathbb{R}^n, \end{array}$$

where b is as in (10). Similarly to [6, Lemma 3.2], the vector field $X(\cdot, t)$ is a C^{∞} -diffeomorphism form Ω onto $\Omega(t)$ and $X \in C^{\infty}([0, \infty) \times \mathbb{R}^n)$. Let us denote the inverse of $X(\cdot, t)$ by $Y(\cdot, t)$. Then, $Y \in C^{\infty}([0, \infty) \times \mathbb{R}^n)$. Moreover, it can be shown that for any T > 0 and $|\alpha| + k > 0$ there exists $C_{k,\alpha,T} > 0$ such that

(13)
$$\sup_{y \in \mathbb{R}^n, 0 \le t \le T} \left| \frac{\partial^k}{\partial t^k} \frac{\partial^\alpha}{\partial y^\alpha} X(y, t) \right| + \sup_{x \in \mathbb{R}^n, 0 \le t \le T} \left| \frac{\partial^k}{\partial t^k} \frac{\partial^\alpha}{\partial x^\alpha} Y(x, t) \right| \le C_{k, \alpha, T_0}$$

Setting

(14)

$$v(x,t) = J_X(Y(x,t),t)w(Y(x,t),t), \quad x \in \Omega, \ t \ge 0$$

where J_X denotes the Jacobian of $X(\cdot, t)$ and

$$p(x,t) = q(Y(x,t),t), \quad x \in \Omega, \ t \ge 0,$$

similarly to [6, Prop. 3.5] and [17], we obtain the following set of equations which are equivalent to (1).

$$\partial_t v - \mathcal{L}v + \mathcal{M}v + \mathcal{N}v + \mathcal{G}p = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot v = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \\ v(x,t) = \mathcal{M}x, \qquad \text{on } \Gamma \times \mathbb{R}_+, \\ v(x,0) = w_0(x), \qquad \text{in } \Omega.$$



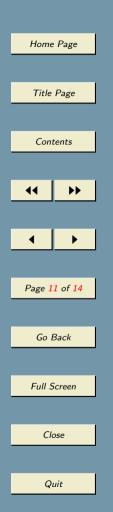
Here

$$\begin{aligned} (\mathcal{L}v)_i &= \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(g^{jk} \frac{\partial v_i}{\partial x_k} \right) + 2 \sum_{j,k,l=1}^n g^{kl} \Gamma_{jk}^i \frac{\partial v_j}{\partial x_l} \\ &+ \sum_{j,k,l=1}^n \left(\frac{\partial}{\partial x_k} (g^{kl} \Gamma_{jl}^i) + \sum_{m=1}^n g^{kl} \Gamma_{jl}^m \Gamma_{km}^i \right) v_j, \\ (\mathcal{N}v)_i &= \sum_{j=1}^n v_j \frac{\partial v_i}{\partial x_j} + \sum_{j,k=1}^n \Gamma_{jk}^j v_j v_k, \\ (\mathcal{M}v)_i &= \sum_{j=1}^n \frac{\partial X_j}{\partial t} \frac{\partial v_i}{\partial x_j} + \sum_{j,k=1}^n \left(\Gamma_{jk}^i \frac{\partial X_k}{\partial t} + \frac{\partial X_i}{\partial x_k} \frac{\partial^2 Y_k}{\partial x_j \partial t} \right) v_j \\ (\mathcal{G}p)_i &= \sum_{j=1}^n g^{ij} \frac{\partial p}{\partial x_j} \end{aligned}$$

with

$$g^{ij} = \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k} \frac{\partial X_j}{\partial y_k}, \quad g_{ij} = \sum_{k=1}^{n} \frac{\partial Y_k}{\partial x_i} \frac{\partial Y_k}{\partial x_j} \text{ and}$$
$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{ij}}{\partial x_l} \right).$$

The obvious advantage of this approach is that we do not have to deal with an unbounded drift term since all coefficients appearing in \mathcal{L} , \mathcal{N} , \mathcal{M} and \mathcal{G} are smooth and bounded on finite time intervals by (13). However, we have to consider a non-autonomous problem. Setting u = v - b, we obtain the following problem with homogeneous boundary conditions which is



equivalent to (14).

$$\begin{array}{rcl} \partial_t u - \mathcal{L}u + \mathcal{M}u + \mathcal{N}u + \mathcal{G}p &= F_b, & \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot u &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0, & \text{on } \Gamma \times \mathbb{R}_+, \\ u(x,0) &= w_0(x) - b(x), & \text{in } \Omega. \end{array}$$

Here,

(17)

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$$(\mathcal{B}u)_i = \sum_{j=1}^n \left(u_j \frac{\partial b_i}{\partial x_j} + b_j \frac{\partial u_i}{\partial x_j} \right) + 2 \sum_{j,k=1}^n \Gamma^i_{jk} u_j b_k, \quad F_b = \mathcal{L}b - \mathcal{M}b - \mathcal{N}b.$$

Since g^{ij} is smooth and $g^{ij}(\cdot,0) = \delta_{ij}$ by definition, it follows from (13) that

(16)
$$\|g^{ij}(\cdot,t) - \delta_{ij}\|_{L^{\infty}(\Omega)} \to 0, \quad t \to 0.$$

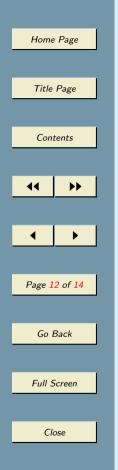
In other words, \mathcal{L} is a small perturbation of Δ and G is a small perturbation of ∇ for small times t. This motivates to write (15) in the following form.

$$\partial_t u - \Delta u + \nabla p = F(u, p), \qquad \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot u = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \\ u = 0, \qquad \text{on } \Gamma \times \mathbb{R}_+, \\ u(x, 0) = w_0(x) - b(x), \qquad \text{in } \Omega, \end{cases}$$

where $F(u, p) := (\mathcal{L} - \Delta)u - \mathcal{M}u - \mathcal{N}u + (\nabla - \mathcal{G})p - Bu + F_b$. We will use maximal L^p -regularity of the Stokes operator and a fixed point theorem to show the existence of a unique strong solution (u, p) of (15). More precisely, let

 $X_T^{p,q} := W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; D(A_q)) \times L^p(0,T; \widehat{W}^{1,p}(\Omega)),$

where $D(A_q) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_{\sigma}(\Omega)$ is the domain of the Stokes operator. Then, by maximal L^p -regularity of the Stokes operator, Hölder's inequality and Sobolev's embedding



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theorems $\Phi: X_T^{p,q} \to X_T^{p,q}, \Phi((\tilde{u}, \tilde{p})) := (u, p)$ where (u, p) is the unique solution of

 ∂_t

$$\begin{array}{rcl} u - \Delta u + \nabla p &=& F(\tilde{u}, \tilde{p}), & & & & & & & & \\ \nabla \cdot u &=& 0, & & & & & & & \\ u &=& 0, & & & & & & & \\ u(x, 0) &=& w_0(x) - b(x), & & & & & & & \\ \end{array}$$

is well-defined for $1 < p, q < \infty$ with $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$ and T > 0. Here, the restriction on p and q comes from the nonlinear term \mathcal{N} .

Finally, let $X_{T,\delta}^{p,q} := \{(u,p) \in X_T^{p,q} : ||(u,p) - (\hat{u},\hat{p})||_{X_T^{p,q}} \le \delta, u(0) = w_0 - b\}$ with $(\hat{u},\hat{p}) = \Phi(\Phi(0,0))$. Then by (16), Hölder's inequality and Sobolev's embedding theorems, it can be shown that for small enough $\delta > 0$ and T > 0, $\Psi|_{X_T^{p,q}}$ is a contraction.

We summarize our considerations in the next theorem which is proved in [7]. Note that the cases n = 2, 3 and p = q = 2 were already proved in [6].

Theorem 3.1. Let $1 < p, q < \infty$ such that $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$ -boundary. Assume that tr M = 0 and that $w_0 - b \in (L^q_{\sigma}(\Omega), D(A_q))_{1-\frac{1}{p}, p}$. Then there exist T > 0 and a unique solution $(u, p) \in X^{T,q}_T$ of problem (15).

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