A REMARK ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF VISCOUS HAMILTON-JACOBI EQUATIONS

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1. Introduction and main result

Consider the viscous Hamilton-Jacobi equation

$$(1.1)\qquad \begin{cases} u_t - \Delta u = |\nabla u|^q, & t > 0, \quad x \in \mathbb{R}^N \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $q > 0$ and $u_0 \in C_b(\mathbb{R}^N)$. It is known [6] that (1.1) admits a unique classical solution, global for $t > 0$.

The large time behavior of solutions of problem (1.1) has been studied recently by several authors, see [1]–[5], [7, 8] and the references therein. In particular it was shown by Gilding [5] that the large time limits

$$\underline{\omega} := \liminf_{t\to\infty} v(x,t) \leq \overline{\omega} := \limsup_{t\to\infty} v(x,t)$$

are independent of $x \in \mathbb{R}^N$. One of the main results of [5] is the following.

Theorem A. *Assume $0 < q < 2$ and $u_0 \in C_b(\mathbb{R}^N)$. Then $\underline{\omega} = \overline{\omega}$.*

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It was known that Theorem A fails for the linear heat equation and, moreover, Gilding observed that it fails for $q = 2$. The aim of this short note is to show that the assumption $q < 2$ in Theorem A is actually necessary.

Theorem 1. *Assume $q \geq 2$. Then there exists $u_0 \in C_b(\mathbb{R}^N)$ such that $\underline{\omega} < \overline{\omega}$.*

Proof. It is known (see e. g. [5, Proposition H1]) that there exists $v_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that the solution v of the heat equation

$$\begin{cases} v_t - \Delta v = 0, & t > 0, \quad x \in \mathbb{R}^N \\ v(0,x) = v_0(x), & \qquad\quad x \in \mathbb{R}^N \end{cases} \tag{1.2}$$

satisfies

$$\underline{\omega}^* := \liminf_{t\to\infty} v(x,t) < \overline{\omega}^* := \limsup_{t\to\infty} v(x,t), \quad x \in \mathbb{R}^N. \tag{1.3}$$

Moreover, upon replacing v_0 by $\lambda v_0 + \mu$ for suitable constants λ, μ, one can assume that

$$\underline{\omega}^* = 0 \tag{1.4}$$

and

$$\|v_0\|_\infty \leq 1/2, \qquad \|\nabla v_0\|_\infty \leq 1/2.$$

Now, set

$$u_0(x) := \mathrm{e}^{v_0(x)} - 1. \tag{1.5}$$

The function $w := \mathrm{e}^v - 1$ satisfies

$$\begin{cases} w_t - \Delta w = |\nabla w|^2, & t > 0, \quad x \in \mathbb{R}^N \\ w(0,x) = u_0(x), & \qquad\quad x \in \mathbb{R}^N. \end{cases} \tag{1.6}$$

Let u be the solution of (1.1) with initial data u_0 defined by (1.5). We note that

$$\|\nabla u_0\|_\infty \leq \|\nabla v_0\|_\infty \|\,\mathrm{e}^{v_0}\,\|_\infty \leq (1/2)\,\mathrm{e}^{1/2} < 1.$$

Since it is known (see e.g. [5, Lemma 2]) that $|\nabla u|$ satisfies a maximum principle, it follows that

$$|\nabla u| \leq \|\nabla u_0\|_\infty < 1 \quad \text{in } Q := (0,\infty)\times\mathbb{R}^N.$$

Due to $q \geq 2$, we deduce that

$$u_t - \Delta u = |\nabla u|^q \leq |\nabla u|^2 \quad \text{in } Q.$$

In view of (1.6), it follows from the comparison principle that

$$u \leq w = \mathrm{e}^v - 1 \quad \text{in } Q.$$

In particular, there holds

$$\underline{\omega} \leq \mathrm{e}^{\underline{\omega}^*} - 1 = 0. \tag{1.7}$$

But on the other hand, we have $u_0 \geq v_0$ due to (1.5). In view of (1.2), the maximum principle implies that $u \geq v$, hence

$$\overline{\omega} \geq \overline{\omega}^*. \tag{1.8}$$

Combining (1.3), (1.4), (1.7) and (1.8), we conclude that

$$\overline{\omega} \geq \overline{\omega}^* > \underline{\omega}^* = 0 \geq \underline{\omega}$$

and the proof of Theorem 1 is complete. □

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