Proceedings of Equadiff-11
2005, pp. 415–425
ISBN 978-80-227-2624-5

ANALYTIC SOLUTIONS FOR THE CLASSICAL TWO-PHASE STEFAN PROBLEM*

JAN PRÜSS , JÜRGEN SAAL†, AND GIERI SIMONETT

Abstract. A survey of the results obtained in [22] is presented. In [22] the authors prove the existence of a local-in-time solution for the classical two-phase Stefan problem that is analytic in space and time. The result is based on L_p maximal regularity, which is proved first, and the implicit function theorem.

Key words. Classical Stefan problem, free boundary problem, phase transition, maximal regularity, analytic solutions.

AMS subject classifications. Primary: 35R35, 35B65, 80A22. Secondary: 35K55, 35K20.

1. Introduction and main result. Consider a family $\Gamma = \{\Gamma(t) : t \geq 0\}$ of hypersurfaces in $\mathbb{R}^{n+1}$, where each individual hypersurface is assumed to be a graph over $\mathbb{R}^n$, that is, $\Gamma(t) = \mathrm{graph}(\rho(t))$ for some $\rho(t) : \mathbb{R}^n \to \mathbb{R}$. Moreover, let $\Omega^+(t)$ and $\Omega^-(t)$ denote the domain above and below $\Gamma(t)$, respectively, that is,

$$\Omega^{\pm}(t) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \pm y > \pm\rho(t, x)\}.$$

We set $\Omega(t) := \Omega^+(t) \cup \Omega^-(t)$ and consider the following problem: Given $\Gamma_0 = \mathrm{graph}(\rho_0)$ and $u_0 : \Omega(0) \to \mathbb{R}$, determine a family $\Gamma = \{\Gamma(t) : t \geq 0\}$ and a function $u : \bigcup_{t\geq 0}\big(\{t\} \times \Omega(t)\big) \to \mathbb{R}$ such that

$$\begin{cases} (\partial_t - c\Delta)u = 0 & \text{in } \bigcup_{t>0}(\{t\} \times \Omega(t)), \\ \gamma u = 0 & \text{on } \bigcup_{t>0}(\{t\} \times \Gamma(t)), \\ V = -[c\partial_\nu u] & \text{on } \bigcup_{t>0}(\{t\} \times \Gamma(t)), \\ u(0) = u_0 & \text{in } \Omega(0), \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{1.1}$$

where γ stands for the trace operator, V denotes the normal velocity of Γ, and ν is the unit normal vector, pointing into $\Omega^+(t)$. Given any function $v : \Omega(t) \to \mathbb{R}$, we write v^+ and v^- for the restriction of v to $\Omega^+(t)$ and $\Omega^-(t)$, respectively. Moreover, we admit the possibility of two different diffusion coefficients in $\Omega^{\pm}(t)$, i.e. c is given as

$$c(t, x, y) = \begin{cases} c_+, & (x, y) \in \Omega^+(t), \\ c_-, & (x, y) \in \Omega^-(t), \end{cases} \tag{1.2}$$

where c_+, c_- are strictly positive constants. Using this notation, let $[c\partial_\nu u]$ denote the jump of the normal derivatives of u across $\Gamma(t)$, that is,

$$[c\partial_\nu u] := c_+\gamma\partial_\nu u^+ - c_-\gamma\partial_\nu u^-.$$

*Supported by Deutsche Forschungsgemeinschaft (DFG).

†Vanderbilt University Nashville, Department of Mathematics, 1326 Stevenson Center, Nashville, TN 37240, USA saal@atlas.math.vanderbilt.edu

Of course, u_0 is a given initial value for u and Γ_0 describes the initial position of Γ.

Problem (1.1) is called the *classical two-phase Stefan problem* which is a model for phase transitions in liquid-solid systems and accounts for heat diffusion and exchange of latent heat in a homogeneous medium. In a typical physical situation the domain Ω is occupied by a liquid and a solid phase, say water and ice, that are separated by the interface Γ. Due to melting or freezing, the corresponding regions occupied by water and ice will change and, consequently, the interface Γ will also change its position and shape, which leads to the free boundary problem (1.1).

In the classical Stefan problem one assumes that the temperatures u^+ and u^- coincide at the interface Γ (where the two phases are in contact), that is, one requires

$$u^+ = u^- = 0 \quad \text{on} \quad \Gamma, \tag{1.3}$$

where 0 is the melting temperature.

The Stefan problem has been studied in the mathematical literature for over a century, see [23, 20] and [26, pp. 117–120] for a historic account, and has attracted the attention of many prominent mathematicians, see e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24].

To formulate our main result, let $W_p^s(\mathbb{R}^n)$, $s \geq 0$, $p \in (1,\infty)$, denote the Sobolev-Slobodeckij spaces, cf. [25]. Then we have

Theorem 1.1. *Let $p > n+3$. Then there is a number $\eta > 0$ such that the following holds: Given $(u_0, \rho_0) \in W_p^{2-2/p}(\Omega(0)) \times W_p^{2-2/p}(\mathbb{R}^n)$ with*

$$\gamma u_0^\pm = 0, \quad \pm u_0^\pm > 0 \quad \text{on} \quad \Omega^\pm(0), \quad \alpha_\pm := \partial_\nu u_0^\pm(0, \rho_0(0)) > 0, \tag{1.4}$$

and

$$\|\rho_0\|_{\mathrm{BUC}^1(\mathbb{R}^n)} + \left\|\partial_\nu u_0^\pm - \alpha_\pm\right\|_{\mathrm{BUC}(\Gamma_0)} \leq \eta, \tag{1.5}$$

there exists $T = T(u_0, \rho_0)$ and a unique solution (u, Γ), where $\Gamma(t) = \operatorname{graph}(\rho(t))$, for the Stefan problem (1.1) that is analytic in space and time. More precisely, we have that

$$M = \bigcup_{t \in (0,T)} (\{t\} \times \Gamma(t)) \quad \text{is a real analytic manifold}$$

and that $u^\pm \in C^\omega(\overline{\Omega}_T^\pm, \mathbb{R})$, with $\overline{\Omega}_T^\pm := \{(t,(x,y)) \in (0,T) \times \mathbb{R}^{n+1} : (x,y) \in \overline{\Omega}^\pm(t)\}$.

In the subsequent sections of this note we will give an outline of the proof of this result presented in [22]. In Section 2 we first transform (1.1) into a quasilinear problem in a fixed domain consisting of the union of two halfspaces. Then, in Section 3 we will provide the maximal regularity of a suitable linearization, which will be the basis for the treatment of the quasilinear system in the last two sections. The existence of a unique local-in-time solution by employing the contraction mapping principle is sketched in Section 4, whereas in Section 5 the analyticity of these solutions is proved by an application of the implicit function theorem.

2. The transformed problem. Let $T > 0$ and set $\dot{\mathbb{R}} := \mathbb{R}\backslash\{0\}$ and $\dot{\mathbb{R}}^{n+1} := \mathbb{R}^n \times \dot{\mathbb{R}}$. Analogously to the definition of $u^\pm : \Omega(t)^\pm \to \mathbb{R}$ for a function u on $\Omega(t)$, we denote v^+ and v^- for the restriction of a function $v : \dot{\mathbb{R}}^{n+1} \to \mathbb{R}$ to $\mathbb{R}_+^{n+1}$ and $\mathbb{R}_-^{n+1}$ respectively, where $\mathbb{R}_\pm^{n+1} := \{x \in \mathbb{R}^{n+1} : \pm x_{n+1} > 0\}$. We intent to transform the equations in $\Omega(t)$

into a problem in $\dot{\mathbb{R}}^{n+1}$. For this purpose we define

$$\Theta : (0,T) \times \dot{\mathbb{R}}^{n+1} \to Q_T := \bigcup_{t\in(0,T)} \{t\} \times \Omega(t),$$
$$\Theta(t,x,y) := (t,x,y+\rho_E(t,x,y)),$$

where $\Gamma(t) = \mathrm{graph}(\rho(t))$ as defined in the last section and ρ_E is a suitable extension of ρ to $(0,T) \times \dot{\mathbb{R}}^{n+1}$. We denote by

$$u = \Theta_* v = v \circ \Theta^{-1} \quad \text{and} \quad v = \Theta^* u = u \circ \Theta$$

the push-forward and pull-back respectively, and set $J = (0,T)$. Then it can be shown that (1.1) is formally equivalent to the system

$$\begin{cases} \partial_t v - c\Delta v = F(v,\rho_E), & \text{in } J \times \dot{\mathbb{R}}^{n+1}, \\ \gamma v^{\pm} = 0, & \text{on } J \times \mathbb{R}^n, \\ \partial_t \rho + [c\gamma\partial_y(v - a\rho_E)] = H(v,\rho_E), & \text{on } J \times \mathbb{R}^n, \\ v(0) = v_0, & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \rho(0) = \rho_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{2.1}$$

and we will require that the function ρ_E satisfies the equation

$$\begin{cases} (\partial_t - c\Delta)\rho_E = 0 & \text{in } J \times \dot{\mathbb{R}}^{n+1}, \\ \gamma\rho_E^{\pm} = \rho & \text{on } J \times \mathbb{R}^n, \\ \rho_E(0) = e^{-|y|(1-\Delta_x)^{\frac{1}{2}}}\rho_0 & \text{in } \dot{\mathbb{R}}^{n+1}. \end{cases} \tag{2.2}$$

Then the nonlinearities F and H are given by

$$\begin{aligned} F(v,\rho_E) &= c\left(\frac{1+|\nabla_x\rho_E|^2}{(1+\partial_y\rho_E)^2} - 1\right)\partial_y^2 v - c\frac{2\langle\nabla_x\rho_E|\nabla_x\partial_y v\rangle}{1+\partial_y\rho_E} \\ &\quad - c\left[\left(\frac{1+|\nabla_x\rho_E|^2}{(1+\partial_y\rho_E)^2} - 1\right)\partial_y^2\rho_E - \frac{2\langle\nabla_x\rho_E|\nabla_x\partial_y\rho_E\rangle}{1+\partial_y\rho_E}\right]\frac{\partial_y v}{1+\partial_y\rho_E} \end{aligned}$$

and

$$H(v,\rho_E) = H_+(v,\rho_E) - H_-(v,\rho_E) \tag{2.3}$$

with

$$H_{\pm}(v,\rho_E) = c_{\pm}\left[\left(1 - \frac{1+|\gamma\nabla_x\rho_E^{\pm}|^2}{1+\gamma\partial_y\rho_E^{\pm}}\right)\gamma\partial_y v^{\pm} - a_{\pm}\gamma\partial_y\rho_E^{\pm}\right]. \tag{2.4}$$

Here $\langle\cdot|\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$, and ∇_x the gradient with respect to x. Furthermore, observe that in slight abuse of notation we also denote the pull-back $\Theta^* c$ of the diffusion coefficient c introduced in (1.2) by c, that is, we set

$$c(x,y) = \begin{cases} c_+, & y > 0, \\ c_-, & y < 0, \end{cases} \tag{2.5}$$

whereas

$$a(x,y)=\begin{cases} a_+, & y>0,\\ a_-, & y<0,\end{cases} \qquad \text{with} \quad a_\pm := \frac{\partial_y v_0^\pm(0,0)}{1+\partial_y \rho_E^\pm(0,0,0)}. \tag{2.6}$$

Observe that $\alpha_\pm > 0$, given in (1.4), implies that also

$$a_\pm = \frac{\partial_y v_0^\pm(0,0)}{1+\partial_y \rho_E^\pm(0,0,0)} = \left(1+|\nabla_x \rho_0(0)|^2\right)^{-1/2} \partial_\nu u_0^\pm(0,\rho_0(0)) > 0.$$

If F and H are replaced by functions belonging to suitable function spaces, then system (2.1)–(2.2) represents the linearization admitting maximal regularity as will be proved in the next section. Note that the additional term '$a\rho_E$' appearing in the linearization of the Stefan condition

$$\begin{aligned}[][c\gamma\partial_y(v-a\rho_E)] &= c_+\gamma\partial_y(v-a\rho_E)^+ - c_-\gamma\partial_y(v-a\rho_E)^- \\ &= c_+\gamma\partial_y(v^+-a_+\rho_E^+) - c_-\gamma\partial_y(v^- - a_-\rho_E^-).\end{aligned}$$

is necessary in order to get sufficient regularity for the function ρ.

3. Maximal regularity for the linearized problem. First let us introduce suitable function spaces. Let $\Omega \subseteq \mathbb{R}^n$ be open and X be an arbitrary Banach space. By $L_p(\Omega;X)$ and $H_p^s(\Omega;X)$, for $1\le p\le\infty$, $s\in\mathbb{R}$, we denote the X-valued Lebegue space and the Bessel potential space of order s, respectively. We will also frequently make use of the Sobolev-Slobodeckij spaces $W_p^s(\Omega;X)$, $1\le p<\infty$, $s\in\mathbb{R}\setminus Z$, with norm

$$\|g\|_{W_p^s(\Omega;X)} = \|g\|_{H_p^{[s]}(\Omega;X)} + \left(\int_\Omega\int_\Omega \frac{\|g(x)-g(y)\|^p_{L_p(\mathbb{R}^n)}}{|x-y|^{n+(s-[s])p}}\mathrm{d}x\mathrm{d}y\right)^{1/p},$$

where $[s]$ denotes the largest integer smaller than s. Let $T\in(0,\infty]$ and $J=(0,T)$. We set

$$ {}_0W_p^s(J;X) := \begin{cases} \{u\in W_p^s(J;X) : u(0)=u'(0)=\ldots=u^{(k)}(0)=0\}, \\ \text{if} \quad k+\frac1p < s < k+1+\frac1p,\ k\in N\cup\{0\}, \\ W_p^s(J;X), \quad \text{if} \quad s<\frac1p. \end{cases}$$

The spaces ${}_0H_p^s(J;X)$ are defined analogously.

In this section we consider the linearized two-phase problem

$$\begin{cases} (\partial_t - c\Delta)v &= f & \text{in } J\times\dot{\mathbb{R}}^{n+1},\\ \gamma v &= 0 & \text{on } J\times\mathbb{R}^n,\\ \partial_t\rho + [c\gamma\partial_y(v-a\rho_E)] &= h & \text{on } J\times\mathbb{R}^n,\\ v(0) &= v_0 & \text{in } \dot{\mathbb{R}}^{n+1},\\ \rho(0) &= \rho_0 & \text{in } \mathbb{R}^n,\end{cases} \tag{3.1}$$

with c, a as defined in (2.5) and (2.6). In the following, we will always assume that the function ρ_E satisfies the equation

$$\begin{cases} (\partial_t - c\Delta)\rho_E &= 0 & \text{in } J\times\dot{\mathbb{R}}^{n+1},\\ \gamma\rho_E^\pm &= \rho & \text{on } J\times\mathbb{R}^n,\\ \rho_E(0) &= e^{-|y|(1-\Delta_x)^{\frac12}}\rho_0 & \text{in } \dot{\mathbb{R}}^{n+1}.\end{cases} \tag{3.2}$$

Remarks

(a) (3.1)–(3.2) constitutes a coupled system of equations, with the functions (v,ρ,ρ_E) to be determined. We will in the sequel often just refer to a solution (v,ρ) of (3.1) with the understanding that the function ρ_E also has to be determined.

(b) Suppose $\rho\in W_p^{1-1/2p}(J;L_p(\mathbb{R}^n))\cap L_p(J;W_p^{2-1/p}(\mathbb{R}^n))$ and $\rho_0\in W_p^{2-2/p}(\mathbb{R}^n)$ is given. Then the diffusion equation (3.2) admits a unique solution

$$\rho_E\in H_p^1(J;L^p(\dot{\mathbb{R}}^{n+1}))\cap L_p(J;H_p^2(\dot{\mathbb{R}}^{n+1})).$$

(c) The solution $\rho_E(t,\cdot)$ of equation (3.2) provides an extension of $\rho(t,\cdot)$ to $\dot{\mathbb{R}}^{n+1}$. We should remark that there are many possibilities to define such an extension. The chosen one is the most convenient one for our purposes.

The main result in this section is

Theorem 3.1. *Let $3<p<\infty$, $T\in(0,\infty)$, $J=(0,T)$.*

(i) *There exists a unique solution (v,ρ,ρ_E) to (3.1)–(3.2)) with*

$$\begin{aligned} v&\in H_p^1(J;L^p(\dot{\mathbb{R}}^{n+1}))\cap L_p(J;H_p^2(\dot{\mathbb{R}}^{n+1}))\\ \rho&\in W_p^{3/2-1/2p}(J;L_p(\mathbb{R}^n))\cap H_p^1(J;W_p^{1-1/p}(\mathbb{R}^n))\cap L_p(J;W_p^{2-1/p}(\mathbb{R}^n)),\\ \rho_E&\in H_p^1(J;L^p(\dot{\mathbb{R}}^{n+1}))\cap L_p(J;H_p^2(\dot{\mathbb{R}}^{n+1}))\end{aligned}$$

if and only if the data satisfy

(a) $f\in L_p(J;L_p(\dot{\mathbb{R}}^{n+1}))$,
(b) $h\in W_p^{1/2-1/2p}(J;L_p(\mathbb{R}^n))\cap L_p(J;W_p^{1-1/p}(\mathbb{R}^n))$,
(c) $v_0\in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})$,
(d) $\rho_0\in W_p^{2-2/p}(\mathbb{R}^n)$,
(e) $\gamma v_0=0$.

(ii) *If $(h(0),v_0,\rho_0)=(0,0,0)$, then the norm of the solution operator*

$$S_T:(f,h)\mapsto(v,\rho,\rho_E) \tag{3.3}$$

is independent of the length of $J=(0,T)$ for any $T\le T_0$, with T_0 arbitrary, but fixed.

We split the outline of the proof of this result in several steps, and remark that the 'only if' part follows from the trace results in [8, Section 5].

(i) In the first step we reduce the problem to the case $(h(0),v_0,\rho_0)=(0,0,0)$. In fact, this can be done by constructing suitable extensions u_1[1] and η in the regularity classes of v and ρ such that $u_1(0)=v_0$ and

$$(\eta(0),\partial_t\eta(0)):=\left(\rho_0,h(0)-[c\gamma\partial_y(v_0-ae^{-|y|(1-\Delta_x)^{\frac12}}\rho_0)]\right)$$

(for the existence see [22]). Then, if η_E is the solution of (3.2), with ρ replaced by η, it follows that

$$(v-u_1,\rho-\eta,\rho_E-\eta_E)$$

[1] Actually, u_1 is chosen as the solution of the homogeneous heat equation in $\dot{\mathbb{R}}^{n+1}$ with initial value v_0.

solves (3.1) and (3.2) with right hand sides $(f, g, 0, 0)$ and $(0, \rho, 0)$ respectively, in the right classes and such that $g(0) = 0$.

(ii) It is also not difficult to see that by the shift $u \mapsto e^{-t}u$ and by the use of a suitable extension operator E that maps functions $f : J \to X$ into functions $Ef : \mathbb{R}_+ \to X$, it suffices to consider the problems

$$\begin{cases} (\partial_t + 1 - c\Delta)u &= f \quad \text{in } (0,\infty) \times \dot{\mathbb{R}}^{n+1}, \\ \gamma u^{\pm} &= 0 \quad \text{on } (0,\infty) \times \mathbb{R}^n, \\ (\partial_t + 1)\rho + [c\gamma\partial_y(u - a\rho_E)] &= h \quad \text{on } (0,\infty) \times \mathbb{R}^n, \\ u(0) &= 0 \quad \text{in } \dot{\mathbb{R}}^{n+1}, \\ \rho(0) &= 0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{3.4}$$

and

$$\begin{cases} (\partial_t + 1 - c\Delta)\rho_E &= 0 \quad \text{in } (0,\infty) \times \dot{\mathbb{R}}^{n+1} \\ \gamma\rho_E^{\pm} &= \rho \quad \text{on } (0,\infty) \times \mathbb{R}^n, \\ \rho_E(0) &= 0 \quad \text{in } \dot{\mathbb{R}}^{n+1}. \end{cases} \tag{3.5}$$

Applying the Fourier-Laplace transform in (t, x), denoted by $\hat{}$, to (3.4) and (3.5) this system can be solved explicitly to the result

$$\hat{u}^+(y) = \int_0^\infty k_+(y, s)\hat{f}^+(s)\mathrm{d}s, \qquad y > 0, \tag{3.6}$$

$$\hat{u}^-(y) = \int_{-\infty}^0 k_-(-y, -s)\hat{f}^-(s)\mathrm{d}s, \quad y < 0, \tag{3.7}$$

$$\widehat{\rho_E}(y) = \mathrm{e}^{-\frac{w}{\sqrt{c}}|y|}\hat{\rho}, \tag{3.8}$$

$$\hat{\rho} = \frac{1}{m}\left(\hat{h} - \int_0^\infty \mathrm{e}^{-\omega_+ s/\sqrt{c_+}}\hat{f}^+(s)\mathrm{d}s - \int_{-\infty}^0 \mathrm{e}^{\omega_- s/\sqrt{c_-}}\hat{f}^-(s)\mathrm{d}s\right). \tag{3.9}$$

Here we used the abbreviations

$$k_\pm(|y|, |s|) := \frac{1}{2\omega_\pm\sqrt{c_\pm}}(\mathrm{e}^{-\omega_\pm||y|-|s||/\sqrt{c_\pm}} - \mathrm{e}^{-\omega_\pm(|y|+|s|)/\sqrt{c_\pm}}),$$

$$\omega = \omega(\lambda, \xi, y) = \sqrt{\lambda + 1 + c(y)|\xi|^2},$$

$$\omega_\pm = \omega_\pm(\lambda, \xi) = \sqrt{\lambda + 1 + c_\pm|\xi|^2},$$

$$m = \lambda + 1 + a_+\sqrt{c_+}\omega_+ + a_-\sqrt{c_-}\omega_-.$$

(iii) The desired regularity for u follows by the fact that it solves the heat equation in $\dot{\mathbb{R}}^{n+1}$. The function $1/m$ represents the principal symbol of the linearization for the classical Stefan problem. We denote by $Op(1/m)$ the associated operator, i.e. $\widehat{Op(1/m)u} = (1/m)\hat{u}$. Note also that $f^\pm \in L_p(\mathbb{R}_+; L_p(\mathbb{R}_+^{n+1}))$ implies that the inverse Fourier-Laplace transform of $\int_0^\infty \mathrm{e}^{-\omega_+ s/\sqrt{c_+}}\hat{f}^+(s)\mathrm{d}s$ and $\int_{-\infty}^0 \mathrm{e}^{\omega_- s/\sqrt{c_-}}\hat{f}^-(s)\mathrm{d}s$ belongs to the space

$${}_0W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W^{1-1/p}(\mathbb{R}^n))$$

(see [8, pages 15–16]). Thus, if we can show that $Op(1/m)$ is an isomorphism between the right spaces, then the regularity for ρ and ρ_E is also clear.

LEMMA 3.2. *Let $1 < p < \infty$. Then $Op(1/m)$ maps the space*

$$ {}_0W_p^{1/2-1/2p}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W_p^{1-1/p}(\mathbb{R}^n)) $$

continuously into the space

$$ {}_0W_p^{3/2-1/2p}(\mathbb{R}_+; L_p(\mathbb{R}^n)) \cap {}_0H_p^1(\mathbb{R}_+; W_p^{1-1/p}(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W_p^{2-1/p}(\mathbb{R}^n)). $$

The proof of this Lemma is based on an abstract result of Kalton and Weis [13, Theorem 4.4]. It essentially follows by the facts that both, the Poisson operator $(-\Delta)^{1/2}$, corresponding to the symbol $|\xi|$, and the operator $\partial_t + 1$, corresponding to $\lambda + 1$, admit a bounded $\mathcal{H}^\infty$-calculus on ${}_0H_p^r(\mathbb{R}_+; \mathcal{K}_p^s(\mathbb{R}^n))$ with H^∞-angles $\phi^\infty_{(-\Delta)^{1/2}} = 0$ and $\phi^\infty_{\partial_t+1} = \pi/2$, respectively, where $1 < p < \infty$ and $r, s \in \mathbb{R}$. Here $\mathcal{K} \in \{H, W\}$, i.e. by $\mathcal{K}_p^s$ we mean either the space H_p^s or W_p^s. We refer to [22] for the details.

As a consequence we obtain that ρ and therefore also ρ_E possess the regularity we claimed in Theorem 3.1, and the proof of Theorem 3.1 is completed.

4. The two-phase problem, local existence. Let $\mathbf{E}_T = \mathbf{E}_T^1 \times \mathbf{E}_T^2$ denote the regularity class of the solution (v, ρ) and let $\mathbf{F}_T = \mathbf{F}_T^1 \times \mathbf{F}_T^2$ denote the class of the data (f, h). By ${}_0\mathbf{E}_T$, ${}_0\mathbf{F}_T$ we mean the corresponding spaces with zero time trace. It will be convenient to split the solution in a part with zero time trace plus a remaining part taking care of the non zero traces. For this purpose we employ Theorem 3.1, which gives us a solution (v^*, ρ^*) for the linear problem (3.1) with given data

$$ (f, h, v_0, \rho_0) = (0, h^*, v_0, \rho_0) \quad \text{where} \quad h^*(t) := \mathrm{e}^{t\Delta_x} H(v_0, w_0). $$

It is a consequence of the assumptions on the initial data that the data in the line above satsify the assumptions (a)–(e) of Theorem 3.1. Thus, $(v^*, \rho^*) \in \mathbf{E}_T$ is well-defined and it suffices to study the reduced nonlinear problem

$$ \left\{ \begin{array}{rcll} (\partial_t - c\Delta)\bar{v} & = & F_0(\bar{v}, \bar{w}) & \text{in } J \times \dot{\mathbb{R}}^{n+1}, \\ \gamma\bar{v}^{\,\pm} & = & 0 & \text{on } J \times \mathbb{R}^n, \\ \partial_t\bar{\rho} + [c\gamma\partial_y(\bar{v} - a\bar{w})] & = & H_0(\bar{v}, \bar{w}) & \text{on } J \times \mathbb{R}^n, \\ \bar{v}(0) & = & 0 & \text{in } \dot{\mathbb{R}}^{n+1}, \\ \bar{\rho}(0) & = & 0 & \text{in } \mathbb{R}^n, \end{array} \right. \tag{4.1} $$

with

$$ F_0(\bar{v}, \bar{w}) := F(\bar{v} + v^*, \bar{w} + w^*), \quad H_0(\bar{v}, \bar{w}) := H(\bar{v} + v^*, \bar{w} + w^*) - h^*, \tag{4.2} $$

where $\bar{w}$ and w^* are extensions of $\bar{\rho}$ and ρ^*, respectively, satisfying equation (3.2). Here we observe that

$$ H_0(\bar{v}, \bar{w}) \in {}_0\mathbf{F}_T^2 $$

for all functions $(\bar{v}, \bar{w}) \in {}_0\mathbf{E}_T^1 \times {}_0\mathbf{E}_T^1$ with $\|\partial_y(\bar{w} + w^*)\|_\infty \le 1/2$. Thanks to this and Theorem 3.1(ii), the reduced nonlinear problem (4.1) can now be rephrased as a fixed point equation

$$ (\bar{v}, \bar{\rho}) \;=\; K_0(\bar{v}, \bar{\rho}) := S_T\big(F_0(\bar{v}, \bar{w}), H_0(\bar{v}, \bar{w})\big) \quad \text{in } {}_0\mathbf{E}_T, \tag{4.3} $$

where S_T is the solution operator of the linear problem defined in (3.3).

The advantage of applying the fixed point argument in the zero trace space ${}_0\mathbf{E}_T$ lies in the fact that the embedding constant of the embedding

$$ {}_0\mathbf{E}_T \hookrightarrow {}_0\mathrm{BUC}(J;\mathrm{BUC}^1(\dot{\mathbb{R}}^{n+1})) $$

does not depend on the length of the time interval $J=(0,T)$. Moreover, according to Theorem 3.1(ii), the norm of the solution operator S_T is independent of T as well. This enables us to choose T as small as we wish for without having the constants blowing up.

In order to show that K_0 is a contraction, mapping a small Ball of radius r into itself, we have to provide suitable estimates of the nonlinearities F and H as defined in (2.3) and (2.3). By an inspection of the single terms appearing in the expressions of F and H we see that there are basically three different kind of terms:

- Terms that will become small by choosing r small,
- terms that will become small by choosing T small,
- terms that will become small by the assumptions on the initial data.

This allows us to apply the contraction mapping principle in order to deduce the following result.

THEOREM 4.1. *Fix $p>n+3$. Then there is a number $\eta>0$ such that the following holds: Given $(v_0,\rho_0)\in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1})\times W_p^{2-2/p}(\mathbb{R}^n)$ with*

$$ \gamma v_0^{\pm}=0, \quad \pm v_0^{\pm}>0 \text{ on } \mathbb{R}^{n+1}_{\pm}, \quad a_{\pm}>0, \qquad \text{and} \tag{4.4} $$

$$ \|\rho_0\|_{\mathrm{BUC}^1(\mathbb{R}^n)}+\left\|\frac{\gamma\partial_y v_0}{1+\gamma\partial_y w_0}-a\right\|_{\mathrm{BUC}(\mathbb{R}^n)}\le\eta, \qquad \text{where} \tag{4.5} $$

$$ w_0:=e^{-|y|(1-\Delta)^{\frac12}}\rho_0, \qquad a_{\pm}:=\frac{\gamma\partial_y v_0^{\pm}(0,0)}{1+\gamma\partial_y w_0^{\pm}(0,0)}, \tag{4.6} $$

there exists $T=T(v_0,\rho_0)$ and a unique solution $(v,\rho)\in\mathbf{E}_T$ for the nonlinear problem (2.1).

5. Analyticity. To prove the analyticity of the solutions to (2.1) we employ a scaling argument and the implicit function theorem. Roughly speaking, this means we consider the translated and dilated solution

$$ \tau_{\lambda,\mu}v(t,x,y):=v(\lambda t,x+t\mu,y), \qquad \tau_{\lambda,\mu}\rho(t,x):=\rho(\lambda t,x+t\mu) $$

for $(\lambda,\mu)\in(1-\delta,1+\delta)\times\mathbb{R}^n$ with $\delta>0$ sufficiently small, and show by an application of the implicit function theorem that the dependence of $\tau_{\lambda,\mu}v$ and $\tau_{\lambda,\mu}\rho$ on λ and μ is analytic. In order to apply this method to a quasilinear system such as the Stefan problem requires the following three ingredients:

(i) Maximal regularity for the linearization.

(ii) The nonlinearities are real analytic maps, that is in our situation

$$ (F,H)\in C^{\omega}(\mathbf{G}_T,\,\mathbf{F}_T) $$

for an appropriate open subset $\mathbf{G}_T\subseteq\mathbf{E}_T^1\times\mathbf{E}_T^1$.

(iii) The nonlinearities commute with translations in space and dilations in time, i.e.

$$ \tau_{\lambda,\mu}F=F\tau_{\lambda,\mu}, \qquad \tau_{\lambda,\mu}H=H\tau_{\lambda,\mu}. $$

In our situation (i) is an immediate consequence of Theorem 3.1. Condition (ii) follows essentially from an inspection of the representations (2.3) and (2.3) for F and H, respectively. Indeed, it can be shown that (ii) is satisfied for

$$\mathbf{G}_T = \{(v, w) \in \mathbf{E}_T^1 \times \mathbf{E}_T^1 : \ \|\partial_y w\|_\infty \leq 1/2\}.$$

On the other hand (iii) can be easily seen by these representations, since there does not appear a time derivative.

Here we also employ the splitting

$$(v, \rho) = (\bar{v}, \bar{\rho}) + (v^*, \rho^*)$$

with $(\bar{v}, \bar{\rho}) \in {}_0\mathbf{E}_T^1 \times_0 \mathbf{E}_T^2$ and (v^*, ρ^*) taking care of the non zero traces, as introduced in the previous section. We focus on the first summand and suppose that the analyticity of (v^*, ρ^*) is already proved, which, for instance, can be obtained as well by an application of the implicit function theorem.

Next, let $\Lambda \subseteq (1-\delta, 1+\delta) \times \mathbb{R}^n$ and ${}_0\mathbf{B}_T^1(0,r) \times {}_0\mathbf{B}_T^2(0,r) \subseteq {}_0\mathbf{E}_T^1 \times {}_0\mathbf{E}_T^2$, where ${}_0\mathbf{B}_T^1(0,r)$ denotes the ball with center 0 and radius r. Further, we introduce the nonlinear map

$$\Psi_0 \ : {}_0\mathbf{B}_T^1(0,r) \times {}_0\mathbf{B}_T^2(0,r) \times \Lambda \to {}_0\mathbf{F}_T$$

$$\Psi_0((u,\sigma),(\lambda,\mu)) := \begin{pmatrix} (\partial_t - \lambda c\Delta)u - F_{\lambda,\mu}(u,\sigma) \\ \partial_t \rho + \lambda[c\gamma \partial_y(u - \mathcal{T}(\lambda,\mu)\sigma))] - H_{\lambda,\mu}(u,\sigma) \end{pmatrix},$$

where

$$F_{\lambda,\mu}(u,\sigma) := \lambda F(u + \tau_{\lambda,\mu} v^*, \mathcal{T}(\lambda,\mu)\sigma + \tau_{\lambda,\mu} w^*) + (\mu|\nabla u),$$
$$H_{\lambda,\mu}(u,\sigma) := \lambda H(u + \tau_{\lambda,\mu} v^*, \mathcal{T}(\lambda,\mu)\sigma + \tau_{\lambda,\mu} w^*) - \lambda\tau_{\lambda,\mu} h^* + (\mu|\nabla\sigma),$$

and $\mathcal{T}(\lambda,\mu)\sigma := \tau_{\lambda,\mu}(\tau_{1/\lambda,-\mu}\,\sigma)_E$. The analyticity of F and H implies that also

$$\Psi_0 \in C^\omega({}_0\mathbf{B}_T^1(0,r) \times {}_0\mathbf{B}_T^2(0,r) \times \Lambda, {}_0\mathbf{F}_T).$$

It readily follows that, if $(\bar{v}, \bar{\rho})$ solves (2.1) then

$$(u,\sigma) = (\tau_{\lambda,\mu}\bar{v}, \tau_{\lambda,\mu}\bar{\rho})$$

satisfies $\Psi_0((u,\sigma),(\lambda,\mu)) = 0$. Therefore, by utilizing the results of the last section it can be shown that for $r, \delta, T > 0$ small enough Ψ_0 is well defined. It turns out that the Fréchet derivative of Ψ_0 with respect to $(\bar{v}, \bar{\rho})$ at $(\lambda,\mu) = (1,0)$ is given by

$$D_1\Psi_0((\bar{v},\bar{\rho}),(1,0))[\tilde{u},\tilde{\sigma}] = U[\tilde{u},\tilde{\sigma}] - (DF_0(\bar{v},\bar{\rho}_E), DH_0(\bar{v},\bar{\rho}_E))[\tilde{u},\tilde{\sigma}_E]$$

for $(\tilde{u},\tilde{\sigma}) \in {}_0\mathbf{E}_T$, where

$$U[\tilde{u},\tilde{\sigma}] := \big((\partial_t - c\Delta)\tilde{u}, \partial_t\tilde{\sigma} + [c\gamma\partial_y(\tilde{u} - a\tilde{\sigma}_{{}_E})]\big)$$

and F_0 and H_0 are defined in (4.2). The proof of the existence also shows that the respective norms of the Fréchet derivatives of F_0 and H_0 are small for $(\bar{v}, \bar{\rho}_E) \in {}_0\mathbf{B}_T^1(0,r) \times$

${}_0\mathbf{B}^1_T(0,r)$ if we suppose that r, T are sufficiently small. This fact and since U represents exactly the linearization given in (3.1), Theorem 3.1 implies

$$D_1\Psi_0((\bar{v},\bar{\sigma}),(1,0)) \in \mathrm{Isom}({}_0\mathbf{E}_T, {}_0\mathbf{F}_T).$$

Thus the analyticity of Ψ_0 and the implicit function theorem imply the existence of an open neighborhood of $(1,0)$ in $(1-\delta, 1+\delta)\times\mathbb{R}^n$, again denoted by Λ, such that

$$[(\lambda,\mu)\mapsto(\tau_{\lambda,\mu}\bar{v},\tau_{\lambda,\mu}\bar{\sigma})] \in C^\omega(\Lambda,\ {}_0\mathbf{E}_T). \tag{5.1}$$

The analyticity of the solution (u,Γ) of the classical Stefan problem (1.1) is now essentially a consequence of (5.1). This completes the proof of Theorem 1.1.

Acknowledgments. The work of the second author was mainly developed during his stay at the Department of Mathematics at Vanderbilt University. He would like to express his gratitude to the Department, and especially to Gieri Simonett for giving him the opportunity of this stay, for the hospitality, and for always fruitful discussions during his stay.

REFERENCES

[1] I. Athanasopoulos, L. Caffarelli, and S. Salsa, *Regularity of the free boundary in parabolic phase-transition problems*, Acta Math. **176**(2) (1996), 245–282.
[2] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math., **139** (1977), 155–184.
[3] L. A. Caffarelli and L. C. Evans, *Continuity of the temperature in the two-phase Stefan problem*, Arch. Rational Mech. Anal., **81** (1983), 199–220.
[4] L. A. Caffarelli and A. Friedman, *Continuity of the temperature in the Stefan problem*, Indiana Univ. Math. J., **28** (1979), 53–70.
[5] E. DiBenedetto, *Regularity properties of the solution of an n-dimensional two-phase Stefan problem*, Boll. Un. Mat. Ital. Suppl., (1980), 129–152.
[6] G. Duvaut, *Résolution d'un problème de Stefan (fusion d'un bloc de glace à zéro degré)*, C.R. Acad. Sci. Paris Sér. A-B, **276** (1973), 1461–1463.
[7] J. Escher and G. Simonett, *Analyticity of the interface in a free boundary problem*, Math. Ann., **305** (1996), 439–459.
[8] J. Escher, J. Prüss, and G. Simonett, *Analytic solutions for a Stefan problem with Gibbs-Thomson correction*, J. Reine Angew. Math., (2003), 1–52.
[9] A. Friedman, *The Stefan problem in several space variables.* Trans. Amer. Math. Soc., **133** (1968), 51–87.
[10] A. Friedman, *Analyticity of the free boundary for the Stefan problem*, Arch. Rational Mech. Anal., **61** (1976), 97–125.
[11] A. Friedman and D. Kinderlehrer, *A one phase Stefan problem*, Indiana Univ. Math. J., **24** (1975), 1005–1035.
[12] E. I. Hanzawa, *Classical solutions of the Stefan problem*, Tôhoku Math. J., **33** (1981), 297–335.
[13] N. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann., **321** (2001), 319–345.
[14] S. L. Kamenomostskaja, *On Stefan's problem*, Math. Sbornik, **53** (1965), 485–514.
[15] D. Kinderlehrer and L. Nirenberg, *The smoothness of the free boundary in the one phase Stefan problem*, Comm. Pure Appl. Math., **31** (1978), 257–282.
[16] D. Kinderlehrer and L. Nirenberg, *Analyticity at the boundary of solutions of nonlinear second-order parabolic equations*, Comm. Pure Appl. Math., **31** (1978), 283–338.
[17] H. Koch, *Classical solutions to phase transition problems are smooth*, Comm. Partial Differential Equations, **23** (1998), 389–437.
[18] H. Matano, *Asymptotic behavior of the free boundaries arising in one phase Stefan problems in multi-dimensional spaces*, Lecture Notes in Num. Appl. Anal., **5** Kinokuniya, Tokyo, 1982, 133–151.
[19] A. M. Meirmanov, *On the classical solution of the multidimensional Stefan problem for quasilinear parabolic equations*, Math. Sb., **112** (1980), 170–192.
[20] A. M. Meirmanov, *The Stefan Problem*, De Gruyter, Berlin, 1992.
[21] R. H. Nochetto, *A class of nondegenerate two-phase Stefan problems in several space variables*, Comm. Partial Differential Equations, **12** (1987), 21–45.

[22] J. Prüß, J. Saal, and G. Simonett, *Existence of analytic solutions for the classical Stefan problem*, preprint.

[23] L. I. Rubinstein, *The Stefan Problem*, Translations of Mathematical Monographs, Vol. 27, American Mathematical Society, Providence, R.I. 1971.

[24] V. A. Solonnikov and E. V. Frolova, *L_p-theory for the Stefan problem*, Journal of Mathematical Sciences, **99**(1) (2000), 989–1006.

[25] H. Triebel, *Theory of Function Spaces*, Volume 78 of Monographs in Mathematics. Birkhäuser, Basel, 1983.

[26] A. Visintin, *Models of Phase Transitions*. Progress in Nonlinear Differential Equations and Their Applications, Volume 28, Birkhäuser, Boston, 1996.