TWO SIMPLE PROOFS OF ANALYTICITY OF THE CENTRAL PATH IN LINEAR PROGRAMMING

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Abstract

Several papers have appeared recently establishing the analyticity of the interior point paths at the boundary point for both linear programs and linear complementarity problems. While the proofs for LP are large, proceeding from limiting properties of the corresponding derivatives, the proofs for LCP are very simple, consisting in an application of the implicit function theorem to a certain system of equations. Inspired by the approach for LCP, this paper gives two simple ways of proving the analyticity of the central path for LP. One follows the idea for LCP, the other consists in a proper partition of the system defining the central path.

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1. Introduction

Consider the following pair of dual linear programming problems

$$\text{(P)} \qquad \min\{\ c^T x \mid Ax = b,\ x \geq 0\ \},$$

$$\text{(D)} \qquad \max\{\ b^T y \mid A^T y + s = c, s \geq 0\ \},$$

where A is an $(m \times n)$ matrix, $\operatorname{rank}(A) = m < n$, $c, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$. We will restrict our attention to the problems satisfying the assumption

$$\text{(AS)} \qquad \text{there exist } x > 0, \quad s > 0 \text{ such that } Ax = b, \quad A^T y + s = c.$$

In the context of interior point methods for solving (P) and (D) it is important to study the properties of certain interior point paths. We recall here the definition and some properties of these paths, called ω- weighted central paths.

Given any fixed $\omega > 0$, $\omega \in \mathbb{R}^n$, define the following μ parametrized system

$$\begin{aligned} Ax = b, &\quad x \geq 0, \\ A^T y + s = c, &\quad s \geq 0, \\ Xs = \mu\omega, & \end{aligned} \tag{1}$$

where $X :=\operatorname{diag}(x)$ and $\mu \in \mathbb{R}$, $\mu \geq 0$. Note that for $\mu = 0$ the system (1) gives necessary and sufficient conditions for optimality of both (P) and (D). Under the assumption (AS) for any $\mu > 0$ there exists the unique solution $(x(\mu), y(\mu), s(\mu))$ of (1) such that $x(\mu) > 0$, $s(\mu) > 0$. The set of points $\{x(\mu), y(\mu), s(\mu)\}_{\mu > 0}$ is called the (ω- weighted) central path. Due to the rank condition for A we have a one-to-one correspondence between y and s in (1). This enables us to leave out $y(\mu)$ from the definition of the (ω- weighted) central path. Given any $\omega > 0$, we will study the properties of the (ω- weighted) central path under its natural parametrization given in (1). That is, we are interested in the properties of $(x(\mu), s(\mu))$ as the function of $\mu > 0$.

It is easy to see that the central paths have nice analytical properties. Actually, the function

$$G(x, y, s; \mu) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ Xs - \mu\omega \end{bmatrix}$$

is real analytic and its Jacobian with respect to (x, y, s) is non-singular at the points where none of the components of either x or s vanishes. Moreover, the central path satisfies

$$G(x(\mu), y(\mu), s(\mu), \mu) = 0 \tag{2}$$

for each $\mu > 0$. Thus by the implicit function theorem, the central path is analytic in $\mu > 0$. That is, it is infinitely differentiable and the Taylor series of $(x(\mu), s(\mu))$ for any $\mu_0 > 0$ converges to $(x(\mu), s(\mu))$ at a neighbourhood of μ_o. (More analytical properties of the central path for $\mu > 0$ were studied in Zhao and Zhu [17] and Wechs [15].)

The basic limiting property of the ω- weighted central path is that there exists the finite limit of $(x(\mu), s(\mu))$ as $\mu \downarrow 0$ and this limit value forms a strictly complementary solution of (P) and (D). For the proof of this assertion see e.g. McLinden [9], Güler at al. [6], or Jansen at al.[8]. Therefore, we can extend the definition domain of the central paths (as the function of $\mu > 0$) to the closed interval $[0, \infty)$ by

$$(x(0), s(0)) := \lim_{\mu \downarrow 0} (x(\mu), s(\mu)). \tag{3}$$

Now, the equation (2) holds even for $\mu = 0$, but the study of the analytical properties of the central paths becomes more complicated since some information contained in the system (1) vanishes at $\mu = 0$. In fact, the system (1) can have more solutions at $\mu = 0$ and the Jacobian, $G'(x, y, s; 0)$, at these solutions is singular. Consequently, the implicit function theorem can not be generally used directly to (2) at $\mu = 0$. Thus the effort was concentrated on the analysis of limiting properties of k-th derivatives $(x^{(k)}(\mu), s^{(k)}(\mu))$ as $\mu \downarrow 0$. The existence of such limits was established in Adler and Monteiro [1] and Witzgall at al.[16] for $k = 1$ and, in Güler [4] for any $k \geq 1$.

Recently, almost simultaneously several papers have appeared proving the analyticity of the central paths at the boundary (Halická [7], Wechs [15], Stoer and Wechs [12, 13], Stoer at al. [14]). While two of them deal with linear programming (LP) problems [7, 15], the others treat linear complementarity problems (LCP) where the linear programming problem is regarded as a special case. The proofs for LP by Halická [7] and Wechs [15] proceed from the investigation of the limits of explicit formulas for high-order derivatives at $\mu > 0$ and also establish the geometric growth of the derivatives at the boundary. Although these proofs for LP are constructive and insightful, they are rather technical and quite long. On the contrary, the proof by Stoer and Wechs [12] (see also [13]) for the monotone LCP is, surprisingly, very simple. It consists of an application of the implicit function theorem to a system of equations, the existence of which was properly deduced from the system of equations defining the central path. Let us note that the construction of that system is described in the follow-up paper Stoer at al.[14].

Inspired by the approach for LCP, this paper gives two simple ways of proving the analyticity of the central path for LP. The main idea of both proof techniques is common: to rewrite the system of equations describing the path, so that the corresponding Jacobian remains non-singular as $\mu \downarrow 0$. One proof technique follows the ideas for LCP from [12,13,14] and allows geometric interpretation. The other uses a partition of A, first described in [7]. Here, the system is rewritten in a different form and the resulting Jacobian exhibits some (skew-)symmetric properties. A more detailed comparison of both proof techniques is given at the end of Section 4.

2. Preliminaries

The main tool in treating the limiting properties is the concept of the optimal partition (B, N) of (P) and (D). We remind that the optimal partition is a partition of the index set $\{1, ..., n\}$ such that every strictly complementary optimal solution (x, s) of (P) and (D) has the following property: $x_i > 0,\ s_i = 0$ for $i \in B$ and $x_i = 0,\ s_i > 0$ for $i \in N$. For further details of the optimal partition concept see Goldman and Tucker [2], or Roos at al. [11].

We will use the notation x_B and x_N to refer to the restriction of any vector $x \in \mathbb{R}^n$ to the coordinate sets B and N, respectively. The submatrices A_B and A_N are defined in the same way and thus we can write $A = (A_B, A_N)$. By $|B|$ and $|N|$ we denote the cardinality of the index sets B and N, respectively. In this notation we can rewrite the system (1) into

$$
\begin{aligned}
A_B x_B + A_N x_N &= b, \\
A_B^T y + s_B &= c_B, \\
A_N^T y + s_N &= c_N, \\
X_B s_B &= \mu \omega_B, \\
X_N s_N &= \mu \omega_N, \quad \forall \mu \geq 0.
\end{aligned} \tag{4}
$$

Since $(x(0), s(0))$ is a strictly complementary optimal solution of (P), (D) and (B, N) is the optimal partition, we have

$$
\begin{aligned}
x_B(0) > 0, \qquad & x_N(0) = 0, \\
s_B(0) = 0, \qquad & s_N(0) > 0.
\end{aligned} \tag{5}
$$

We now use the following trick also used by Stoer and Wechs [12,13] in the context of LCP's and define the new "tilde" variables

$$
\begin{aligned}
\tilde{s}_B &:= X_B^{-1} \omega_B \Rightarrow s_B = \mu \tilde{s}_B, \\
\tilde{x}_N &:= S_N^{-1} \omega_N \Rightarrow x_N := \mu \tilde{x}_N.
\end{aligned} \tag{6}
$$

Then, instead of $(x_B(\mu), x_N(\mu), s_B(\mu), s_N(\mu))$, we will study the corresponding "tilde" path $(x_B(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu), s_N(\mu))$ for $\mu \geq 0$. It is easy to see that for each $\mu \geq 0$ the "tilde" path is the solution to the system

$$A_B x_B + \mu A_N \tilde{x}_N = b, \tag{7a}$$

$$A_B^T y + \mu \tilde{s}_B = c_B, \tag{7b}$$

$$A_N^T y + s_N = c_N, \tag{7c}$$

$$X_B \tilde{s}_B = \omega_B, \tag{7d}$$

$$\tilde{X}_N s_N = \omega_N. \tag{7e}$$

More precisely, for $\mu > 0$ there is even the unique solution to (7) with the property $x_B > 0,\ s_N > 0$. Moreover, by (5) and (6), we have $\tilde{s}_B(0) > 0,\ \tilde{x}_N(0) > 0$.

The Jacobian of (7)

$$
J(x_B, y, s_N, \tilde{x}_N, \tilde{s}_B; \mu) = \begin{pmatrix}
A_B & 0 & 0 & \mu A_N & 0 \\
0 & A_B^T & 0 & 0 & \mu I_N \\
0 & A_N^T & I_N & 0 & 0 \\
\tilde{S}_B & 0 & 0 & 0 & X_B \\
0 & 0 & \tilde{X}_N & S_N & 0
\end{pmatrix}
$$

is non-singular at $\mu > 0$, however its non-singularity at $\mu = 0$ depends on A_B. It is easy to see that $J(x_B(0), y(0), s_N(0), \tilde{x}_N(0), \tilde{s}_B(0); 0)$ is non-singular if and only if $|B| = m$ and $\operatorname{rank}(A_B) = m$. Note that these conditions are equivalent to the uniqueness of the optimal solutions of (P) and (D) (see Güler at al.[5] and also Roos, Terlaky and Vial [10]).

3. The first approach

As mentioned above, Stoer and Wechs proved the analyticity of the interior-point paths for the monotone LCP in [12] and for the sufficient LCP in [13]. The main idea of their proof resides in adding some equations to the original system of equations for the path. Then, the enlarged (overdetermined) system of equations has the Jacobian with the full column rank and the existence of a subsystem with the non-singular Jacobian is ensured. Thus the analyticity follows by the application of the implicit function theorem. The study of this problem was completed in [14], where the authors describe explicitly how to find a selection of the enlarged system with a non-vanishing Jacobian at $\mu = 0$.

It is well known that LP problems can be formulated in the LCP form. So, the results about the analyticity of the central path obtained for LCP, are valid also for LP. As mentioned above, the analyticity of the central path was proven also directly for LP in [7, 15]. Thus, the value of the results [12, 13, 14] for LP is especially in the fact that the analyticity can be proved in a very simple manner and that the system of equations for the central path may be rewritten, so that the corresponding Jacobian is non-vanishing at $\mu = 0$. Obviously, the proof technique and the resulting system as described in [12,13,14] depend on the general form of LCP, so it is not clear what the proof and the resulting system would look like in terms of LP. Because of this, in this section the idea of the proof by Stoer and Wechs for LCP is applied directly to the LP problem (without the reformulation of LP to LCP). Moreover, while the procedure of Stoer and Wechs is described in the two different steps (first adding some equations to the original system, then the selection of the subsystem), here the result will be obtained in one step.

Denote

$$P := \begin{pmatrix} A_B & 0 \\ 0 & A_B^T \end{pmatrix}, \quad Q := \begin{pmatrix} A_N & 0 \\ 0 & I_B \end{pmatrix} \text{ and } q := \begin{pmatrix} b \\ c_B \end{pmatrix}.$$

Then (7a) and (7b) can be rewritten

$$P\begin{pmatrix} x_B \\ y \end{pmatrix} + \mu Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = q. \tag{8}$$

Let $\mathrm{rank}(P) = r < m + |B|$. Then there exists a non-singular $(m + |B|) \times (m + |B|)$ matrix W such that

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \text{ and } W_2 P = 0.$$

Here W_2 is an $(m + |B| - r) \times (m + |B|)$ matrix. It is easy to verify that

$$W_2 w = 0 \Longleftrightarrow \exists u: \quad w = Pu. \tag{9}$$

Actually, $W_2 P = 0$ implies $\mathrm{Col}(P) \subseteq \mathrm{Null}(W_2)$. However, since W is non-singular, $\mathrm{rank}(W_2) = m + |B| - r$, which implies $\dim(\mathrm{Null}(W_2)) = r$. Now, since $\mathrm{rank}(P) = r$, we get $\mathrm{Col}(P) = \mathrm{Null}(W_2)$, from which (9) follows.

Multiplying (8) by W from the left we obtain

$$W_1 P \begin{pmatrix} x_B \\ y \end{pmatrix} + \mu W_1 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = W_1 q \tag{10a}$$

$$\mu W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = W_2 q. \tag{10b}$$

From (10b) at $\mu = 0$ we get $W_2 q = 0$. Thus instead of (10b) we have

$$W_2 Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = 0. \tag{11}$$

Now, it is evident that for any $\mu \geq 0$ the "tilde"path is a solution of the system of equations (10a), (11) and (7cde).

Theorem 1. *For any $\mu \geq 0$ the "tilde"path $(x_B(\mu), \tilde{x}_N(\mu), y(\mu)\tilde{s}_B(\mu), s_N(\mu))$ is the solution of the system (10a), (11) and (7cde) of $2n + m$ equations and the Jacobian $J(x_B, y, s_N, \tilde{x}_N, \tilde{s}_B; \mu)$ of the system at the values of the path is non-singular.*

Proof. Denote $J(x_B(\mu), y(\mu), s_N(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu); \mu) =: J(\mu)$ It suffices to prove that if $J(0).dh = 0$ for some vector dh, then $dh = 0$.

Let $dh = (dx_B, dy, ds_N, d\tilde{x}_N, d\tilde{s}_B)$, then the equations of $J(0).dh = 0$ are

$$
\begin{aligned}
W_1 P \begin{pmatrix} dx_B \\ dy \end{pmatrix} &= 0, && (12a)\\
W_2 Q \begin{pmatrix} d\tilde{x}_N \\ d\tilde{s}_B \end{pmatrix} &= 0, && (12b)\\
A_N^T dy + ds_N &= 0, && (12c)\\
\tilde{S}_B(0) dx_B + X_B(0) d\tilde{s}_B &= 0, && (12d)\\
\tilde{X}_N(0) ds_N + S_N(0) d\tilde{x}_N &= 0, && (12e)
\end{aligned}
$$

First, we consider (12a). Since $W_2 P = 0$, we have $WP \begin{pmatrix} dx_B \\ dy \end{pmatrix} = 0$ and due to the non-singularity of W we obtain

$$P \begin{pmatrix} dx_B \\ dy \end{pmatrix} = 0. \tag{13a}$$

Further, we consider (12b). According to (9) applied to (12b), there exist u_1 and u_2 such that

$$P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + Q \begin{pmatrix} d\tilde{x}_n \\ d\tilde{s}_B \end{pmatrix} = 0. \tag{13b}$$

Moreover, for u_2 there exists u_3 such that

$$A_N^T u_2 + I_N u_3 = 0. \tag{14}$$

From (13a) and (13b) we obtain

$$P \begin{pmatrix} u_1 + \lambda dx_B \\ u_2 + \lambda dy \end{pmatrix} + Q \begin{pmatrix} d\tilde{x}_n \\ d\tilde{s}_B \end{pmatrix} = 0, \quad \forall \lambda \in \mathbb{R}. \tag{15}$$

Thus, from (15), (12c) and (14) it follows

$$
\begin{aligned}
A_B(u_1 + \lambda dx_B) + A_N d\tilde{x}_N &= 0\\
A_B^T(u_2 + \lambda dy) + I_B d\tilde{s}_B &= 0\\
A_N^T(u_2 + \lambda dy) + I_N(u_3 + \lambda ds_N) &= 0 \quad \forall \lambda \in \mathbb{R}.
\end{aligned} \tag{16}
$$

Since the vectors $\begin{pmatrix} u_1 + \lambda dx_B \\ d\tilde{x}_N \end{pmatrix}$ and $\begin{pmatrix} d\tilde{s}_B \\ u_3 + \lambda ds_N \end{pmatrix}$ are orthogonal for each λ, we have

$$dx_B^T d\tilde{s}_B + d\tilde{x}_N^T ds_N = 0. \tag{17}$$

From (12d) and (12e) we now obtain

$$d\tilde{s}_B = -X_B(0)^{-1}\tilde{S}_B(0)dx_B \tag{18a}$$

and

$$ds_N = -\tilde{X}_N(0)^{-1}S_N(0)d\tilde{x}_N. \tag{18b}$$

Substituting (18a) and (18b) into (17) yields

$$-dx_B^T X_B(0)^{-1}\tilde{S}_B(0)dx_B - d\tilde{x}_N^T\tilde{X}_N(0)^{-1}S_N(0)d\tilde{x}_N = 0$$

and since $X_B(0)^{-1}\tilde{S}_B(0)$ and $\tilde{X}_N(0)^{-1}S_N(0)$ are positive definite, we obtain $dx_B = 0$ and $d\tilde{x}_N = 0$. Consequently, (18a) and (18b) yield $d\tilde{s}_B = 0$ and $ds_N = 0$.

Finally, by (12c) and $ds_N = 0$, we get $A_N^T dy = 0$ and from (13a) we obtain $A_B^T dy = 0$. Now, the full row rank of A yields $dy = 0$. Thus $dh = 0$ and the theorem is proved. □

Now, let us give a geometric interpretation of the equations developed above. It is known that the central path at each $\mu > 0$ can be interpreted as the analytic center of a certain set described by Gonzaga [3] as the constant-cost slice of the feasible set. It is obvious that this property is contained in the original system (1) while $\mu > 0$. It is also known that the limit of the central path as $\mu \downarrow 0$ is the analytic center of the set of optimal solutions (see e.g. [1], or [8]). Unfortunately, this information for $\mu = 0$ is lost in the system (1) when $\mu = 0$.

In the proof presented above, the lost information is identified as (10b) and it is replaced by (11). As will be shown below, (11) really expresses the fact that at each $\mu \geq 0$ the ω-weighted central path $(x(\mu), y(\mu), s(\mu))$ is the ω-weighted analytic center of a certain set $P(\mu)$.

The set of optimal solutions can be described by

$$P(0) = \{(x_B, x_N, y, s_B, s_N) |\ x_N = 0,\ s_B = 0,$$

$$x_B \geq 0,\ s_N \geq 0,\ A_B x_B = b,\ A_B^T y = c_B,\ A_N^T y + s_N = c_N\}.$$

The ω-weighted analytic center of $P(0)$ is the unique solution of the problem

$$\max\{\sum_{i\in B}\omega_i \ln x_i + \sum_{i\in N}\omega_i \ln s_i |\quad x_B > 0,\ s_N > 0,\ A_B x_B = b,\ A_B^T y = c_B, A_N^T y + s_N = c_N\}.$$

and the necessary and sufficient conditions for the optimal solution of this problem are

$$\exists \quad v_1, v_2 : \quad X_B^{-1}\omega_B - A_B^T v_2 = 0, \quad -A_B v_1 + A_N S_N^{-1}\omega_N = 0$$

or, in the block-matrix formulation

$$\exists \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} : \quad \begin{pmatrix} A_N & 0 \\ 0 & I_B \end{pmatrix} \begin{pmatrix} S_N^{-1}\omega_N \\ X_B^{-1}\omega_B \end{pmatrix} = \begin{pmatrix} A_B & 0 \\ 0 & A_B^T \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{19a}$$

Now, (6) and (19a) yield

$$\exists \quad v_1, v_2 : \quad Q \begin{pmatrix} \tilde{x}_N \\ \tilde{s}_B \end{pmatrix} = P \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{19b}$$

According to (9), (19b) is equivalent to (11) at $\mu = 0$.

At $\mu > 0$ for fixed $x_N(\mu)$ and $s_B(\mu)$ taken from the central path, we define $P(\mu)$=

$$\{(x_B, y, s_N) | x_B \geq 0, s_N \geq 0, A_B x_B + A_N x_N(\mu) = b, A_B^T y + s_B(\mu) = c_B, A_N^T y + s_N = c_N\}$$

The necessary and sufficient conditions for the analytic center of the set $P(\mu)$ yield (19b) which is equivalent with (11).

Note that the analytic center of $P(0)$ is defined uniquely. Thus the interpretation of (11) presented above implies the uniqueness of the (positive) solutions of (10a), (11) and (7cde) at $\mu = 0$.

4. The second approach

We now present another procedure which yields the equations for the central path with non-vanishing Jacobian as well, but the form of the equations is different.

Let $rank(A_B) = r$. From the rank conditions on A and A_B one has that A can be partitioned in the form (by reordering the columns in A_B or in A_N, if necessary,)

$$A = [\, A_B \vdots A_N \,] =: \begin{bmatrix} A_{B11} & A_{B12} & \vdots & A_{N11} & A_{N12} \\ \underbrace{A_{B21}}_{r} & \underbrace{A_{B22}}_{|B|-r} & \vdots & \underbrace{A_{N21}}_{m-r} & \underbrace{A_{N22}}_{|N|-m+r} \end{bmatrix} \begin{matrix} \}r \\ \}m-r \end{matrix},$$

where

$$R := \begin{bmatrix} A_{B11} & A_{N11} \\ A_{B21} & A_{N21} \end{bmatrix}$$

is a nonsingular $m \times m$ matrix. Multiplying A by R^{-1} from the left we obtain

$$\bar{A} := R^{-1}A = R^{-1}[\,A_B \,\vdots\, A_N\,] =: [\,\bar{A}_B \,\vdots\, \bar{A}_N\,]$$

and also

$$\bar{A} = [\,\bar{A}_B \,\vdots\, \bar{A}_N\,] =: \begin{bmatrix} I_r & \bar{A}_{B12} & \vdots & 0_{r\times(m-r)} & \bar{A}_{N12} \\ 0_{(m-r)\times r} & \bar{A}_{B22} & \vdots & I_{m-r} & \bar{A}_{N22} \end{bmatrix}.$$

Note that $\bar{A}_{B22} = 0$, since rank(A_B) =rank$(\bar{A}_B) = r$ and thus

$$\bar{A}_B = \begin{bmatrix} I_r & \bar{A}_{B12} \\ 0 & 0 \end{bmatrix}, \qquad \bar{A}_N = \begin{bmatrix} 0 & \bar{A}_{N12} \\ I_{m-r} & \bar{A}_{N22} \end{bmatrix} \tag{20}$$

Define $\bar{b} := R^{-1}b$ and $\bar{y} := R^T y$. Using the corresponding partition of all vector variables the equations (7) can be written in the form

$$\bar{A}_B \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix} + \mu \bar{A}_N \begin{bmatrix} \tilde{x}_{N1} \\ \tilde{x}_{N2} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} \tag{21}$$

$$\bar{A}_B^T \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} + \mu \begin{bmatrix} \tilde{s}_{B1} \\ \tilde{s}_{B2} \end{bmatrix} = \begin{bmatrix} c_{B1} \\ c_{B2} \end{bmatrix} \tag{22}$$

$$\bar{A}_N^T \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} + \begin{bmatrix} s_{N1} \\ s_{N2} \end{bmatrix} = \begin{bmatrix} c_{N1} \\ c_{N2} \end{bmatrix} \tag{23}$$

$$X_{B1}\tilde{s}_{B1} = \omega_{B1} \tag{24}$$

$$X_{B2}\tilde{s}_{B2} = \omega_{B2} \tag{25}$$

$$\tilde{X}_{N1}s_{N1} = \omega_{N1} \tag{26}$$

$$\tilde{X}_{N2}s_{N2} = \omega_{N2}. \tag{27}$$

Substituting (20) into (21)-(23) yields

$$x_{B1} + \bar{A}_{B12}x_{B2} + \mu\bar{A}_{N12}\tilde{x}_{N2} = \bar{b}_1 \tag{28}$$

$$\mu\tilde{x}_{N1} + \mu\bar{A}_{N22}\tilde{x}_{N2} = \bar{b}_2 \tag{29}$$

$$\bar{y}_1 + \mu\tilde{s}_{B1} = c_{B1} \tag{30}$$

$$\bar{A}_{B12}^T\bar{y}_1 + \mu\tilde{s}_{B2} = c_{B2} \tag{31}$$

$$\bar{y}_2 + s_{N1} = c_{N1} \tag{32}$$

$$\bar{A}_{N12}^T\bar{y}_1 + \bar{A}_{N22}^T\bar{y}_2 + s_{N2} = c_{N2} \tag{33}$$

Here vectors with indices $B1$, $B2$, $N1$ and $N2$ are the vectors of dimensions r, $|B|-r$, $m-r$ and $|N|-m+r$, respectively. Since we consider (29) for any $\mu \geq 0$, $\bar{b}_2 = 0$ and so the equation can be divided by μ to give

$$\tilde{x}_{N1} + \bar{A}_{N22}\tilde{x}_{N2} = 0. \tag{29'}$$

The equations (30) and (32) can be written in the form

$$\bar{y}_1 = c_{B1} - \mu\tilde{s}_{B1} \tag{30'}$$

$$\bar{y}_2 = c_{N1} - s_{N1} \tag{32'}$$

and thus the variables $\bar{y}_1$ and $\bar{y}_2$ can be eliminated from (31) and (33) to obtain

$$\bar{A}_{B12}^T(c_{B1} - \mu\tilde{s}_{B1}) + \mu\tilde{s}_{B2} = c_{B2} \tag{31'}$$

$$\bar{A}_{N12}^T(c_{B1} - \mu\tilde{s}_{B1}) + \bar{A}_{N22}^T(c_{N1} - s_{N1}) + s_{N2} = c_{N2}. \tag{33'}$$

The equation (31') holds for any $\mu \geq 0$ and thus $\bar{A}_{B12}^T c_{B1} = c_{B2}$ and

$$-\bar{A}_{B12}^T\tilde{s}_{B1} + \tilde{s}_{B2} = 0. \tag{31''}$$

Now (28), (31"), (29'), (33') together with (24)- (27) form the full system of $2n$ equations for the $2n$ variables $x_{B1}, \tilde{s}_{B2}, \tilde{x}_{N1}, s_{N2}, \tilde{s}_{B1}, x_{B2}, s_{N1}, \tilde{x}_{N2}$.

The Jacobian $J(\mu)$ of this system with respect to the variables in the given order is

$$J(\mu) = \left(\begin{array}{cccc:cccc} I_r & 0 & 0 & 0 & 0 & \bar{A}_{B12} & 0 & \mu\bar{A}_{N12} \\ 0 & I_{|B|-r} & 0 & 0 & -\bar{A}_{B12}^T & 0 & 0 & 0 \\ 0 & 0 & I_{m-r} & 0 & 0 & 0 & 0 & \bar{A}_{N22} \\ 0 & 0 & 0 & I_{|N|-m+r} & -\mu\bar{A}_{N12}^T & 0 & -\bar{A}_{N22}^T & 0 \\ \hdashline \tilde{S}_{B1} & 0 & 0 & 0 & X_{B1} & 0 & 0 & 0 \\ 0 & X_{B2} & 0 & 0 & 0 & \tilde{S}_{B2} & 0 & 0 \\ 0 & 0 & S_{N1} & 0 & 0 & 0 & \tilde{X}_{N1} & 0 \\ 0 & 0 & 0 & \tilde{X}_{N2} & 0 & 0 & 0 & S_{N2} \end{array}\right) = \left(\begin{array}{c:c} I_n & S \\ \hdashline D_1 & D_2 \end{array}\right)$$

Theorem 2. *For any $\mu \geq 0$ the "tilde"path $(x_B(\mu), \tilde{x}_N(\mu), \tilde{s}_B(\mu), s_N(\mu))$ is the solution of the system (28), (29'), (31"), (33'), (24)- (27) of $2n$ equations and the Jacobian $J(\mu)$ of the system at the values of the path is non-singular.*

Proof. It suffices to prove the non-singularity of $J(\mu)$ at $\mu = 0$. To this aim we use the following formula for the determinant of the block matrix

$$det \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} = det(Z_1).det(Z_4 - Z_3 Z_1^{-1} Z_2), \tag{34}$$

where Z_1 and Z_4 are square matrices and Z_1 is non-singular. According to (34), $det(J(\mu)) = det(M)$, where $M := M(\mu) = D_2 - D_1 S$. Through simple matrix manipulations we obtain

$$M(\mu) = \left(\begin{array}{cc:cc} X_{B1} & -\tilde{S}_{B1}\bar{A}_{B12} & 0 & -\mu\tilde{S}_{B1}\bar{A}_{N12} \\ X_{B2}\bar{A}_{B12}^T & \tilde{S}_{B2} & 0 & 0 \\ \hdashline 0 & 0 & \tilde{X}_{N1} & -S_{N1}\bar{A}_{N22} \\ \mu\tilde{X}_{N2}\bar{A}_{N12}^T & 0 & \tilde{X}_{N2}\bar{A}_{N22}^T & S_{N2} \end{array}\right) = \left(\begin{array}{c:c} M_1(\mu) & M_2(\mu) \\ \hdashline M_3(\mu) & M_4(\mu) \end{array}\right).$$

Since $M_2(0) = 0$ and $M_3(0) = 0$, we have $det(M(0)) = det(M_1(0)).det(M_4(0))$. We now apply formula (34) to $M_1(0)$ and $M_4(0)$. Thus we obtain

$$\begin{aligned} det(M_1(0)) &= det(X_{B1})det(\tilde{S}_{B2} + X_{B2}\bar{A}_{B12}^T X_{B1}^{-1}\tilde{S}_{B1}\bar{A}_{B12}) \\ &= det(X_{B1})det(X_{B2})det(X_{B2}^{-1}\tilde{S}_{B2} + \bar{A}_{B12}^T X_{B1}^{-1}\tilde{S}_{B1}\bar{A}_{B12}) \neq 0, \end{aligned}$$

where the last relation holds because X_{B1}, X_{B2}, $X_{B2}^{-1}\tilde{S}_{B2}$ are positive definite and $\bar{A}_{B12}^T X_{B1}^{-1}\tilde{S}_{B1}\bar{A}_{B12}$ is positive semidefinite. Analogously, $det(M_4(0)) \neq 0$ and thus also $det(J(0)) \neq 0$. The theorem is proved. $\square$

We have presented two proof techniques to obtain the system of equations describing the central path, so that the corresponding Jacobian is non-singular as $\mu \downarrow 0$. The resulting systems of both approaches are described in different forms, nevertheless, they share some common features. First, they use some matrices (W and R^{-1}, resp.) to obtain proper partition of the system of equations describing the path. Then, they identify the parts of the equations that vanish at $\mu = 0$ ((10b) in the first approach and, (29) and (31') in the second approach). Finally, they repair the vanishing equations (replacing

(10b) by (11) in the first approach, and (29) and (31') by (29') and (31"), resp., in the second approach).

Differences between the two approaches are caused by the first step. At the first approach, the $(m + |B|) \times (m + |B|)$ matrix W operate on the primal-dual system (7a) and (7b). In the second approach, proper partition of the matrix A is first given. Then the $m \times m$ matrix R is constructed and A is multiplied by R^{-1} from the left, and consequently also b and y are multiplied by R^{-1} and R^T from the left. This enables to exclude the variable y from the equations describing the path.

5. Conclusion

In Sections 3 and 4 we have obtained the systems of equations describing the path with non-vanishing Jacobian. It is easy to see that all variables $x_B, \tilde{x}_N, y, \tilde{s}_B, s_N, \mu, \omega$ enter into the both systems analytically and thus the application of the analytic version of the implicit function theorem yield the analyticity of the "tilde"path as the function of μ at $\mu = 0$ (and as the function of ω as well). Since $x_N(\mu) = \mu\tilde{x}_N(\mu)$, $s_B(\mu) = \mu\tilde{s}_B(\mu)$, the analyticity of $x_N(\mu)$ and $s_B(\mu)$ is evident.

Obviously, the analyticity of the weighted central path implies some weaker results on the limiting properties of its derivatives. In fact those limiting properties have applications to the analysis and construction of polynomial-time algorithms which are also locally superlinearly or quadratically convergent. (The direction opposite to the first derivative at a point on a weighted central path coincides with the primal-dual affine scaling direction at that point.) In this connection it is worth noting that the proofs of analyticity presented here are simpler not only than the proofs of analyticity by Halická [7] and Wechs [15] but also than the proofs of some weaker results on limiting properties of derivatives by Adler and Monteiro [1], Witzgall at al. [16] and Güler [4]. So the simple proofs, presented here for LP, can be efficient both as a teaching and as a research tool.

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