On sustainability constraint in models with non-renewable resources

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Abstract

This paper systematizes main results on economic models concerning renewable and non-renewable capital goods where the criterion is to sustain the utility on some minimal level over whole time horizon. Using the framework of the multidimensional Dasgupta-Heal-Solow model, it sheds light on the relation of two different approaches: the discounted utility approach with sustainability constraint, which is historically older and the maximin approach which has been introduced only recently. In both approaches, we deal especially with the Hartwick's rule and formulate assumptions when this rule (or its generalized version) constitutes either necessary or sufficient condition for constant utility.

Keywords: Sustainability, Hartwick rule, maximin paths, optimal control

1 *INTRODUCTION*

In the second half of the 20^{th} century a number of articles dealt with the problem of optimal consumption in an economy endowed with some capital goods which are non-renewable but unavoidable for production (cake-eating economy). The prevalent approach based on maximizing the discounted utility established by Ramsey (1928) was abandoned as unethical. The ground for this criticism (represented mainly by John Rawls) lied in the fact that the current utility has a higher weight at the expense of future generations who cannot rise any objections. A new requirement of sustainability arose by which we seek the maximal well-being of each generation that leaves the economy with the capacity to generate the same well-being in each ensuing period. This requirement is also called sustainability constraint.

As a response to this remarks, Solow (1974) formulated a problem of maximizing the level of consumption which can be maintained forever, even if one of the essential inputs of the production function is a non-renewable (exhaustible) capital, given that the renewable and exhaustible capital goods are sufficiently substitutable. However, he only *assumed* the optimal consumption to be constant but he did not formulate any condition for this. This result was further extended by Hartwick (1977) who introduced the Hartwick's rule which prescribes to invest all the revenues from exhaustible capital depletion into the renewable capital (i.e. zero net investment). He formulated this rule as a sufficient condition for constant consumption paths, provided that the path satisfies the necessary conditions for optimality[1] and some other assumptions are met.

In the existing literature, two main approaches of dealing with sustainability constraint requiring to preserve the capacity of economy can be identified. The discounted utility approach assumes that agents maximize their discounted utility subject to the constraint of

[1]This is referred to as *efficient path* in the prevalent literature as this path minimizes the quantity of resource used.

constant utility over the time horizon. This approach was adopted among other by Dixit et al. (1980), Withagen and Asheim (1998), Cairns and Yang (2000), Mitra (2002), Withagen et al. (2003), Asheim et al. (2003) and Buchholz et al. (2005). Assumptions regarding no technology or population growth and exponential discount factor were released by Hartwick and Long (1999), Asheim et al. (2005) and Farzin (2006). On the other hand, the maximin approach supposes maximizing the level of utility of the least advantaged generation. Unlike the discounted utility approach that can be found in majority of literature on this topic, the maximin approach was not subject of interest until recently. This approach and the relationship to the first approach was addressed by Cairns and Tian (2002) and Cairns (2003).

The aim of this paper is to summarize the most important results on Hartwick's rule related to the sustainability constraint in general and to present these results in a unified framework. Regarding the discounted utility approach, we present a new proof of the converse of Hartwick's result. We also shed light on the relationship between the discounted utility approach and the maximin approach by extending the results that were introduced by Cairns and Tian (2002).

The structure of this paper is as follows: In section 2 the main model of economy with renewable and non-renewable resources with sustainability constraint is presented. Section 3 introduce the discounted utility approach and present the main results regarding this approach. The maximin approach is described in Section 4. Section 5 is devoted to comparison of the two approaches. It extends the results that has been known so far. Finally, Section 6 concludes.

2 FORMULATION OF THE MODEL

Consider the following model of an economy with renewable and non-renewable resources:

$$\max_{\{c(t),r(t)\}} \inf_{t\geq 0} U(c(t)), \tag{1}$$

$$\dot{k}(t) = f(k(t), r(t)) - \delta k(t) - c(t), \quad k(0) = k_0 > 0 \text{ given}, \quad \delta > 0 \text{ given}, \tag{2}$$

$$\dot{s}(t) = -r(t), \quad s(0) = s_0 > 0 \text{ given}, \tag{3}$$

$$k(t) \geq 0, \quad s(t) \geq 0, \quad r(t) \geq 0, \quad c(t) \geq 0. \tag{4}$$

This is a problem of maximizing the level of utility that can be sustained forever. The amount of production depends on the amount of the renewable capital in individual sectors ($k \in \mathbb{R}^n_+$)[2] and on the rate of extraction ($r \in \mathbb{R}^m_+$) of the non-renewable capital ($s \in \mathbb{R}^m_+$). The production can be partitioned into investment in the capital in order to increase its volume in the future and current consumption ($c \in \mathbb{R}^n_+$). In this model we assume that there is a positive rate of amortization of the capital in each sector (δ). The population and technology is supposed to be constant over the time horizon.

Further assumptions of the model:

(P1) The production function $f : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \to \mathbb{R}$ is increasing, strictly concave, unbounded and twice continuously differentiable w.r.t. both variables and it holds $f(k, r) = 0$

[2] We use the following notation: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_i \geq 0,\ i = 1, \ldots, n\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n;\ x_i > 0,\ i = 1, \ldots, n\}$.

whenever any component of the vectors k or r is zero. In addition it holds

$$\lim_{k_i \to 0^+} \frac{\partial f(k,r)}{\partial k_i} = \infty \quad \forall k \in \mathbb{R}^n_{++} \text{ and } r \in \mathbb{R}^m_{++},\ i = 1, \dots, n,$$
$$\lim_{r_j \to 0^+} \frac{\partial f(k,r)}{\partial r_j} = \infty \quad \forall k \in \mathbb{R}^n_{++} \text{ and } r \in \mathbb{R}^m_{++},\ j = 1, \dots, m.$$

(P2) The utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ is increasing, strictly concave and twice continuously differentiable on $\mathbb{R}^n_+$ and it holds

$$\lim_{c_i \to 0^+} \frac{\partial U(c)}{\partial c_i} = \infty, \quad \lim_{c_i \to \infty} \frac{\partial U(c)}{\partial c_i} = 0 \qquad \forall\, c \in \mathbb{R}^n_{++},\ i = 1, \dots, n.$$

This formulation of the model which directly include the sustainability constraint in the objective function fits to the original formulation of this problem (cf. Solow (1974)). Nevertheless, it is common in the literature that the objective function (1) is substituted by

$$\max_{\{c(t), r(t)\}} \int_0^\infty \pi(t) U(c(t))\, \mathrm{d}t \tag{5}$$

with additional requirement that $U(c(t))$ has to be constant for all $t \geq 0$[3] and we assume that

(P3) the discount factor $\pi(t)$ is positive, non-increasing and $\int_0^\infty \pi(t)\, \mathrm{d}t < \infty$.

Two special cases of this problem often used in the literature on Hartwick's rules are as follows:

- Ramsey model, where $m = 0$ and $n = 1$ (i.e. only one renewable but no non-renewable resource is considered) and
- Dasgupta-Heal-Solow model, where $m = 1$ and $n = 1$ (i.e. one renewable and one non-renewable capital is considered).

3 DISCOUNTED UTILITY APPROACH

In this section, we briefly summarize the main results regarding the discounted utility approach. For more comprehensive summary, see e.g. Asheim et al. (2003). Firstly, we present the necessary condition for optimality of the solution to (2) – (5) which will be used in these results.

The Hamiltonian function is defined by

$$H(k, s, c, r, \psi^0, \psi, \varphi) = \psi^0 \pi U(c) + \psi^T (f(k, r) - \delta k - c) + \varphi^T (-r)$$

and the Lagrangian is defined by

$$L(k, s, c, r, \psi^0, \psi, \varphi, \mu_r, \mu_c, \nu, \eta) =$$
$$= \psi^0 \pi U(c) + (\psi + \nu)^T (f(k, r) - \delta k - c) + (\varphi + \eta)^T (-r) + \mu_r^T r + \mu_c^T c.$$

[3] Note that although this is common formulation in the literature, it suffices that $U(c(t))$ does not decrease under some level.

Further, let us denote

$$H^*[t] := H\Big(k^*(t), s^*(t), c^*(t), r^*(t), \psi^0, \psi(t), \varphi(t)\Big),$$

where (k^*, s^*, c^*, r^*) is an optimal solution to (2) – (5).

Now we formulate the necessary conditions of optimality according to Seierstad and Sydsæter (1987) [Theorem 6.9]: If (k^*, s^*, c^*, r^*) is an optimal solution to the problem (2) – (5), then there exist a constant $\psi^0 = 0$ or $\psi^0 = 1$, vector functions $\psi(t) : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) : \mathbb{R} \to \mathbb{R}^m$, $\mu_c(t) : \mathbb{R} \to \mathbb{R}^n$ and $\mu_r(t) : \mathbb{R} \to \mathbb{R}^m$ and non-increasing vector functions $\nu(t) : \mathbb{R} \to \mathbb{R}^q$ and $\eta(t) : \mathbb{R} \to \mathbb{R}^m$ such that $(\psi^0, \psi(t^-), \varphi(t^-)) \neq (0,0,0)$ and $(\psi^0, \psi(t^+), \varphi(t^+)) \neq (0,0,0)$ for all $t > 0$ all having one-sided limits everywhere, such that following conditions are met:

$$H^*[t] \geq H\Big(k^*(t), s^*(t), c(t), r(t), \psi^0, \psi(t), \varphi(t)\Big) \text{ for all } (c,r) \in \mathbb{R}_+^{m+n} \tag{6}$$

$$\frac{\partial L}{\partial c} = \psi^0 \pi \frac{\mathrm{d}U}{\mathrm{d}c} - \psi^T - \nu^T + \mu_c^T = 0 \tag{7}$$

$$\frac{\partial L}{\partial r} = (\psi + \nu)^T \frac{\partial f}{\partial r} - \varphi^T - \eta^T + \mu_r^T = 0 \tag{8}$$

$$\psi(t) + \nu(t) \text{ and } \varphi(t) + \eta(t) \text{ are continuous everywhere} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi^T + \nu^T) = -\frac{\partial L}{\partial k} = \delta(\psi + \nu)^T - (\psi + \nu)^T \frac{\partial f}{\partial k} \text{ almost everywhere} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^T + \eta^T) = -\frac{\partial L}{\partial s} = 0 \text{ almost everywhere} \tag{11}$$

$$\nu \text{ and } \eta \text{ are constant on any interval where } k^* > 0, \text{ resp. } s^* > 0, \tag{12}$$

$$\nu \text{ is continuous if } k^* = 0 \text{ and } f(k^*, r^*) - \delta k^* - c^* \text{ is discontinuous,} \tag{13}$$

$$\eta \text{ is continuous if } s^* = 0 \text{ and } r^* \text{ is discontinuous,} \tag{14}$$

$$\mu_c^T c^* = 0, \quad \mu_c \geq 0, \quad \text{and} \quad \mu_r^T r^* = 0, \quad \mu_r \geq 0. \tag{15}$$

Theorem 1 (Hartwick's result) *Let (k, s, c, r) be an admissible[4], solution to problem (2) – (5) and suppose that there exist (ψ, φ) such that the following conditions are met:*

(i) the quadruple (k, s, c, r) fulfills conditions (6) – (15) together with $\psi^0 = 1$, $\mu_c = 0$, $\mu_r = 0$, $\nu = 0$, $\eta = 0$ and (ψ, φ),

(ii) for all $t \geq 0$ it holds $\psi^T \dot{k} = \varphi^T r$ (Hartwick's rule).

Then $U(c) \equiv$ const. for all $t \geq 0$.

Remark 1 (Economic interpretation of Hartwick's rule) The Hartwick's rule means that the net investment has to be zero, i.e. that any decrease in non-renewable capital or renewable has to be compensated by increase in (other) renewable capital. In the special case of Ramsey model, it means that the level of capital has to be constant.

Remark 2 (Generalized Hartwick's result) Theorem 1 holds true even if we replace the Hartwick's rule by its generalized version in form $\psi^T \dot{k} - \varphi^T r =$ const. The first who proved the generalized Hartwick's result were Dixit et al. (1980).

[4] Admissible solution to problem (2) – (5) is a quadruple (k, s, c, r) such that k and s are continuous functions, conditions (2) – (5) are satisfied and the integral in (5) is convergent.

Remark 3 (An existence result) Hartwick's result does not address the question of existence of paths satisfying the assumptions of Theorem 1. Said differently, the discount factor π has to be chosen properly such that condition (ii) is met in addition to the necessary conditions (6) – (15). This artificial ex ante setting of preferences was an object of critics, see e.g. Martinet and Rotillon (2005).

Remark 4 (Optimal level of utility) Theorem 1 does not state whether the utility level is the highest among all utility levels that can be sustained forever. Nevertheless, considering the case $m = n = 1$, Cairns and Yang (2000) proved that this is true if there exists $\alpha > 0$ such that $r\frac{\partial^2 f}{\partial r \partial k} > \alpha$ for all $t \geq 0$.

The converse of Hartwick's result is not quite straightforward, as seen in the next theorem: It does not suffice that the quadruple (k, s, c, r) meets conditions (6) – (15); we have to assume that it is optimal solution. In the proof of the converse of Hartwick's result which is presented below we use this optimality to derive an additional condition formulated in Lemma 1 which is used to prove this result.

Theorem 2 (Converse of Hartwick's result) *Let (k^*, s^*, c^*, r^*) be an optimal solution to problem (2) – (5) which satisfies the conditions (6) – (15) with $(\psi^0, \psi, \varphi, \mu_r, \mu_c, \nu, \eta)$. Then it holds: If $U(c^*(t)) \equiv const.$ for all $t \geq 0$, then $\psi(t)^T \dot{k}^*(t) = \varphi(t)^T r^*(t)$.*

This result has been broadly cited in the literature. It was formulated among other by Withagen et al. (2003) [Proposition 2], Asheim et al. (2003) [Proposition 4], Withagen and Asheim (1998), Cairns and Yang (2000) [Theorem 1 (i)], Martinet (2004) [Proposition 3] and Buchholz et al. (2005) [Theorem 1]. In addition, several proofs of the converse of Hartwick's result can be found in the literature. Withagen and Asheim (1998) introduced a proof based on the transformation to a problem on finite time horizon with free final time. However, they neglected the scrap value function that should be in the objective functional since the final time is free (cf. (29)). Another proof was given by Mitra (2002) who used a new condition of terminal cost minimization, but only for a special discount factor $\pi(t) = 1$. Aronsson et al. (1995) proved this theorem for the special case of Ramsey model (i.e. n=1, m=0) and $\pi(t) = e^{-\rho t}$, where $\rho > 0$. The proof by Buchholz et al. (2005) is restricted only to interior solutions. Finally, Farzin (2006) showed that this result is valid in the general framework as considered in this paper extending the well-known result of Michel (1982) to general discount factor. However, he neglected the control and state constraints. Moreover, his proof is based on the additional assumption that the Hamiltonian is differentiable w.r.t. t which was not verified. In this paper, we present a new proof of the converse of Hartwick's result for the general problem (2)–(5) using the extension of the result proved by Michel (1982) considering also pure state constraints. In the proof, we use the following lemma:

Lemma 1 *Let (k^*, s^*, c^*, r^*) be an optimal solution to (2)–(5) which satisfies the conditions (6) – (15) with $(\psi^0, \psi, \varphi, \mu_r, \mu_c, \nu, \eta)$. Then it holds for all $t \geq 0$:*

$$H^*[t] = -\psi^0 \int_t^\infty \pi'(s) U(c^*(s))\, ds.$$

Proof. The proof is given in the Appendix. ■

Proof of the Theorem 2. From the Lemma 1 we obtain that it holds for all $t \geq 0$

$$\psi^0 \pi(t) U(c^*(t)) + \psi(t)^T \dot{k}^*(t) - \varphi(t)^T r^*(t) = -\psi^0 \int_t^\infty \pi'(s) U(c^*(s)) \, \mathrm{d}s. \tag{16}$$

By integrating the right-hand side per-partes we have

$$-\psi^0 \int_t^\infty \pi'(s) U(c^*(s)) \, \mathrm{d}s =$$

$$= -\psi^0 \Big[\pi(s) U(c^*(s))\Big]_t^\infty + \int_t^\infty \psi^0 \pi(s) \frac{\mathrm{d}U(c^*(s))}{\mathrm{d}s} \, \mathrm{d}s. \tag{17}$$

The necessary condition for the convergence of the integral in the objective function in (5) together with the fact that $U(c^*(t)) = \text{const.}$ and $\pi(t)$ is non-increasing (see (P3)) implies

$$\lim_{t \to \infty} \pi(t) U(c^*(t)) = 0.$$

Moreover,

$$\frac{\mathrm{d}U(c^*(t))}{\mathrm{d}t} = 0$$

as $U(c^*(t)) = \text{const.}$ Using these two equalities we obtain that the right-hand side of (17) is simply

$$\psi^0 \pi(t) U(c^*(t)). \tag{18}$$

We complete the proof by combining (16) and (18) which implies $\psi(t)^T \dot{k}^*(t) - \varphi(t)^T r^*(t) = 0$ for all $t \geq 0$. ■

Theorem 3 (Converse of generalized Hartwick's rule) *Let (k, s, c, r) be an admissible solution to (2) – (5) which satisfies the conditions (6) – (15) with $(\psi^0, \psi, \varphi, \mu_r, \mu_c, \nu, \eta)$. Then it holds: If $U(c(t)) \equiv$ const. for all $t \geq 0$, then $\psi(t)^T \dot{k}(t) - \varphi(t)^T r(t) \equiv$ const. $=: \gamma$ for all $t \geq 0$.*

This theorem was proved e.g. by Mitra (2002) [Theorem 2]. In addition, Mitra (2002) also presented an example that γ can be different from zero.

4 MAXIMIN APPROACH

In the previous section we have dealt with the discounted utility approach to the sustainability constraint. Although the problem which has been solved in this approach is certainly not the same as the original problem (1) – (4), it is prevalent in the relevant literature. However, the maximization of the discounted value of utility is rather artificial in the original context. This is confirmed also by the fact that in the Hartwick's result (Theorem 1) and in the Converse of Hartwick's result (Theorem 2), the discount factor has to be chosen properly.

Hence, it is reasonable to study directly the problem (1) – (4). This is not a standard optimal control problem. Using a new variable w, we can transform the problem into form

$$\max_{\{c(t), r(t)\}} w, \tag{19}$$

$$\dot{k} = f(k, r) - \delta k - c, \quad k(0) = k_0 > 0 \text{ given}, \quad \delta > 0 \text{ given}, \tag{20}$$

$$\dot{s} = -r, \quad s(0) = s_0 > 0 \text{ given}, \tag{21}$$

$$k(t) \geq 0, \quad s(t) \geq 0, \quad r(t) \geq 0, \quad c(t) \geq 0, \tag{22}$$

$$U(c) \geq w. \tag{23}$$

The problem (1) is formulated as an optimal control problem with parameter w. As it was suggested e.g. by Leonard and Long (1992), we can transform this problem as a standard optimal control problem (without parameter) considering w as a new state variable. Hence, (19) can be rewritten as follows

$$\max_{\{c(t),r(t)\}} w(0), \qquad \text{where } \dot{w} = 0, \quad w(0) \text{ free} \tag{24}$$

The problem (20) – (24) is an autonomous optimal control problem with infinite time horizon, with constrained control region and pure state constraints and with an initial scrap value function. Note that there is no initial condition imposed on $w(0)$.

Define the Hamiltonian by

$$\bar{H}(k,s,c,r,w,\psi,\varphi,\lambda) = \psi^T(f(k,r) - \delta k - c) - \varphi^T r + \lambda\, 0$$

and the Lagrangian by

$$\bar{L}(k,s,c,r,w,\psi,\varphi,\lambda,\mu_c,\mu_r,\mu_u,\nu,\eta) =$$
$$= (\psi+\nu)^T(f(k,r) - \delta k - c) - (\varphi+\eta)^T r + \lambda\, 0 + \mu_c^T c + \mu_r^T r + \mu_u(U(c) - w),$$

Again we can formulate the necessary conditions for optimality based on Seierstad and Sydsæter (1987) [Theorem 6.9]: If $(k^*, s^*, w^*, c^*, r^*)$ is an optimal solution to the problem (20) – (24), then there exist a constant $\psi^0 = 0$ or $\psi^0 = 1$, vector functions $\psi(t) : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) : \mathbb{R} \to \mathbb{R}^m$, $\lambda(t) : \mathbb{R} \to \mathbb{R}$, $\mu_c(t) : \mathbb{R} \to \mathbb{R}^n$, $\mu_r(t) : \mathbb{R} \to \mathbb{R}^m$ and $\mu_u(t) : \mathbb{R} \to \mathbb{R}$ and non-increasing vector functions $\nu(t) : \mathbb{R} \to \mathbb{R}^q$ and $\eta(t) : \mathbb{R} \to \mathbb{R}^m$ such that $(\psi^0, \psi(t^-), \varphi(t^-), \lambda(t^-)) \neq (0,0,0,0)$ and $(\psi^0, \psi(t^+), \varphi(t^+), \lambda(t^+)) \neq (0,0,0,0)$ for all $t > 0$ all having one-sided limits everywhere, such that conditions (8) – (15) provided that L is replaced by $\bar{L}$ and following conditions are met:

$$\psi^T(f(k^*,r^*) - \delta k^* - c^*) - \varphi^T r^* \geq \psi^T(f(k^*,r) - \delta k^* - c) - \varphi^T r$$
$$\text{for all } (c,r) \in \mathbb{R}_+^{m+n} \text{ and } U(c) \geq w^*, \tag{25}$$

$$\frac{\partial \bar{L}}{\partial c} = \mu_u \frac{\mathrm{d}U}{\mathrm{d}c} - \psi^T - \nu^T + \mu_c^T = 0, \tag{26}$$

$$\dot{\lambda} = -\frac{\partial \bar{L}}{\partial w} = \mu_u, \quad \lambda(0) = -1, \tag{27}$$

$$\mu_u(U(c) - w) = 0, \quad \mu_u \geq 0. \tag{28}$$

Considering the Hartwick's rule in the maximin framework, the following theorem can be proved:

Theorem 4 (Converse of Hartwick's result in the maximin framework) *Assume that $(k^*, s^*, c^*, r^*, w^*)$ is an optimal solution to the problem (8) – (15) which fulfills the conditions (8) – (15) and (25) – (28) together with $(\psi^0, \psi, \varphi, \lambda, \mu_c, \mu_r, \mu_u, \nu, \eta)$. Then $\psi(t)^T \dot{k}^*(t) = \varphi(t)^T r^*(t)$ for all $t \geq 0$.*

Proof. The theorem follows immediately from the fact that the Hamiltonian for the problem (20) – (24) is zero everywhere. We will not prove this; the proof that the Hamiltonian is zero is analogous to the proof of Lemma 1 used formally with $\pi(t) = 1$ and $U(c) = 0$.[5] ■

[5] Note that $\pi(t) = 1$ and $U(c) = 0$ does not meet assumptions (P2) and (P3). However, these assumptions are not needed in the proof of Lemma 1.

5 DISCOUNTED UTILITY APPROACH VS. MAXIMIN APPROACH: A COMPARISON

The aim of this section is to find a relationship between the discounted utility approach and the maximin approach which were described in the previous sections.

This relationship was addressed by Cairns and Tian (2002). However, they used only the framework of Ramsey model. Their main conclusion was that the discount factor in the discounted utility approach can be interpreted as a shadow value of the constraint that the utility is constant in the maximin approach. Further similarities and differences of these two approaches were also commented by Cairns (2003).

To extend the interpretation found by Cairns and Tian (2002), we use that the necessary conditions to the problem (20) – (24) are very similar to the necessary conditions to the problem (2) – (5). Actually, this similarity helps us to find the relationship between the solution to the problem (2) – (5) representing the discounted utility approach and the solution to the problem (20) – (24) which was formulated using the maximin approach. Indeed, the following theorem holds:

Theorem 5 [6] *Let $(\bar{k}, \bar{s}, \bar{c}, \bar{r})$ be a feasible solution to the problem (2) – (5) that satisfies the necessary conditions (6) – (15) together with $(\psi^0, \psi, \varphi, \mu_c, \mu_r, \nu, \eta)$, where $\psi^0 = 1$, $\mu_c = 0$, $\mu_r = 0$, $\nu = 0$, $\eta = 0$ and it holds $\psi^T \dot{\bar{k}} = \varphi^T \bar{r}$. Then there exist $\bar{w}$, λ and μ_u such that $(\bar{k}, \bar{s}, \bar{c}, \bar{r}, \bar{w})$ satisfies the necessary conditions (8) – (15) with L replaced by $\bar{L}$ and (25) – (28) to the problem (20) – (24) together with $(\psi, \varphi, \lambda, \mu_c, \mu_r, \mu_u, \nu, \eta)$.*

Proof. Under the assumptions of this theorem, Theorem 1 implies that $U(\bar{c}(t)) =$ const. Denote this constant by $\bar{w}$. Further define

$$\lambda(t) = -1 + \int_0^t \pi(t)\,\mathrm{d}t$$

and

$$\mu_u(t) = \psi^0 \pi(t).$$

Then it is straightforward to verify that $(\bar{k}, \bar{s}, \bar{w}, \bar{c}, \bar{r})$ together with $(\psi, \varphi, \lambda, \mu_c, \mu_r, \mu_u, \nu, \eta)$ satisfy the conditions (7) – (15) with L replaced by $\bar{L}$ and (26) – (28). Hence, we only show in detail that (6) implies (25). Recall that (6) reads

$$\psi^0 \pi U(\bar{c}) + \psi^T(f(\bar{k}, \bar{r}) - \delta\bar{k} - \bar{c}) - \varphi^T \bar{r} \geq \psi^0 \pi U(c) + \psi^T(f(\bar{k}, r) - \delta\bar{k} - c) - \varphi^T r$$

for all (c, r) such that $c \geq 0$ and $r \geq 0$. If $U(c) \geq \bar{w} = U(\bar{c})$, one has

$$\psi^T(f(\bar{k}, \bar{r}) - \delta\bar{k} - \bar{c}) - \varphi^T \bar{r} \geq \psi^T(f(\bar{k}, r) - \delta\bar{k} - c) - \varphi^T r$$

again for all (c, r) such that $c \geq 0$, $r \geq 0$ and $U(c) \geq \bar{w}$. This proves (25). ■

Remark 5 (Interpretation of the relationship between both approaches) The importance of Theorem 5 is based on the fact that it links the discounted utility approach and the maximin approach. Actually, it highlights the artificiality behind the discounted utility approach: This discount factor can be considered as implicitly included in the maximin

[6] This theorem is original result of this paper. Cairns and Tian (2002) [Proposition 3] introduced the converse of this theorem but only for the case $n = 1$ and $m = 0$. Moreover, they neglected the nonnegativity constraints on state and control variables and conditions (6) and (25), respectively.

approach. Actually, it is the shadow value of the constraints $U(c) \geq w$. Moreover, as Cairns (2003) noted, the relationship between these two approaches can be interpreted also from another point of view: The appropriate chosen discount factor $\pi(t)$ can be considered as a coefficient in the infinitely-dimensional hyperplane given by equation

$$\int_0^\infty \pi(t)U(c^*(t))\mathrm{d}t = w^*.$$

Besides the optimal control theory framework, also other approaches to handle the sustainability constraint can be used. For example, papers by Martinet and Doyen (2003) and Martinet (2004) are based on the viable control analysis. On the other hand, Martinet and Rotillon (2005) examined invariants that are preserved endogenously without a priori choosing of preferences as in the discounted utility approach.

6 CONCLUSION AND DISCUSSION

In this paper, we have summarized the main results related to the problem with sustainability constraint in the optimal control framework under two approaches: the discounted utility approach and the maximin approach. In the discounted utility approach, we have provided a new proof of Theorem 2, i.e that in the multidimensional Dasgupta-Heal-Solow model, Hartwick's rule is a necessary condition for optimal solutions to have constant utility. We have also extended the results introduced by Cairns and Tian (2002) on the relationship between the two approaches in Theorem 5.

Let us mention that the converse of Theorem 4 has not been proved nor disproved yet. In addition, although we have confirmed in Remark 5 the economical interpretation of the discount factor as the shadow value of the sustainability constraint, it has not been verified that the standard assumptions lied on a discount factor (see (P3)) are satisfied for this shadow value. These issues could be subjects of further research.

7 APPENDIX: PROOF OF LEMMA 1

Proof. Choose an arbitrary sequence $\{T_n\}$, where $T_n > 0$ and $T_n \to \infty$. For each T_n define a free final time problem with a scrap value function as follows

$$\max \int_0^T \pi(t)U(c(t))\,\mathrm{d}t + \Phi(T - T_n), \tag{29}$$

where

$$\Phi(\xi) = \pi(\xi) \int_{T_n}^\infty \pi(t)U(c^*(t))\,\mathrm{d}t$$

subject to (2) – (4). It can be easily shown that T_n and (k^*, s^*, c^*, r^*) on $\langle 0, T_n\rangle$ is an optimal solution for this problem. If we transform this problem to an autonomous one considering t as a new state variable

$$\tau := t, \text{ i.e. } \dot{\tau}(t) = 1, \ \tau(0) = 0, \ \tau(T) \text{ free},$$

the augmented solution $(k^*, s^*, \tau^*, c^*, r^*, T_n)$ will satisfy the necessary conditions for optimality formulated by Seierstad and Sydsæter (1987) [Theorem 5.2]. Denoted the Hamiltonian by $\tilde{H}_n^*[t]$ and the Lagrangian by $\tilde{L}_n^*$. In particular, there exists a piecewise continuous function $\lambda_n : \mathbb{R} \to \mathbb{R}$ solving the differential equation

$$\dot{\lambda}_n = -\frac{\partial \tilde{L}_n^*}{\partial \tau} \tag{30}$$

almost everywhere and the terminal condition

$$\lambda_n(T_n) = \psi_n^0 \frac{\partial \Phi(\tau(T_n) - T_n)}{\partial \tau}. \tag{31}$$

Since the only functions dependent on τ directly are $\pi(\tau)$ and $\Phi(\tau)$, (30) can be reduced to

$$\dot{\lambda}_n = -\psi_n^0 \frac{\partial \pi(\tau) U(c)}{\partial \tau} = -\psi_n^0 \pi'(\tau) U(c). \tag{32}$$

Further it holds that λ_n is continuous everywhere. This is implied by Equation (5.90) stated by Seierstad and Sydsæter (1987) using the fact that all constraints in the given problem are autonomous.

We also know that (32) holds almost everywhere, hence

$$\lambda_n(T_n) - \lambda_n(t) = -\psi_n^0 \int_t^{T_n} \pi'(s) U(c^*(s)) \, \mathrm{d}s. \tag{33}$$

Using the definition of ϕ, (31) implies

$$\lambda_n(T_n) = \psi_n^0 \frac{\partial \Phi(\tau(T_n) - T_n)}{\partial \tau} =$$

$$= \psi_n^0 \left. \frac{\partial \left(\pi(\tau(T - T_n)) \int_{T_n}^{\infty} \pi(s) U(c^*(s)) \, \mathrm{d}s \right)}{\partial \tau} \right|_{T = T_n} =$$

$$= \psi_n^0 \, \pi'(0) \int_{T_n}^{\infty} \pi(s) U(c^*(s)) \, \mathrm{d}s. \tag{34}$$

Putting (33) and (34) together we obtain

$$\lambda_n(t) = \psi_n^0 \int_t^{T_n} \pi'(s) U(c^*(s)) \, \mathrm{d}s + \psi_n^0 \, \pi'(0) \int_{T_n}^{\infty} \pi(s) U(c^*(s)) \, \mathrm{d}s =$$

$$= \psi_n^0 \int_t^{T_n} \pi'(s) U(c^*(s)) \, \mathrm{d}s + C(T_n), \tag{35}$$

where $\lim_{n \to \infty} C(T_n) = 0$.

Let us denote by $H_n^*[t]$ the Hamiltonian of the original (non-autonomous) problem. Then one has

$$\tilde{H}_n^*[t] = H_n^*[t] + \lambda_n(t).$$

Now we can exploit the advantage of the transformation of the original problem to an autonomous one and use three results formulated by Seierstad and Sydsæter (1987) (see notes in parentheses for detailed references):

(i) $\dfrac{\mathrm{d}}{\mathrm{d}t} \tilde{H}_n^*[t] = 0$ for almost all $t \in (0, T_n)$ (Note 5.3(f)),

(ii) $\tilde{H}_n^*[t]$ is continuous for all $t \in (0, T_n)$ (Note 5.3(c)),

(iii) $\tilde{H}_n^*[T_n^-] = 0$ (Theorem 6.16).

Combining these three results we obtain that

$$H_n^*[t] + \lambda_n = \tilde{H}_n^*[t] = 0 \tag{36}$$

for all $t \in (0, T_n)$. Note that the set of necessary conditions (6) – (15) does not include such a type of on Hamiltonian except (6). Indeed, (36) is additional to these conditions. This is the reason why the optimality of the solution is needed in the formulation of Lemma 1.

Since for all n, $\psi_n^0 = 0$ or $\psi_n^0 = 1$, a convergent subsequence $\{\psi_{n_i}^0\}_{i=1}^{\infty}$ exists; denote the limit of this subsequence ψ^0. Thus there exists i_0 such that for all $i > i_0$ it holds $\psi_{n_i}^0 = \psi^0$.

Let us return now to (36), but consider it only for the subsequence $\{n_i\}$. Using (35) and taking limit for $i \to \infty$ we obtain

$$H^*[t] = -\psi^0 \int_t^{\infty} \pi'(s) U(c^*(s)) \, \mathrm{d}s. \blacksquare$$

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Stochastic Programming Problems with Recourse via Multiobjective Optimization Problems*

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Abstract

Stochastic programming problems with recourse are a composition of two (outer and inner) optimization problems. A solution of the outer problem depends on an "underlying" probability measure while a solution of the inner problem depends on the solution of the outer problem and on the random element realization. The optimal solution is sought with respect to the mathematical expectation of the outer problem. Of course, to be the problem "well" defined a finite optimal value of the inner problem has to exist for every feasible solution of the outer problem and every "possible" realization of the random element; moreover, a finite mathematical expectation of the optimal value of the outer problem has to exist for every feasible solution of the outer problem. To this end, sufficient assumptions are well known in a linear case. The aim of the note is to deal with (generally) nonlinear case. To obtain new results, the multiobjective deterministic theory is employed.

Keywords: Stochastic programming problems with recourse, inner and outer problem, multiobjective optimization problems, efficient points, properly efficient points, concave functions.

1 INTRODUCTION

Stochastic programming problems with recourse correspond to many applications. Historically, first a sense of this type of the problems has been to compensate unfulfilled constraints with a random element. Consequently, the original aim has been a "generalization" of stochastic programming problems with a penalty. It is known that problems with simple recourse can be (under some assumptions) equivalent just to the problems with penalty. However, at present these problems correspond to many real life situations in which it is possible to correct a solution determined before a random element realization. We can recall e.g. the applications: Financial problems (investment problem, portfolio revision problem, see e.g. [2]), farmer's problem (see e.g. [1]), melt control problem (see e.g. [2]), power–station planning (see e.g. [5], [14]), aircraft allocation problem, transportation problem (see e.g. [12]), location problem (see e.g. [15]), production planning (see e.g. [7], [8]). Evidently, most of this applications are connected with a loss or a profit by some economic activities. Some other applications can be found in [11].

Let $\xi := \xi(\omega)\,(m \times 1)$ be a random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $\xi = (\xi_1, \ldots, \xi_m)$. We denote by F^{ξ}, $P_{F^{\xi}}$ the distribution function and the probability measure corresponding to the random vector ξ; $Z_{F^{\xi}}$ the support of $P_{F^{\xi}}$. Let, moreover, g_0, g_0^1 be

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