

# Chapter 1

## Introduction

### 1.1 Definition of financial risks

In general, financial risk might be defined e.g. as a potential future financial loss related to a given financial instrument or to a given portfolio of financial instruments. However, one may argue that this definition is neither fully comprehensive nor unambiguous.

**Example 1.1.** Examples of typical losses incurred by a bank:

- Losses stemming from loans granted to clients which are not repaid.
- A decline in prices of equities held in the bank's portfolio.
- An increase in interest rates causing a drop in bond prices.
- Depreciation of foreign currency decreasing the value of assets denominated in this currency.

The definition of financial risks is far from being uniformly agreed. Nevertheless, it is crucial for a risk manager to understand in details the concept of financial risk under his management system. In particular, it is necessary to know which aspects of the financial risk are covered. Let us hence briefly mention some areas which attract particular attention:

- **Unexpected profits:** Sometimes the definition of a risk is considered in a broader sense: It not only includes downside risks (losses) but also upside risks (e.g. profits larger than expected or better than expected improvement of a situation).
- **Unexpected vs expected loss:** In some cases (e.g. in retail loan portfolio), the risk manager might be fairly sure about some level of losses due to loans which are not

repaid by the borrowers. These losses might be estimated from the past experience with a high degree of confidence. What is more uncertain are the losses above this level, i.e. unexpected losses. It is important to know in advance whether the risk concerns only unexpected losses or both type of losses.

- **Opportunity costs:** Imagine that I have a mortgage with a ten-year interest rate fixation, but the interest rates start to decline after the first year. Clearly this will have no direct negative impact on my financial situation in the course of the next nine years. However, I could start to feel unhappy that I have not decided for the one-year fixation.
- **Incurred loss vs reported loss:** Assume that I have a strong historical evidence that a jump in unemployment rate yields higher losses in retail loan portfolio. In such a case, losses might be recognized immediately when the unemployment rises, or only later when loans are not repaid in higher extent than initially expected. It should be clear when the concept of risk management recognizes the losses.
- **Appropriate loss benchmark:** Assume that the bank incurred losses due to a jump in unemployment rate, although lower than expected. Should this be still considered as a loss?

It is important to distinguish two distinct characteristics of a financial risk:

- a) probability of an event,
- b) size of the potential impact.

**Example 1.2.** Examples of different types of risks regarding their probability and size:

- Large probability, low impact: Individual case of a non-repaid mortgage loan
- Large probability, high impact: General decrease of the demand for loans due to a global economic slowdown in a crisis period
- Low probability, high impact: damage caused by an earthquake, losses stemming from a significant open position caused by a trader circumventing the internal system of limits

The type of model applied to the particular risk significantly depends on these characteristics.

Notwithstanding the exact definition, a risk is always related to an uncertainty. Hence, it is crucial for a risk manager to be able to apply techniques from financial

mathematics, probability theory, statistics, time series analysis, actuarial mathematics and theory of stochastic<sup>1</sup> processes.

## 1.2 Basic principles of bank regulation

In principle, the current bank regulation does not directly prevent banks from investing into more risky assets. Neither it imposes direct limits for risky positions. Hence the bank regulation is significantly different e.g. from the regulation of Slovak pension funds, which contains a number of hard limits related to the amount and structure of investments of these funds.

Contrary to pension funds, the basic principles of bank regulation are significantly different. Banks can take any size of risk, provided that all the risks are sufficiently covered by their own funds (i.e. capital).

The aim of the capital is to create a buffer which should absorb reasonably high amount of potential losses. In other words, if a bank suffers a loss, it should be borne by the owners of the bank and not by its clients.

How to calculate the amount of capital which should be required? Clearly, this should be somehow related to the riskiness of its activities.

This principle is expressed by so-called **capital adequacy ratio**:

$$\frac{\text{own funds}}{\text{risk-weighted assets}} \geq 8\%^2$$

Note that there is one obvious issue with this application of the basic principle: The required amount of capital should always be preserved because it is needed to absorb losses in a crisis. This is fair enough. The problem is that "always be preserved" means that it actually cannot be depleted in the crisis. To illustrate this on a more practical problem, consider a city which adopts a local law that there always has to be a taxi on the local train station to be prepared to pick up customers. You arrives to this city and ask the (only) taxi driver to take you to Šturák. She replies that as she is currently the only taxi driver on the station, she has to stay and wait for customers. Leaving the station would be a violation of the local law.

**Example 1.3.** Examples of risk weights associated with different types of claims:

- Traditional corporate loan: 8% covered by own funds (risk weight: 100%)

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<sup>1</sup>In this context, it is quite interesting to recall that the term "stochastic" comes from Greek: *Stochazesthai* means to aim, to guess. *Stochastikos* means good at aiming ( $\sigma\tau\omicron\chi\omicron\varsigma$  = aim).

<sup>2</sup>Let us note that the European banking authority currently expects at least 9%.

- Equity investment: 12% covered by own funds
- Mortgage loan: 4% covered by own funds (risk weight 50%)
- Interbank claim: 1.6% covered by own funds (risk weight 20%)
- Slovak government bond: 0% covered by own funds (risk weight 0%)

How are the risk weights determined?

In general, risk weights depend on the debtor's credit quality in case of loan, or on the riskiness of the particular asset. There are two basic approaches which might be used by the bank:

- (i) **Standardized approach:** The risk weights are simply set by the regulation based on the rating given by one of the major rating agencies. The bank does not need to analyse the risk more deeply.
- (ii) **Internal rating based approach:** This approach is based on a bank's proprietary model of the risk assessment. In this case, demanding qualitative criteria applies (mainly for internal processes, data quality etc.). However, unlike the standardised approach, the internal rating based approach can only be used upon a prior approval by the competent regulatory authority (NBS). Theoretically, the main advantage is the lower capital requirement compared to the standardised approach, but this might be conditional on specific cases.<sup>3</sup>

It should be stressed that the regulatory requirements related to the capital adequacy rules have never been meant to replace a well-established risk management process.

A bank is obliged to cover all significant risks. Sometimes this might be quite tricky (the requirement also includes e.g. strategic risk, legal risk, reputational risk, risk of a damage by an earthquake etc.). Naturally, the legislation sets the risk weights only for the most standard risks. Some of these other less typical risks are not covered by the calculation of risk-weighted assets. However, there is general principle that a bank should cover these risks as well by a so-called internal (economic) capital.

The most important document containing the details on the bank regulation is **Basel III**. This is a document prepared by the Basel Committee on Banking Supervision. It is a globally valid document – it is (formally) non-binding international agreement which should be observed by large internationally active banks.

However in most jurisdictions, the principles set in the Basel III are adopted as a binding law. In the EU, the rules on banking regulations are included in the Capital

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<sup>3</sup>Note the interesting opportunity for a mathematician working in the field of risk management.

Requirement Directive<sup>4</sup> (CRD IV<sup>5</sup>) and Capital Requirement Regulation<sup>6</sup> (CRR). Both CRD IV and CRR are in force from 1 January 2014, replacing CRD III which was valid before this date.

Regarding the legal framework in Slovakia, the CRR is directly applicable to all Slovak banks without any transposition into Slovak laws. The CRD IV has to be transposed in the Law on banks (No 483/2001 as amended) and in decrees issued by the NBS.

## 1.3 Process of the risk management

As shown in Figure 1.1, the process of the risk management has three main phases.

The risk identification is mainly focused on the risk factors which the institution is exposed to. It should be noted that any transactions comprises several types of risks, and mainly the "residual risks" (e.g. counterparty risk) can prove to be important during crisis. The risk identification will be described in more details in the rest of this chapter.

Risk measurement can be based on different types of metrics, from a simple monitoring of open positions to comprehensive statistical models and Value-at-Risk, which will be discussed in details in the next chapter.

Risk management comprises setting the risk limits based on the determined risk appetite, back testing (testing the quality of the models on the historical data) and stress testing (assessing the impact of severe but still plausible adverse scenarios).

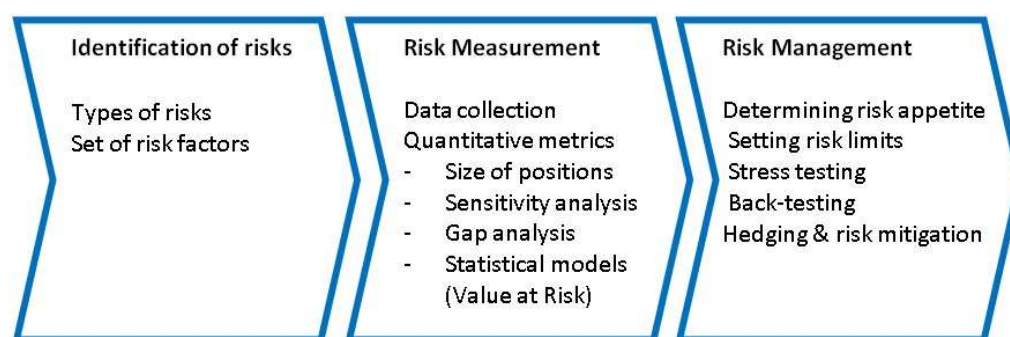


Figure 1.1: Process of the risk management.

<sup>4</sup>Directive 2013/36/EU of the European Parliament and of the Council of 26 June 2013 on access to the activity of credit institutions and the prudential supervision of credit institutions and investment firms.

<sup>5</sup>Without going more deeply into the history of bank regulation, note that as the number of the current version of the documents indicates, some rather important refinements are needed from time to time (mainly shortly after a new crisis, as a rule of thumb).

<sup>6</sup>Regulation (EU) No 575/2013 of the European Parliament and of the Council of 26 June 2013 on prudential requirements for credit institutions and investment firms.

## 1.4 A brief overview of main risk categories

### 1.4.1 Credit risk

The credit risk is a risk that a borrower fails to deliver payments which he or she is obliged to pay in case of different type of claims (bonds, loans, settlement of financial derivatives ...) because he or she is unable or unwilling to do so.

In addition to this "traditional" concept, credit risk also includes some other types of risk:

For example, consider an interest rate swap where the banks pays fixed interest rate and receives floating interest rate. If the interest rates increase, the value of the swap will be positive. However, if the financial situation of the counterparty worsens, the value of the swap decreases.

Other example is the drop of price of a bond if its issuer is downgraded. In such a case, the bond prices decreases even if no breach of contract has occurred.

### 1.4.2 Market risks

Market risk is a risk of an adverse impact of movements in market factors (equity prices, interest rates, FX rates, commodity prices) on the value of financial instruments.

Several types of market risk can be distinguished:

- Interest rate risk
- Equity risk
- FX risk, currency risk
- Commodity risk
- Credit spread risk

All market risks can be further divided into two main groups:

- (a) **General risk** is the risk which is related to worsening of the situation on the whole market. This may be caused for example by rising risk aversion of market participants, general loss of confidence, global economic slowdown or materialization of a systemic risk in the large part of financial sector.
- (b) **Specific risk** is risk which is relevant only to the a particular counterparty. For example, worsening situation of a particular firm may cause a drop in its equity and bond prices.

One of the important issues which should be taken into account when assessing market risk is that the value of financial instruments depends on a number of factors. For example, a price of an equity option is mainly driven by equity prices, but it can be also influenced by changes in volatility of the equity price (even if the price itself stays on the same level), interest rates, dividend policy and financial situation of the underlying counterparty (i.e. its credit spreads).

### Interest rate risk

Even if the definition of interest rate risk is fairly straightforward, its practical measurement rises several important issues which should be thoroughly considered by the risk manager:

- There are various types of interest rates on the financial markets. They differ in many characteristics – type of counterparties, type of contract, common maturities, liquidity of the respective market. When constructing yield curve, the following types of interest rates can be used: EURIBOR, EUR-LIBOR, FRAs, swaps, bond yield, loan rates, deposit rates etc. All these instruments have different liquidity for different maturities. For example, short-term rates are usually represented by EURIBOR. For longer horizon, we can use interest rate swaps or bond yields. However, interest rates might be different also for instruments with the same maturity. For instance, interest rate swaps are less exposed to credit risk than bonds and hence the respective yield might be lower.
- When compiling the yield curve from different sources, there are technical issues as well. For example, day convention might be different for individual financial instruments.
- Interest rates themselves are influenced by many risk factors, e.g. inflation, banks' credibility, sovereign risk, expectations about the development of the economy, monetary policy, banks' interest rate policy etc.
- The interest rate risk should be viewed from two different perspectives:
  - (a) **Bond revaluation risk:** The change in interest rates has a direct impact on the change of real value of bonds and derivatives. Hence, if the bank has to revalue a particular bond to real value<sup>7</sup>, it has to recognize the loss or profit in its profit and loss account or change in its equity. However, this change in real value will gradually vanish as the maturity of the bond becomes closer. Hence, bond revaluation risk is mainly relevant for financial instruments with long duration which are revaluated daily to its real value.
  - (b) **Risk of changes in interest income:** This risk is mainly relevant for bonds with floating coupons or for loans and deposits with floating interest rates.

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<sup>7</sup>Note that this the if the bond is held for trading or for sale. On the other hand, if the bank has decided to hold the bond to maturity, it is not obliged to revalue the bond on daily basis.

If the interest rates change, the interest revenues or interest costs change as well after its refixation. Hence, this risk has no immediate impact. Its impact, however, is important in longer time horizon, causing that the bank's net interest income is different than expected.

**Example 1.4.** Assume that the bank holds a five-year bond with a fixed coupon. In addition, assume that a parallel upward shift of the yield curve occurs. Immediately after this upward shift, the real value of this bond decreases. In case that the bond is revaluated on daily basis, the bank has to recognize the loss. This loss is however later compensated by profits as the real value increases back close to the bond's nominal as the maturity approaches. In case of a bond with floating coupons, the initial loss due to revaluation would be smaller, but the interest revenues in subsequent years would decrease.

### 1.4.3 Operational risk

Operational risk includes:

- Improper or faulty internal processes in the bank
- Human error
- Failure of the banks' internal system
- External or internal fraud (e.g. theft, forgery, bribery)
- External events (e.g. natural disasters, terrorism)

The operational risk is closely associated with the functioning of internal processes. Hence, the operational risk management is mainly focused on the qualitative side of processes, corporate governance.

Compared to other previously mentioned risks, operational risk is difficult to measure. Even if it is present everywhere, the identification of proper loss amounts and their sources is tricky. In addition, some categories of operational risk includes very extreme events with potentially very high impact.

One of the examples of a huge losses was the loss of EUR 4.9 billion due to an unauthorized trading at one of the largest French banks, Société Générale by Jérôme Kerviel in 2008.



### 1.4.4 Liquidity risk

Liquidity risk is a potential loss due to the fact that the bank is unable to meet its obligations on the contractual basis due to a lack of liquid funds.

There are three key aspects of the liquidity risk:

- (i) **Short-term liquidity risk:** Bank for insufficient funds to pay obligations when they are due no liquid assets that could be sold without much loss
- (ii) **Market liquidity risk:** Example: A bank needs to sell its securities to obtain sources of liquidity (fire sales), but the stock market is not liquid enough (only a small volume might be sold at the quoted price, then the price drops) The problem in turbulent market (widening bid-ask spread)
- (iii) **Funding liquidity risk:** Example: The bank is financed by short-term interbank deposits constantly renewed. At the time of distrust between banks, it might be significantly more expensive to continue with these funds

### 1.4.5 Additional risks

**Strategic risk** is the risk of a loss arising from a poor strategic business decision.

**Legal risk** usually refers to a loss due to a legal uncertainty. It may also include losses stemming from incorrect or incomplete legal documentation, unexpected changes in laws and litigation risk.

**Reputational risk** (reputation risk) is a risk of loss resulting from damages to a firm's reputation. It may cause significant drop in revenues or destruction of shareholder value, even if the company is not found guilty of a crime. The reputational risk can be mitigated by good governance practices and transparency.

**Model risk** is the risk of loss resulting from inappropriate decisions based on a model. For example, it may refer to the difference between the model-based value of a complex financial instruments and the price of this instruments which is actually traded on the market.

**Systemic risk** is a risk of collapse of an entire financial system or entire market. It can be defined as instability in the financial system. It might be caused by risks and vulnerabilities in the system, including herd behaviour and interdependencies which may reinforce the contagion risk. It also refers to systemically important financial institutions whose failure can potentially bring down the entire system or market.

### 1.4.6 Counterparty risk

The counterparty risk has been already mentioned as a specific type of credit risk. In this subsection, we would explore this risk in more details.

It should be noted that this risk poses some additional challenges. First of all, hedging against counterparty risk might be difficult. For example, there is often a significant level of concentration of derivative transactions in a single counterparty. In other words, large part of derivative transactions is traded with large international banking groups.

In addition, the existence of the counterparty risk makes traditional hedging strategies imperfect. For instance, traditional hedging of bonds by interest rate swaps (IRS) may still leave large residual risk: If the financial situation of the counterparty in the IRS transaction worsens, the real value of the IRS might decrease without any change in the value of the bond.

The impact of the counterparty risk has been more pronounced after the beginning of the financial and economic crisis in mid-2007. This is visible e.g. on the evolution of interest rates. To illustrate this issue, consider the following two strategies:

- (i) Put money to 6-month deposit yielding  $r_{0,0.5}$ .
- (ii) Put money first to 3-month deposit yielding  $r_{0,0.25}$  and then to other deposit for three months (the interest rate for this deposit is a forward rate for the period starting in 3 months and valid for 3 months ( $r_{0.25,0.5}$ )).

In theory, the yield of these two strategies should be the same, i.e.

$$(1 + r_{0.5})^{0.5} = (1 + r_{0,0.25})^{0.25} (1 + r_{0.25,0.5})^{0.25}.$$

In practice, this was indeed the case before the crisis. During the crisis, the yield of the second strategy was higher. The reason is that this strategy is less exposed to the counterparty risk given the bank can at least retain the yield after the first 3 months if the counterparty defaults between 3 and 6 months. Increasing the difference between the yields of these two strategies signals higher level of uncertainty on the market. This issue is illustrated on Figure 1.2<sup>8</sup>.

### 1.4.7 Interconnectedness between various types of risk

In some cases, the individual risk types might be difficult to distinguish. This is due to a deep interconnectedness between various risks as well as due to the fact that the identification of the original source of risk might be ambiguous. For example, if a loan is not repaid, this might be caused by financial distress of this client (credit risk) as

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<sup>8</sup>Source: Michael Nealon, Pricing IR Derivatives Post-Credit Crunch, SFMV Talk.

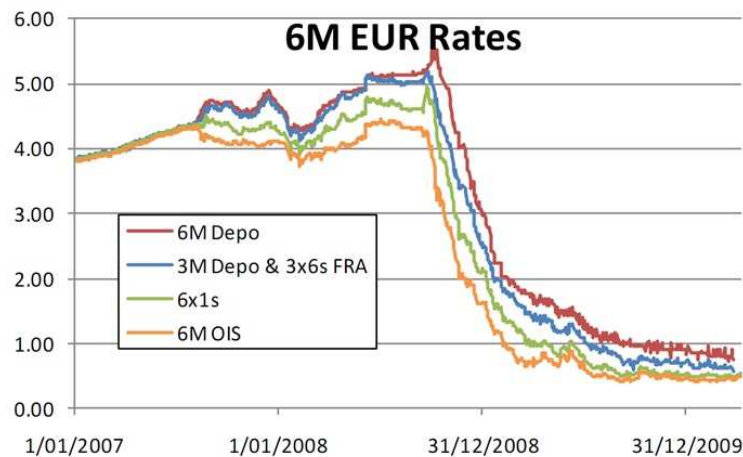


Figure 1.2: Interest rate for the period of 6 months based on financial instruments with different level of counterparty risk.

well as by an internal failure in the loan approval process (operational risk). However, in practice it might be more important to understand all the risks (together with their possible magnitude and negative consequences) than their exact classification.

As an example of risk interconnectedness caused by a possible contagion, consider the following (hopefully imaginary) *story of sorrow*:

- The management of the bank *Looser and Poor's* decides to implement a new (faulty) information system, introducing the possibility of circumventing dealers' limits (operational risk).
- Due to an error, a dealer enters into a significant position in Portuguese bonds (operational risk).
- A large increase in credit spreads on Portuguese bonds causes significant loss (market risk - specific interest rate risk).
- The bank tries to close the position, but the market is illiquid (market liquidity risk).
- The name of the bank and the amounts of the losses leaks to public, causing a reputational damage (reputational risk).
- Deposit withdrawals rise significantly, the bank has a shortage of liquid assets (short-term liquidity risk).
- As a result of these significant additional losses, the bank may become insolvent and subsequently becomes unable to pay its contractual obligations (credit risk to other market participants).

### 1.4.8 The stylised facts of financial data

In this section, we will have a closer look on some properties of financial data which should be duly taken into account when proposing a model for the risk management:

1. **Non-normality:** Many models assume that the underlying distribution of assets returns or risk factors changes is normal. However, real data are usually far from this assumption. In particular, they exhibit **fat tails** (leptokurtosis) and asymmetry (skewness). In addition, the wrong assumption of normality has severe consequences: For example, probabilities of extreme events tend to be significantly underestimated. The implied volatilities calculated from options are not independent from strike price, instead we obtain so-called volatility smile (or volatility smirk).
2. **Non-stationarity:** Data are not stationary, i.e. the parameters of the underlying distribution change over time. For example, volatility often exhibits **volatility clustering**: on the financial markets, we can observe periods of low volatility (quiet periods) which are then followed by a turmoil with high level of uncertainty and risk aversion.
3. **Incomplete markets:** One of the important assumptions in asset pricing theory is that markets are complete, i.e. any financial instrument necessary for hedging strategies exists and is actively traded on the market. This assumption is far from being true, mainly in period of high uncertainty when the liquidity vanishes. Hence, it is not possible to make a perfect hedge which is assumed in the models. This might have significant impact on the deviation between theoretical and real prices of assets. Moreover, the completeness of the market does not occur even in quiet periods. For example, when hedging the interest rate risk of bonds by interest rate swaps, the counterparty risk still remains and is very difficult to hedge. For example, if the bank tries to hedge the counterparty risk with a third counterparty, it is still exposed to the risk of this third counterparty. If it receives collateral, then it is exposed to the market risk of the collateral value. If it trades the interest rate swap with a central counterparty, then it is exposed to margin calls.
4. **Non-existence of risk-free rate:** Many pricing models require a risk-free rate in their pricing formula. A risk-free instrument, however, does not exist on the market and has to be approximated. This approximation might turn to be false, mainly in the crisis period.
5. **Complex dependencies:** First, the dependencies between risk factor might be non-linear and might not be fully captured by correlations. Second, the level of dependencies might be different for small changes and for extreme changes. Third, the structure of dependencies might change in time. In particular, correlations tend to be higher in crisis period which limits the effects of diversification.

### 1.4.9 Exercises

**Example 1.5.** Identification of risks related to a bond. Slovenská Sporiteľňa purchased a bond issued by Slovak Railways Company (ŽSR) with a fixed coupon of 4% and maturity 5 years. This investment has been financed by retail term deposits with remaining average maturity 9 months. Identify the risk factors in this transaction!

In this transaction, the following risk factors should be considered:

- General interest rate risk:
  - An increase in long-term interest rates (a negative revaluation of bond)
  - The increase in short-term interest rates (higher costs to recover deposits)
- Specific interest rate risk:
  - The increase in ŽSR's credit spread (financial difficulties, reputational risk)
- Credit risk:
  - ŽSR will be unable to pay its obligations once they are due
  - Loss of reputation, ŽSR's credit rating downgrading
- Systemic risk:
  - The economic crisis would cause a significant drop in the transportation sector
- Funding liquidity risk:
  - Significant early withdrawal of deposits - when the market is liquid enough, this should not cause any loss except a possible disruption in bank's strategy
- Market liquidity risk:
  - The bank needs to sell the bond due to an early withdrawal of deposits, but there is no buyer for it

**Example 1.6.** Identification of risks related to an equity forward. VÚB entered into a forward contract with Commerzbank related to a purchase of Facebook shares with a maturity of one year. (Recall that the pricing function of a forward contract is  $F_t = S_t e^{r(T-t)}$ .) Which risk factors should the risk manager consider?

Possible risk factors are the following:

- General equity risk:

- Adverse changes in the Facebook share price due to problems in the IT industry
- Adverse changes in the Facebook share price due to the turmoil in the stock market as a whole
- Specific equity risk:
  - Facebook shares change in prices due to changes in their financial situation or perceived market
- General interest rate risk:
  - Changes in interest rates affects the value of the forward
- Counterparty credit risk
  - Commerzbank's default
  - The increase in credit spread of Commerzbank and / or a downgrade of the Commerzbank
- Systemic risk:
  - The general increase of risk aversion on the market

The counterparty credit risk (notably the effect of a possible downgrade of the Commerzbank) would be even more pronounced when the value of the forward contract is positive, due to an increase of the uncertainty whether the Commerzbank will be able to repay its obligation. However, it might also affect the value of the forward even when it is negative, mainly in the case of long remaining maturity when the probability of stepping to the positive territory is not negligible.

Note that if the contract would be a future contract instead of a forward contract<sup>9</sup> the counterparty credit risk would vanish, but it would be replaced by the risk of an additional margin call in case of a VÚB's downgrade.

**Example 1.7.** Stylised facts related to market data. Download stock prices for Microsoft and Citybank from [finance.yahoo.com](http://finance.yahoo.com) and calculate their daily returns. Create a Matlab program which will illustrate the following data properties:

- **Non-normality**
  - Plot the histogram of the data vs normal distribution.
  - Test the normality of the data using the Jarque-Bera test.

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<sup>9</sup>Recall that futures are standardised contracts which are exercised through a central clearing counterparty, whereas forwards are bilateral contracts. When the value of a future decreases, the central clearing counterparty requires a collateral to be posted by the bank (this is so called margin call).

- Display the quantile-quantile (QQ) plot of the sample quantiles of data vs theoretical quantiles from normal distribution.
- Display the QQ plot using theoretical quantile from Student t distribution using an estimated value of the degree of freedom.
- **Fat tails**
  - Compare the right hand tail of the empirical distribution with the fitted normal distribution.
- **Volatility clustering**
  - Using the autocorrelation function, show that the autocorrelation in the returns is rather small, but the autocorrelation in squared returns is significant.
  - Plot the estimated standard deviation calculated for each individual year.
- **Nonlinear dependence**
  - Compare the empirical multivariate distribution of returns for two different stocks with data simulated from multivariate normal as well as multivariate Student t distribution.
  - Plot the estimated correlation calculated for each individual year.
  - Calculate sample exceedance correlations. Compare them with exceedance correlations for data simulated from fitted multivariate normal as well multivariate Student t distribution. Given that  $X$  and  $Y$  are two stock returns, exceedance correlation is defined as

$$\rho(p) = \begin{cases} \text{corr}(X, Y | X \leq Q_X(p) \wedge Y \leq Q_Y(p)) & \text{for } p \leq 0.5, \\ \text{corr}(X, Y | X > Q_X(p) \wedge Y > Q_Y(p)) & \text{for } p > 0.5, \end{cases}$$

where  $Q_X(p)$  and  $Q_Y(p)$  are  $p$ th quantiles of  $X$  and  $Y$ , respectively.