Chapter 5

Credit risk

5.1 Basic definitions

Credit risk is a risk of a loss resulting from the fact that a borrower or counterparty fails to fulfill its obligations under the agreed terms (because he or she either cannot or does not want to pay).

Besides this definition, the credit risk also includes the following risks:

- **Sovereign risk** is the risk of a government or central bank being unwilling or unable to meet its contractual obligations.
- **Concentration risk** is the risk resulting from the concentration of transactions with regard to a person, a group of economically associated persons, a government, a geographic region or an economic sector. It is the risk associated with any single exposure or group of exposures with the potential to produce large enough losses to threaten a bank's core operations, mainly due to a low level of diversification of the portfolio.
- **Settlement risk** is the risk resulting from a situation when a transaction settlement does not take place according to the agreed conditions. For example, when trading bonds, it is common that the securities are delivered two days after the trade has been agreed and the payment has been made. The risk that this delivery does not occur is called settlement risk.
- **Counterparty risk** is the credit risk resulting from the position in a trading instrument. As an example, this includes the case when the counterparty does not honour its obligation resulting from an in-the-money option at the time of its maturity.

It also important to note that the credit risk is related to almost all types of financial instruments. To illustrate this, let us consider the following examples:

- **Loans and receivables:** failure to repay the principal or interests.
- **(Debt) securities:** failure to repay the nominal value or coupons.
- **Guarantees issued:** in case of a failure of the guarantee, the guarantor has to settle the borrower's obligations.
- **Promises for granting a loan / undrawn credit facilities:** if a client is in troubles, he can draw his loan that has been already approved but not yet used and then default on his debt. As an example, people having mortgages might use the limit on their credit cards in order to continue repaying their mortgages (in order not to lose their home) but not repay the credit card obligation.
- **Derivatives:** failure to meet the contractual payments, for example the cash settlement at maturity. Mainly derivatives with currently positive value are subject to credit risk. However, credit risk might decrease the value of derivatives with negative value as well since this value might turn to be positive in the future.

5.2 General principles of credit risk modelling

In the credit risk modelling, one of the most crucial issues is the definition of a credit event which indicates that the loss has occurred. Naturally, the credit event is the failure of payments, bankruptcy or default. The practical question is, however, how to determine that a loan which has not been repaid for some time period is indeed non-performing, implying that the loss should be recognized by the bank. For example, if the client is past due 10 days, this may already indicate that he or she has some troubles with repaying the loan, but it may be only a technical issue as well. Hence, it is important to properly determine the threshold for the past due period. In the credit risk management, the standard assumption is that a loan is considered to be defaulted (or non-performing) when the client is past due at least 90 days (3 months).

When the bank has indications that a loan might not be fully repaid (e.g. due to some past due payments), it should recognize the loss and create provisions (also called reserves or allowances). In the accounting framework, this operation decreases the value of the loan (negative value on the side of asset) and creates costs (which imply a decrease in net profits and hence equity). For this purpose, the bank estimates so called expected loss (EL). The expected loss is based on the value of the loan (i.e. the exposure at default, EAD) multiplied by the probability, that the loan will default (i.e. probability of default, PD). In addition, the bank takes into account that even when the default occurs, it might still get back some part of the loan (e.g. due to the bankruptcy procedure). Hence, the previous figure is further multiplied by the estimation of the part of the loan which will be lost in case that a default occurs (i.e. loss given default, LGD).

Figure 5.1: Distribution of credit losses.

To sum up, the expected loss is calculated as follows:

$$EL = PD \times LGD \times EAD = PD \times (1 - RR) \times EAD,$$

where :

PD = probability of default

LGD = loss given default

EAD = exposure at default

RR = recovery rate ($RR = 1 - LGD$).

Expected loss is covered by revenues (interest rate, fees) and by loan loss provisions (based on the level of expected impairment).

The expected loss corresponds to the mean value of the credit loss distribution. Hence, it is only an average value which can be easily exceeded. Therefore, we define the unexpected loss as difference between a high quantile (i.e. 99 %) and the expected loss. Banks should hold enough capital in order to fully cover the unexpected loss.

When modelling credit risk losses, several important issues should be taken into account:

- Defaults are relatively rare events compared to market losses. The lack of available data is an issue for both calibrating the models as well as backtesting.
- Correlations between failures have a significant impact on the final result. They should not be underestimated.
- Wrong way exposure (growing utilization of credit cards in case of an increase in PD) should be taken into account.

- In case of deterioration of the situation, both the PD and LGD may worsen. An assumption on their mutual independence is not realistic.
- Concentration risk should be taken into account in the loan portfolio, mainly in case of its low granularity.
- Credit exposures are not only subject to idiosyncratic risk of individual borrowers or counterparties, but to the systemic risk as well. For example, a drop in real estate prices will negatively affect the whole construction industry; changes in FX rates can have impact on exporters).
- The loss distribution has fat tails and is not symmetric.

Credit risk models can be divided into two broad categories:

(a) **Structural models:** These models assume that a default can be explained by a specific trigger point, for example it can be caused by a decrease in asset value below some threshold (i.e. the value of debt). The value of assets itself is modelled as a stochastic process.

(b) **Reduced-form models:** These models assumes that defaults are driven by a default intensity. No specific trigger event is assumed, but the default intensity (or default rate) might depend on changes in external factors (GDP growth, inflation, unemployment, interest rates etc.). This relationship is estimated using historical data and econometric techniques.

5.3 Merton and KMV model

5.3.1 Merton model

Merton model is one of the structural models. The model estimates the probability of defaults of a corporate based on a simple structure of its balance sheet. The balance sheet consists of assets (A_t) and liabilities, where the latter can be further divided into debt (D_t) and equity (E_t), i.e. $A_t = D_t + E_t$. As in other structure models, we have one observable binary state variable (default/non-default), which is based on a latent variable. In the case of Merton model, the latent variable is the real value of assets (A_t). This variable is however not directly observable, since we only know the firm's accounting value of assets. In addition, the asset value is not observable daily, but only when accounting reports are published (usually on quarterly basis).

The main assumption of the model is that the firm defaults when the real value of its assets is below the value of its debt at the time of the maturity of the debt (T), i.e. $A_T < D_T$.

In this case, the owner of the firms gets nothing and the value of its equity is zero ($E_T = 0$). The creditor gets back only part of his debt, i.e. his pay-off is only A_T instead of D_T. Note that due to the principle of the limited liability of the owner, he loses all his capital in the firm, but he is not obliged to refund the creditor.

On the contrary, if at time T the real value of assets exceeds the value of debt ($A_T > D_T$), then the debt is fully repaid and the value of the equity equals to $E_T = A_T - D_T$.

The real value of assets is assumed to follow a stochastic process. In particular, we assume that A_T follows geometric Brownian motion

$$dA_t = \mu_A A_t + \sigma_A A_t dW_t,$$

where W_t is Wiener process. This means that $\ln A_t$ is normally distributed, i.e.

$$\ln A_t \sim N\left(\ln A_0 + \left(\mu_A - \frac{1}{2}\sigma_A^2\right)t, \sigma_A^2 t\right).$$

Using the stochastic calculus (Ito's lemma), we can derive the probability of default at time t as follows:

$$P(A_T < D_T) = P(\ln A_T < \ln D_T) = \Phi\left(\frac{\ln\frac{D_T}{A_t} - \left(\mu_A - \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}\right).$$

While the value of debt at the time of its maturity (T) can be found e.g. in accounting reports of the firm, the real value of assets cannot be observed. Even if the drift (μ_A) can be estimated using the data from financial reports, to calculate the olatility (σ_A) we need to know A_t at daily basis. In this case, the quarterly frequency of accounting reports is not sufficient.

Note however that if the equities of the firm are listed and actively traded on a stock exchange, we know the daily value of E_t (as the number of shares issued multiplied by the value of one share). The daily values of A_t can be then estimated from the daily values of E_t using the following procedure.

Given that A_T is random variable, we do not know which of the two states mentioned above (default or non-default) occurs. We can, however, summarize the wealth of the owners at time T as

$$E_T = \max(A_T - D_T, 0).$$

Note that this actually is the pay-off of a call option written on the real value of the firm's assets as an underlying and D_T as strike price. Indeed, it makes an economic sense: At the time of maturity, the equity holders (i.e. the owners) have the right (but not the obligation) to pay-off the creditors and take over the remaining assets of the firm at time T. If they decide to do so, the firm continues to operate. Otherwise, the firm vanishes (goes bankrupt).

Given this, the value of E_t can be computed using the Black-Scholes option pricing formula

$$E_t = \mathrm{call}(A_t, \mu_A, \sigma_A, r, D_T, T), \tag{5.1}$$

where r is the risk-free rate and

$$\mathrm{call}(A_t, \mu_t, \sigma_t, r, D, T) = A_t N(d_1) - De^{r(T-t)} N(d_2),$$

$$d_1 = \frac{\ln \frac{D_T}{A_t} - \left(r - \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A \sqrt{T-t}}, \qquad d_2 = d_1 - \sigma_A \sqrt{T-t}.$$

In addition, we can assume that E_t follows geometric Brownian motion as well:

$$dE_t = rE_t + \sigma_E E_t dW_t.$$

Since

$$E_t = \mathrm{call}(A_t, \mu_t, \sigma_t, r, D, T) = V(A_t, t),$$

the Ito's lemma yields

$$dE_t = \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial A_t} A_t r + \frac{1}{2} \frac{\partial^2 V}{\partial (A_t)^2} (A_t \sigma_A)^2 \right] dt + \frac{\partial V}{\partial A_t} A_t \sigma_A \, dW_t.$$

Comparing the coefficients multiplying the Wiener process in the previous two equations yields

$$\sigma_E E_t = \frac{\partial V}{\partial A_t} A_t \sigma_A \tag{5.2}$$

Taking (5.1) and (5.2) we obtain a system of two non-linear equations with two unknown variables (A_t, σ_A) for all t. Note that E_t is observable daily and σ_E can be calculated using the time series of E_t.

5.3.2 KMV model

KMV model[1] is an upgrade of the Merton model which has been proposed and used by Moody's. In this model, several improvements of the Merton model are implemented, mainly in the following areas:

- **Structure of the debt:** The Merton model in its original setting assumes that the whole debt of a firm is represented by one single zero coupon bond or one single loan which is fully repaid at maturity. This assumption is obviously rather unrealistic. Firms commonly have a complex debt structure, including different types of short-term and long-term debt instruments (mainly loans from banks, liabilities to

[1] KMV = Kealhofer, McQuown & Vašíček

other firms and bonds) with different maturities and different structure of pay-offs (e.g. bonds are mainly repaid at maturity, loans may be repaid in several uniformly distributed payments). Hence, the assessment of whether the value of the firm's assets is sufficient to repay its debt is more complex than in the original setting. One of the possible solutions is to assume that the value of the debt (i.e. the strike price of the call option) equals to the amount of short-term debt + one half of the total amount of the long-term debt.

- **Threshold for default:** It should be taken into account that the real value of the firm's assets is not directly observable on the market; stakeholders have only access to the delayed information on the accounting value of the firm's assets. Hence, in practice, the firm might be able to operate even after the real value of its assets is lower than its payable debt. The reason is that the firm might be able to rise additional debt which can be used to repay the old debt, given that stakeholders do not know that the firm should be actually defaulted.

- **Loss distribution:** The assumption of normal distribution in the Merton model is not required. It can be replaced by a distribution based on the empirical data. In particular, MoodyŠs uses an empirical relationship between so-called distance to default (DD) and the expected default frequency (EDF), where

$$DD_t = \frac{\ln \frac{D_T}{A_t} - \left(\mu_A - \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A \sqrt{T-t}}.$$

5.3.3 Strong points and weaknesses of Merton / KMV model

One of the main advantages of the Merton / KMV framework is that it is able to adopt very promptly to any information related to the worsening of the firm's financial situation, provided that this information is captured by the changes in the equity prices. On the other hand, the model-based PD estimates might be too volatile in periods of significant changes in the equity prices, even if these changes are related more to changes in the general sentiment of market participants than to idiosyncratic changes of the firm's situation itself.

It should be noted, however, that the data requirements of this framework are rather demanding. In particular, the model can only be applied to companies listed and traded on a stock exchange. Hence, this framework cannot be directly applied to the majority of Slovak companies. In addition, the framework requires data from financial statements of the firm. Such data have low frequency and are seriously delayed.

In addition, the main weakness is that model yields reasonable results only after taking into account the adjustments proposed in the KMV model. However, as indicated above, these adjustments are rather arbitrary (i.e. setting the default thresholds). In addition, construction of the empirical EDF requires significant amount of reliable data.

Without incorporating these refinements, the model usually performs purely in its original setting.

5.3.4 Further reading

The following diploma theses (in Slovak) are a good reference for a basic overview of the Merton model framework and its application:

- A. Pišková (2004): Modelovanie portfólia dlhopisov s uvažovaním rizika defaultu
- K. Kadlečíková (2009): Ocenenie Credit default swapov a porovnanie ich vývoja v čase finančnej krízy

5.4 CreditMetrics

CreditMetrics was introduced in 1997 by J. P. Morgan and other co-sponsors banks. The basic principle of this framework is the statistical modeling of changes in credit ratings (external or internal ratings) of individual investments (bonds or loans) in the bank's portfolio. It is actually a value-at-risk approach with a long-term horizon (e.g. 1 year), which is also applicable to non-tradable debt instruments (such as loans or privately placed bonds). It is a reduced-form model, since it is based on the possible credit rating changes, but is does not rely on any latent factors or trigger events driving these changes.

The output of this model is the distribution of the economic value of the portfolio one year ahead (expressed in the present value at that time).

To illustrate the main principle, assume that we have bond (or loan, it does not matter at all) which is currently rated as A. Assume in addition that it is a simple bond with four year maturity and annualy paid coupon 5 %, which corresponds to the actual market 4-year yield. One year ahead (which is the risk horizon), this bond might be still rated A, or the rating might change. The probabilities of the rating changes can be estimated using historical data and are summarized in so-called 1-year *transition matrix*. One example of such a matrix is Table 5.1. Based on this table, the rating of the bond will not change with probability 87.27 %. Alternatively, the bond may be upgraded to AAA (with probability 0.04 %) or AA (1.91 %), or it can be downgraded to BBB (5.44 %), BB (0.38 %) etc. It may even default with probability 0.08 %.

Assuming that the market 4-year yield is the same as the coupon rate, the economic value[2] of the bond one year ahead will be equal to 100. Furthermore, the economic value

[2] If the bond is not held for trading or for sale, the bank holding this bond might keep its accounting value unchanged. The economic value, however, refers to the real value of this bond irrespective of whether it is reported in accounting or real value.

From/to	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	87.19	8.69	0.54	0.05	0.08	0.03	0.05	0.00	3.37
AA	0.56	86.32	8.30	0.54	0.06	0.08	0.02	0.02	04.9
A	0.04	1.91	87.27	5.44	0.38	0.16	0.02	0.08	4.72
BBB	0.01	0.12	3.64	84.87	3.91	0.64	0.15	0.24	6.42
BB	0.02	0.04	0.16	5.24	75.87	7.19	0.75	0.90	9.84
B	0.00	0.04	0.13	0.22	5.57	73.42	4.42	4.48	11.72
CCC/C	0.00	0.00	0.17	0.26	0.78	13.67	43.93	26.82	14.37

Table 5.1: 1-year transition matrix, Standard & PoorŠs, Global Corporate Average Transition Rates (1981-2011), http://www.standardandpoors.com/spf/ratings/DefaultStudy.pdf Table21

of this bond one year ahead increases in case of its upgrade and decreases in case of downgrade. These changes in the economic value is implied by different forward yields (for the period starting one year ahead and ending at the time of maturity of the bond) for different rating categories: Apparently, the yield of a bond with better rating is lower than the yield of a bond with worse rating, since the higher yield has to compensate higher credit risk in this case. Hence, assuming an upgrade to AA, in order to obtain the economic value of the bond one year ahead, we have to discount the cash flows during the period starting one year ahead and ending at the time of maturity with discount rates derived from the forward AA curve. Repeating this computation for all rating categories and combining the results with the transition probabilities yields the distribution of the economic value of this bond one year ahead. Note that in case of default, the economic value one year ahead is based on the recovery rate.

The procedure illustrated on the previous example can be summarized as follows:

(i) Determine the probability of changes in the credit ratings in one year horizon.

(ii) Calculate the economic value one year ahead as the present value of cash flows during the period starting one year ahead and ending at the time of maturity for each individual rating based on the forward interest rates for different ratings.

(iii) Based on the transition probabilities and economic values for different ratings, construct the distribution of the economic value one year ahead. Based on this distribution, we can calculate the expected loss as the difference between current price and the mean of this distribution (for securities priced at market value, this should be close to zero) as well as the unexpected loss based on the chosen quantile (i.e. 99 %).

To write this procedure in mathematical terms, we use the following notations:

n = the number of rating classes

T = remaining maturity

N = nominal value

c = coupon rate (assuming yearly coupons)

P_{ij} = 1-year transition probability from rating i to rating j, $i, j = 1, \ldots, N$

$R_j^{1,t}$ = forward yield (p.a.) of an asset with rating j for the period starting one year ahead and ending at time t

The aim is to estimate the distribution of the economic value of the asset one year ahead provided that its current rating is i, which is constructed as N economic values with assigned probabilities

$$\left\{ P_{ij}, \sum_{k=2}^{T} \frac{cN}{(1+R_j^{1,k})^{k-1}} + \frac{N}{(1+R_j^{1,T})^{T-1}} \right\}_{j=1}^{N}$$

Let us note that the calculation for more assets is based on Monte Carlo simulation based on scenarios of credit ratings for each asset. In this case, the exact details are beyond the scope of this text; an interested reader might consult the CreditMetrics technical document (see Further reading).

5.4.1 Strong points and weaknesses of CreditMetrics

The main advantages of this framework as compared to Merton / KMV model are the following:

- The CreditMetrics has significantly lower sensitivity to short-term turmoil period.
- The application is broader: CreditMetrics can be applied to non-listed companies as well provided that their external or internal rating exist.
- Using the transition matrices estimated over a long period allows us to measure the amount of risk that is largely unchanged over the economic cycle (the approach is less procyclical).

On the other hand, one of the main weaknesses is that changes in ratings are delayed compared to market signals. In other words, the rating-based information is significantly delayed behind the information which can be obtained from equity prices.

In addition, one of the strong assumptions of the model is the Markovian property of credit rating changes: The model is based on the assumption that the probability of a particular rating change depends only on the current rating, irrespective of history of previous changes. This assumption is clearly not correct: In practice, a firm which has recorded several credit rating downgrades in its recent history is more prone to a further downgrade than a firm which has had a stable or improving rating for a long period.

On the other hand, the Markovian property allows us to easily extend the risk horizon (which is originally set to one year): Assuming this property, the n-year transition matrix is simply the n-th power of one-year matrix.

5.4.2 Further reading

A detailed description of this framework can be found in its original documentation:

Gupton, G. M., Finger, Ch. C., Bhatia, M.: CreditMetrics – Technical Document, J. P. Morgan, New York, April 1997.

5.5 CreditPortfolioView (CPV)

The CPV model assumes that the probability of default varies in time and depends on a set of macroeconomic variables. For example, it is reasonable to assume that the default probability increases in a recession which is characterised by a fall in the GDP and rise in unemployment.

Hence, the PD is estimated based on macro variables. Instead of using a simple linear regression, the model employs a so-called logit transformation. The reason is that PD attains value only from the interval $(0,1)$, other values are not feasible.

In particular, the logit model assumes that PD is obtained by a logistic transformation of a macroeconomic index y_t:

$$PD_t = \frac{1}{1+e^{-y_t}}.$$

This macroeconomic index is assumed to be a linear combination of macro variables:

$$y_t = \beta^T X_t + \varepsilon_t.$$

The logit model is obtained by combining these two equations:

$$PD_t = \frac{1}{1+e^{-\beta^T X_t - \varepsilon_t}}.$$

When estimating this model, the historical PDs are replaced by observed default rates.

In order to estimate the PDs for the following period, we need to know the future values of the macroeconomic variables. In practice, the following approach is often used: The individual macroeconomic variables are modeled as AR(2) processes, i.e.

$$X_t^i = \theta_0^i + \theta_1^i X_{t-1}^i + \theta_2^i X_{t-2}^i + u_t^i.$$

Then we used this model to simulate a number of scenarios of the future value of these variables. For each of these scenario, the value of the PD is calculated using the logit model. Finally, we calculate the average of these values.

The main advantage of this model is that it can be easily applied to retail portfolios (e.g. mortgages or consumer loans) as well loans to small and medium enterprises since, unlike the previous models, it does not rely on information like equity prices or credit rating. Hence, this is a framework which commonly serves as a basis of credit models applied by Slovak institutions. In practical applications, the model is often extended by including additional indicators other than macroeconomic variables, e.g. indicators of profitability or balance sheet structure (ROE, leverage, liquidity...) of the corporate sector or indicators on the situation of the households. These indicators might be aggregated for the whole sectors or its sub-segments, or they might be included as firm- or household specific characteristics.

5.6 CreditRisk+

This model is based on the approaches used in the modelling of losses in non-life insurance. It has been proposed by Credit Suisse.

CreditRisk+ is a simulation-based model. The basic principle of this framework is that the number of defaulted loans (*default frequency*) and size of losses (*default severity*) are modelled independently based on some assumptions of these two distributions.

The simulations are based on the following procedure:

1. Divide the loan portfolio to approximately homogeneous groups according to their size and LGD. Denote the number of these groups by N and the average size of the loans in individual groups by L_i, $i = 1, \dots, N$.
2. For each individual groups, estimate the default frequency (i.e. the number of the loans defaulting in one year horizon) using historical data; denote this estimated default frequence by λ_i, $i = 1, \dots, N$.
3. Assume that the actual number of defaults in each group can be simulated from Poisson distribution, where
$$P(\text{number of defaults} = m) = \frac{\lambda_i^m \, e^{-\lambda_i}}{m!}$$
(this might be a reasonable assumption provided that defaults are independent).
4. For each group $i = 1, \dots, N$, draw the number of defaults (m_i) from the respective Poisson distribution and calculate the total loss as
$$\sum_{i=1}^{N} m_i \, L_i \, LGD_i,$$

5. Repeating this calculation for a large number of simulation yields a distibution of total loss. Using this distribution, we can calculate both expected as well as unexpected loss.

5.7 Regulatory approach

As we have already indicated in the description of previous models, in some cases we only know the estimation of the default probability but the whole loss distribution is not known. The calculation of the minimum capital requirement is based, however, on the estimation of unexpected loss.

Therefore, the regulatory approach is a model which allows banks to calculate the unexpected loss based on their internal estimation of probability of default. We will present here the model for the portfolio of retail loans; the models for other asset classes are, however, based on the same principle. For retail exposures, the formula is given directly in the EU law (Article 154 of Regulation No 575/2013 on prudential requirements for credit institutions and investment firms) as follows:

$$RW = 1.06 \times 12.5 \times \left[\underbrace{LGD \times \Phi\left(\frac{1}{\sqrt{1-R}} \Phi^{-1}(PD) + \sqrt{\frac{R}{1-R}} \Phi^{-1}(0.999) \right)}_{total\ expected\ +\ unexpected\ loss} - \underbrace{LGD \times PD}_{expected\ loss} \right], \tag{5.3}$$

where Φ denotes cumulative distribution function of the standard normal distribution and R denotes correlation (further details on this correlation will be given later).

Note this formula yields the risk weight instead of minimum capital requirement. The calculation is based on the unexpected loss, which is obtained as the expected loss ($PD \times LGD$) subtracted from total loss conditional to a significantly negative situation on the market (corresponding to the 99.9% quantile). Recall that according to the general rule mentioned in the introductory lecture, banks have to cover 8% of risk-weighted exposure by capital. Hence, total unexpected loss is mupliplied by 12.5, which is the inverse value to 8%. Furthermore, the scaling factor 1.06 is an ad-hoc factor which is imposed in order to stay on the conservative side.

Now, we will provide a detailed derivation of the loss conditional to a large market drop (stated in parentheses in the formula given above).

First, let us use the following notation:

V_i = nominal value of the i-th loan

W_i = actual value of the i-th loan at the time of maturity (random variable)

In addition, we introduce the following assumption:

Assumption 1: W_i has a lognormal distribution, i.e. $\ln W_i = r + \sigma_i X_i$, where X_i is a random variable from $N(0,1)$.

We assume that the loan i defaults when $W_i < V_i$. We assume the probability of this event is PD, which is one of the inputs to the model. Hence one has

$$\begin{aligned} PD &= P(W_i < V_i) = P(\ln W_i < \ln V_i) = P(r + \sigma_i X_i < \ln V_i) \\ &= P\left(X_i < \frac{\ln V_i - r}{\sigma_i}\right) = P(X_i < a_i) = \Phi(a_i), \end{aligned} \tag{5.4}$$

where

$$a_i = \frac{\ln V_i - r}{\sigma_i}.$$

In other words, a default is caused by a decrease of the random variable X_i below certain threshold a_i.

As an additional assumption, we will decompose the random variable X_i to two components.

Assumption 2: Random variable X_i can be written as follows:

$$X_i = \sqrt{R}\, Y + \sqrt{1-R}\, \varepsilon_i, \tag{5.5}$$

where all random variables $Y, \varepsilon_1, \varepsilon_2, \ldots$ are from $N(0,1)$ and mutually independent.

Note in this setting, the random variable Y can be interpreted as a component corresponding to the systemic risk of the whole market, whereas random variables ε_i correspond to the idiosyncratic risk of each loan. It can be easily verified that the property that X_i has a standardized normal distribution is preserved. Indeed,

$$E[X_i] = E\left[\sqrt{R}\, Y + \sqrt{1-R}\, \varepsilon_i\right] = \sqrt{R}\, E[Y] + \sqrt{1-R}\, E[\varepsilon_i] = 0$$

and

$$Var[X_i] = Var\left[\sqrt{R}\, Y + \sqrt{1-R}\, \varepsilon_i\right] = R\, Var[Y] + (1-R)\, Var[\varepsilon_i] = R + (1-R) = 1,$$

where we have used the independence between Y and ε_i.

In addition, it can be shown that the correlation between X_i and X_j for any $i \neq j$ is R:

$$Cov[X_i, X_j] = E\left[\left(\sqrt{R}\, Y + \sqrt{1-R}\, \varepsilon_i\right)\left(\sqrt{R}\, Y + \sqrt{1-R}\, \varepsilon_j\right)\right] = E[R\, Y^2] = R,$$

where we have used that $E[\varepsilon_i \varepsilon_j] = 0$ for any $i \neq j$ and $E[Y\, \varepsilon_i] = 0$.

Now we will calculate the conditional probability of default of a loan, provided that the systemic component reaches a significantly negative value (e.g. $Y = \Phi^{-1}(0.001)$).

$$p_i := P(W_i < V_i)\Big|_{Y=\Phi^{-1}(0.001)} = P(X_i < a_i)\Big|_{Y=\Phi^{-1}(0.001)}$$

$$
\begin{aligned}
&= P\left(\sqrt{R}\,Y + \sqrt{1-R}\,\varepsilon_i < a_i\right)\Big|_{Y=\Phi^{-1}(0.001)} \\
&= P\left(\sqrt{R}\,\Phi^{-1}(0.001) + \sqrt{1-R}\,\varepsilon_i < a_i\right) \\
&= P\left(\varepsilon_i < \frac{a_i - \sqrt{R}\,\Phi^{-1}(0.001)}{\sqrt{1-R}}\right) = \Phi\left(\frac{a_i - \sqrt{R}\,\Phi^{-1}(0.001)}{\sqrt{1-R}}\right).
\end{aligned}
$$

Notice that $\Phi^{-1}(0.001)$ can be rewritten as $-\Phi^{-1}(0.999)$ and (5.4) implies that $a_i = \Phi^{-1}(PD)$. Hence one has that the default probability of the i-th loans can be written as:

$$
p_i = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{R}\,\Phi^{-1}(0.999)}{\sqrt{1-R}}\right).
$$

Now, we introduce the last assumption:

Assumption 3: The portfolio is sufficiently granular, i.e. it consists of a large number of equal loans with the same probability of default.

Based on this assumption, the value of the loss (conditional on the low value of the systemic component) is calculated as the default probability (p_i) of a single loan multiplied by the value of portfolio (using the law of large numbers). $\square$

Let us note that in the theory of credit risk management, the model which has been described while deriving Equality (5.3) is known as *asymptotic single risk factor approach* and has been introduced by Oldřich Vašíček.

5.7.1 Strong points and weaknesses of the regulatory formula

The main advantages of this regulatory approach is that it is simple (i.e. it represents a simple formula between PD and RW without on the additional requirements on model development) and it exhibits portfolio invariance (i.e. the risk of a particular loan only depends its own default probability, but is independent on characteristics of the portfolio it is added to).

The underlying model, however, relies on several important assumptions, some of which are hardly met in practice:

- normal distribution of asset values,
- infinitely diversified portfolio (high granularity),
- any dependence pattern is only described by correlations (i.e. non-linear dependencies are ignored),
- one-factor model (any risk not covered by the systemic factor is captured by residuals) – an analogy to the CAPM,

- PD and LGD are assumed to be mutually independent.

In fact, this is not the bank's own model. Although banks use their internal ratings (i.e. estimates of PD) which might be derived from some underlying loss distribution, the formula is given based on normal returns of assets. In addition, the assumption of infinitely diversified portfolio tends to underestimate the concentration risk in the portfolio.

These weaknesses are partially compensated by the strict level of the drop of the systemic components (corresponding to its 99.9% quantile) which is significantly higher than e.g. in the market risk (99%).

One of the important concerns is that such a model yields so-called procyclical behaviour of banks, which can be illustrated by the following example: Assume that the economy is hit by a recession. Hence, banks observe an increase in default probabilities and hence Formula (5.3) yields higher capital requirements. This however imply constraints for banks to the ability to grant new loans (or even risk of deleveraging), since it is difficult to attract new capital during a crisis. The flow of credit decreases which further deepens the recession.

5.8 Models comparison

The comparison of different models is summarized in Table 5.2.

Model	Merton	KMV	CreditMetrics	CreditRisk+	CreditPortfolio View	Regulatory model
Author		KMV, Moody's	JP Morgan, RiskMetrics	Credit Suisse	McKinsey	Basel Committee for Banking Supervision
Type of the model	Structural	Structural	Structural	Reduced form	Reduced form	Asymptotic single risk factor approach
Inputs	Daily equity prices, the number of equities issued, the value of assets and debt (quarterly)	Daily equity prices, the number of equities issued, the value of assets and debt (quarterly)	Ratings, transitional matrices, expected returns for individual rating classes	The number of loans and PDs for individual types of loans	Time series (quarterly or yearly) of PDs and macro variables	PD, LGD
Risk factors	Stochastic process for asset value	Stochastic process for asset value	Stochastic process for asset value / changes in ratings	Default intensity	Macro variables	Systemic factor, idiosyncratic variables
Outputs	PD, expected loss	Distance-to-Default, Expected default frequency, expected loss	Empirical loss distribution	Loss distribution, expected and unexpected loss	PD, expected loss	Unexpected loss

Table 5.2: Comparison of credit risk models

5.9 Appendix: Validation of credit risk models

In this appendix, we will briefly address the question how to assess the quality of different credit risk models. In particular, we focus here on *Gini coefficient* which is one of indicators measuring the discriminatory power of rating systems (scorecards) in credit risk management.

Assume that we have a credit risk model which assigns a PD to each loan from a set of different loans based on their individual characteristics. In addition, we already know whether the loans have defaulted or not and based on this information, we want to evaluate the quality of the credit risk model.

The assessment is based on the following simple principle: the model performs well if it assigns low PD to loans which actually have not defaulted and high PD to loans where the default has been observed. Hence, we sort all loans in descending order accroding to their estimated PD (denoted by $\widehat{PD}_i$) assigned by the model. For the first k of loans in such an order, we calculate the number of these loans which have actually defaulted (default of the i-th loan is indicated by $D_i = 1$, non-default by $D_i = 0$). This number is expressed as a percentage of the total number of defaulted loans.

Mathematically, this can be expressed in the following form:

$$F(k) = \frac{\sum_{i=1}^{k} I(D_i = 1)}{\sum_{i=1}^{N} I(D_i = 1)},$$

where I is an indicator function. Function F represents so-called *Lorenz curve*. In case of a model with no discriminatory power, the actual defaults are independent on the estimated values $\widehat{PD}_i$, and hence the defaults are approximately uniformly distributed across this ordered set. Therefore, function F is represented by a line with slope 45°. On the contrary in case of an optimal model with perfect discriminatory power, all defaults are accumulated in the subset of loans of high $\widehat{PD}$ and non-defaults are observed for other loans. Denoting the total number defaulted loans by N_d, the Lorenz curve in this case is an upward-sloping line attaining value 1 at N_d and staying on this value afterwards.

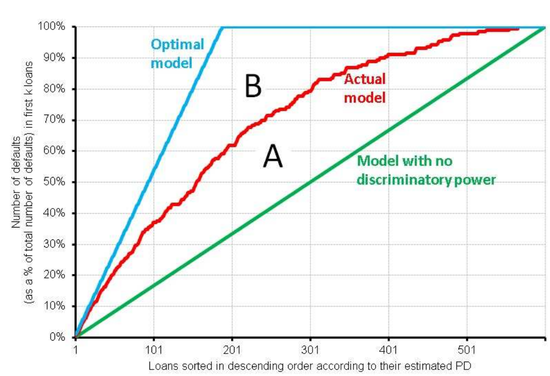

Figure 5.2: Gini coefficient. The total number of loans is $N = 600$, of which the total number of defaults is $N_d = 189$. The value of the Gini coefficient is 58.1 %.

The Gini coefficent measures how close is the actual Lorenz curve to the optimal one. Hence, it is calculated as

$$\text{Gini} = \frac{A}{A+B},$$

where A and B are the surfaced as depicted on Figure 5.9.

Note that in practice it is common to slightly adjust the sample of loans chosen for the validation purposes compared to the original sample. In particular, we obtain more credible results when we increase the share of defaulted loans by leaving some non-defaulted loans asides (even if PD is low). Indeed, if we would calculate the Gini coefficient for a very unbalanced sample with low share of defaulted loans, the model which will assign $PD_i = 0$ to all loans would appear as a very good model (it would correctly predict the default in $(1 - PD)$ % of cases), although such a model is totally useless.