Principal component analysis

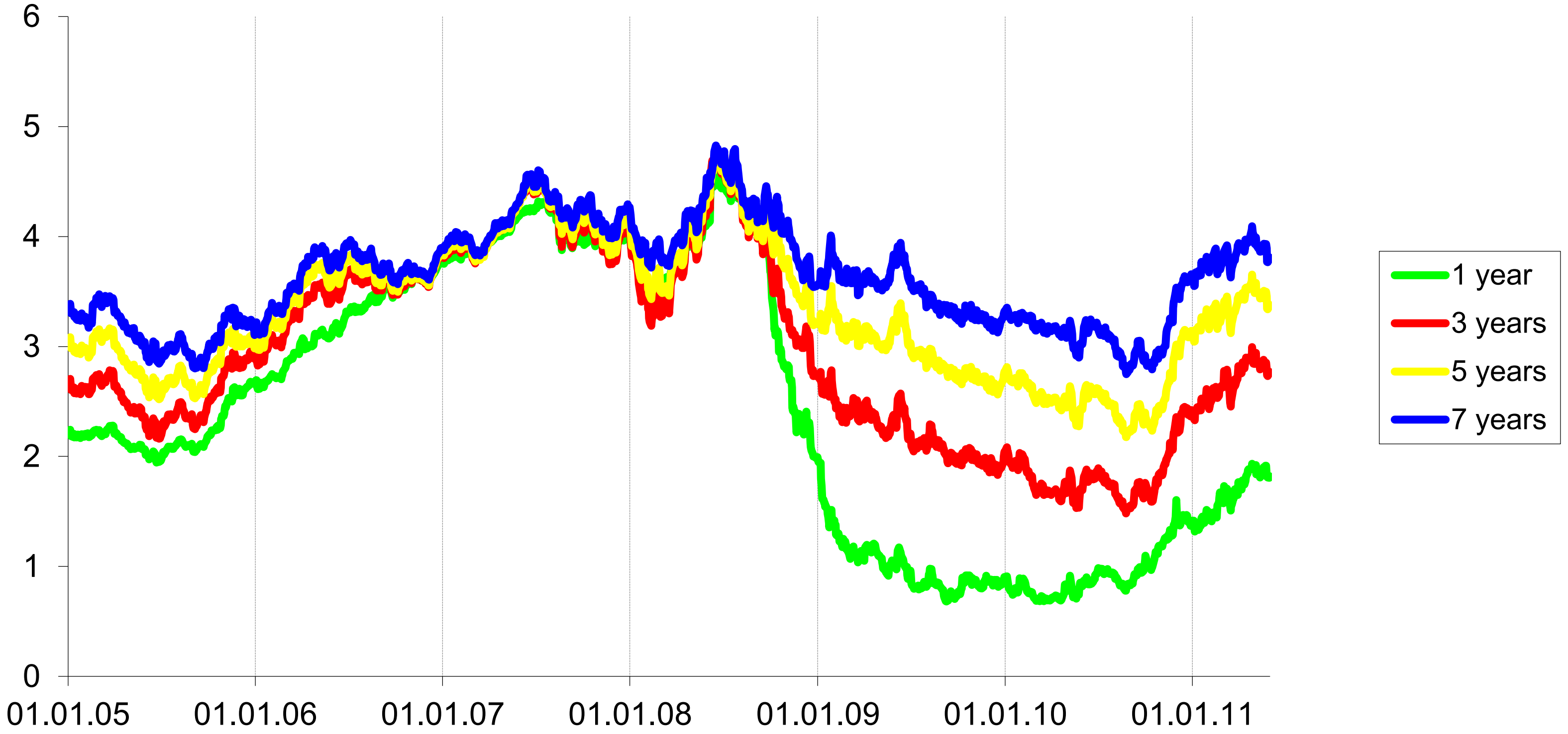
Motivation

- Task: Propose the scenarios of the changes in the yield curve for the purpose of stress testing of the bond portfolio

- Problem: The yield curve comprises a number of points in which the changes are partially correlated

Motivation

- What can be seen in the data:



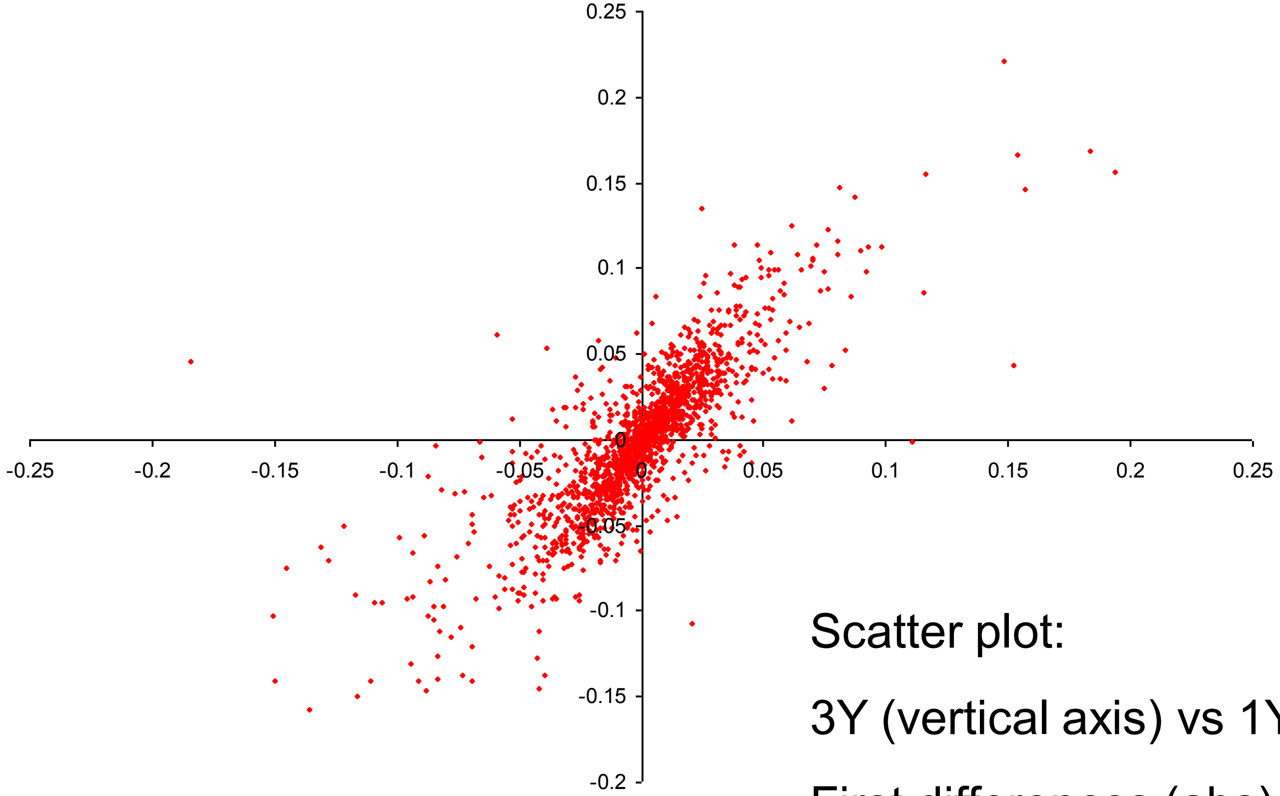
Source: http://www.ecb.int/stats/money/yc/html/index.en.html

Motivation

- What is prescribed by the regulation:
  - Act on banks (§ 33f):
    - The economic value of a bank may not fall by more than 20% of the value of its own funds as a result of a sudden and unexpected change in market interest rates.
  - Decree 13/2010 on risks and on details of risk management system (§ 4):
    - A sudden and unexpected change of interest rates on the market is understood as a parallel shift of the yield curve upwards or downwards by 200 basis points.

Main idea

- In a (hypothetical) case of only two interest rates:
  - The ellipse can be represented by its main axis



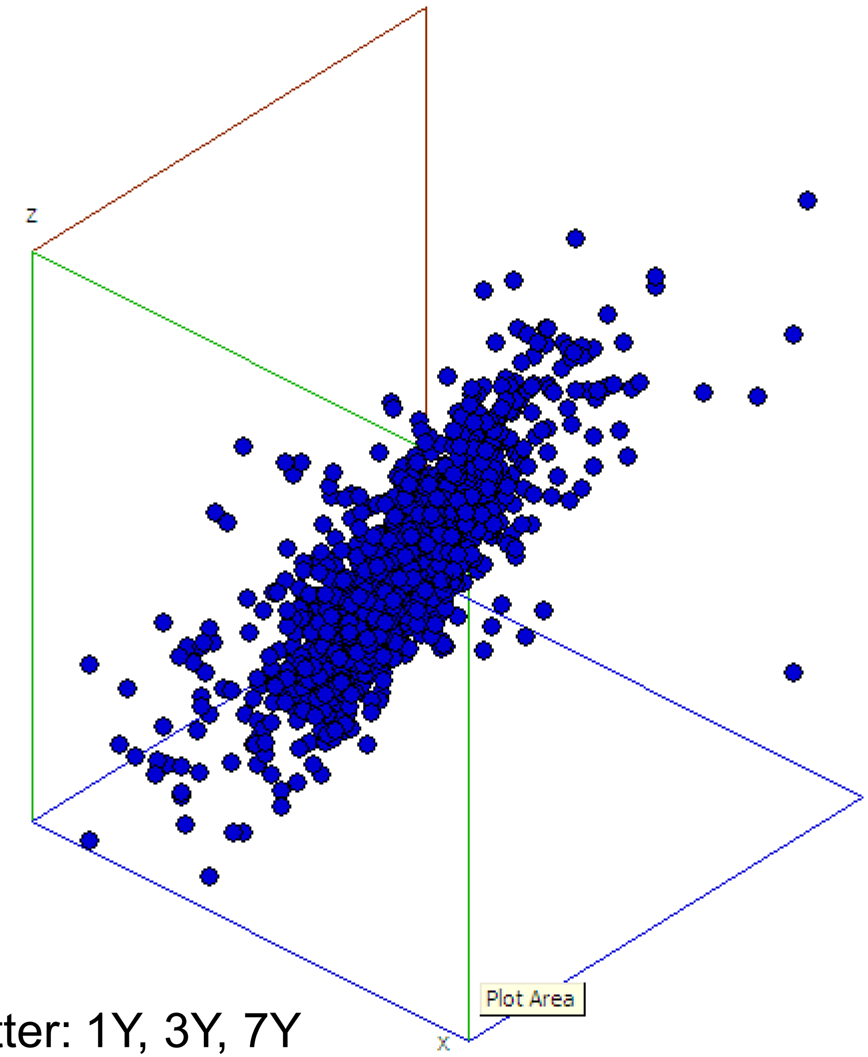
Scatter plot:

3Y (vertical axis) vs 1Y (horizontal axis),

First differences (abs) (2005 – mid-2011)

Main idea

- The case of three interest rates
  - Ellipsoid can be compressed to an ellipse



3D scatter: 1Y, 3Y, 7Y

PCA: a sketch

- PCA = principal component analysis
- In risk management, we often face the problem of a large number of risk factors
  - equity portfolio with a large number of shares
  - yield curve containing a large number of points
  - positions in different currencies
- The aim is to identify the most important risk factors for typical movements
  - identification of the systemic risk
  - designing stress scenarios
- The basic idea: Reduce the dimension!
- The aim is to find a small number of linear combinations of the original variables
  - Sufficiently high share of volatility should be explained
  - These linear combinations should be pairwise uncorrelated

Theoretical background

- Assumption: X is $N \times T$ matrix of the changes in the risk factors (time series are in rows)

- Theoretical background: Spectral decomposition of a matrix

  Each symmetric matrix A can be decomposed as follows:

$$A = P \Lambda P^T,$$

  where Λ is a diagonal matrix of eigenvalues of the matrix A and
  P is an orthonormal matrix of the standardised eigenvectors of the matrix A (rows of the matrix P).

- We can apply the spectral decomposition to the variance-covariance matrix Σ of the original variables: $\Sigma = P \Lambda P^T$

- The calculation of the $N \times T$ matrix of the principal components (PCs)

$$Y = P^T (X - \mu)$$

Properties of the PCs

- One has
  - $E(Y) = 0$
  - $\mathrm{cov}(Y) = P^T \Sigma P = P^T P \Lambda P^T P = \Lambda$

- Consequence: PCs are not correlated and their variances are equal to the eigenvalues

- Transformation of the original data to the PCs corresponds to their centralisation and rotation

- Since the total variability of the original data is

$$\sum_{i=1}^{n} \mathrm{var}(X_i) = \mathrm{trace}(\Sigma) = \sum_{i=1}^{n} \lambda_i$$

the part of total variability corresponding to the j-th PC is $\dfrac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$

Properties of the PCs

- The first PC corresponds to the linear combination of X, which has a maximum variance among all such combinations
  - The direction of greatest variance of the N-dimensional cloud of original data

- Sample principal components:
  - If X has a normal distribution and all the eigenvalues of cov (X) are different, then the sample principal components are estimates of the actual PCs of the random vector X provided that the sample PCs are estimated using the ML.

- PCs are not invariant to the scaling of the original variables
  - It is common to apply the PCA to standardized variables, i.e. using the correlation matrix

How to determine the number of PCs

- Eigenvalues higher than their average (Kaiser)
- Impose a limit on the percentage of the explained variance
- Break in the plot of the sorted eigenvalues (scree plot)

Interpretation of the PCA

- Component loadings:
  - Standardised eigen vectors
  - The coefficients correspond to the weight of the original variables in the respective PC
  - Can be computed by an OLS

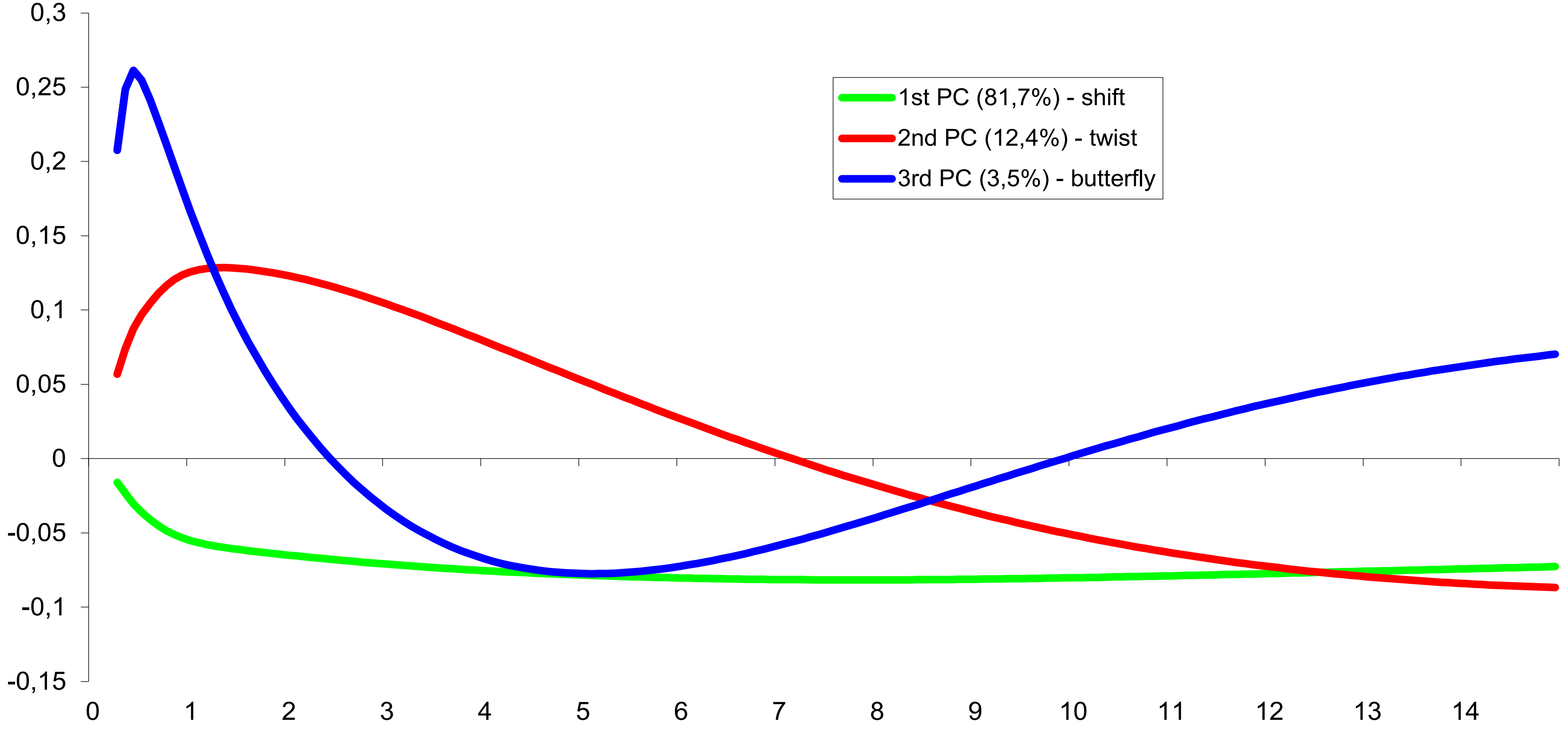
- Component scores
  - Coordinates in the PCs space

- Biplot = scatter plot of the component scores for two different PCs

- Exercise: What if the original variables have not been centralised?

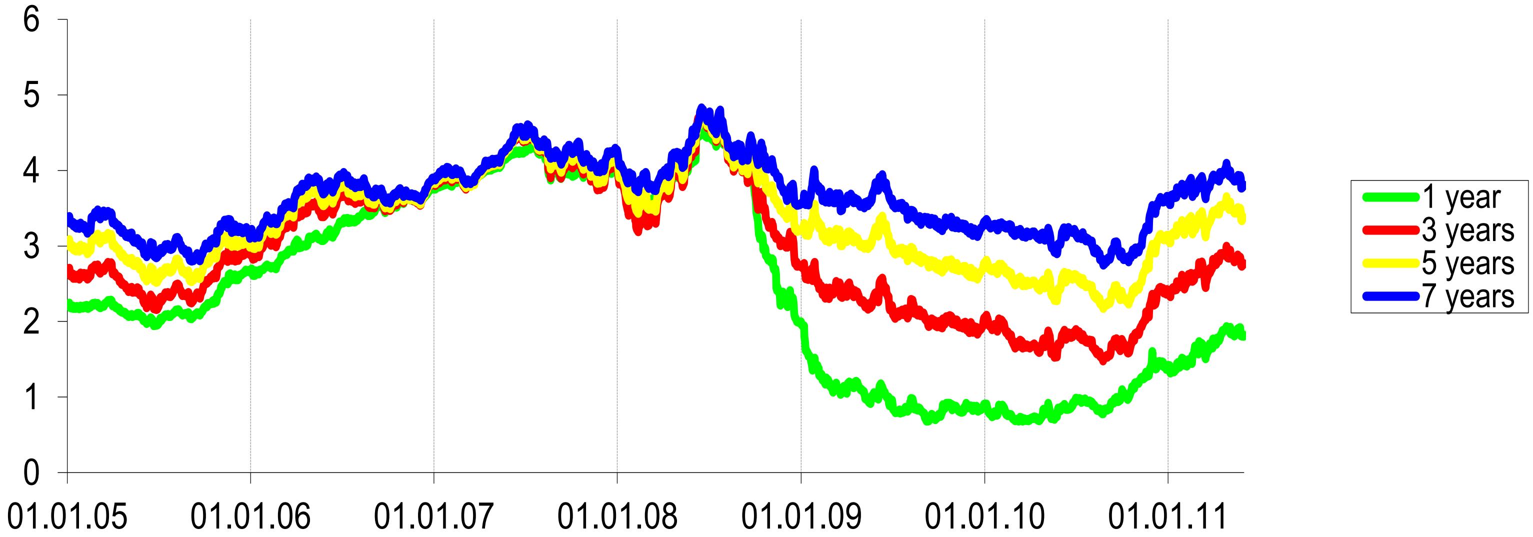
Results

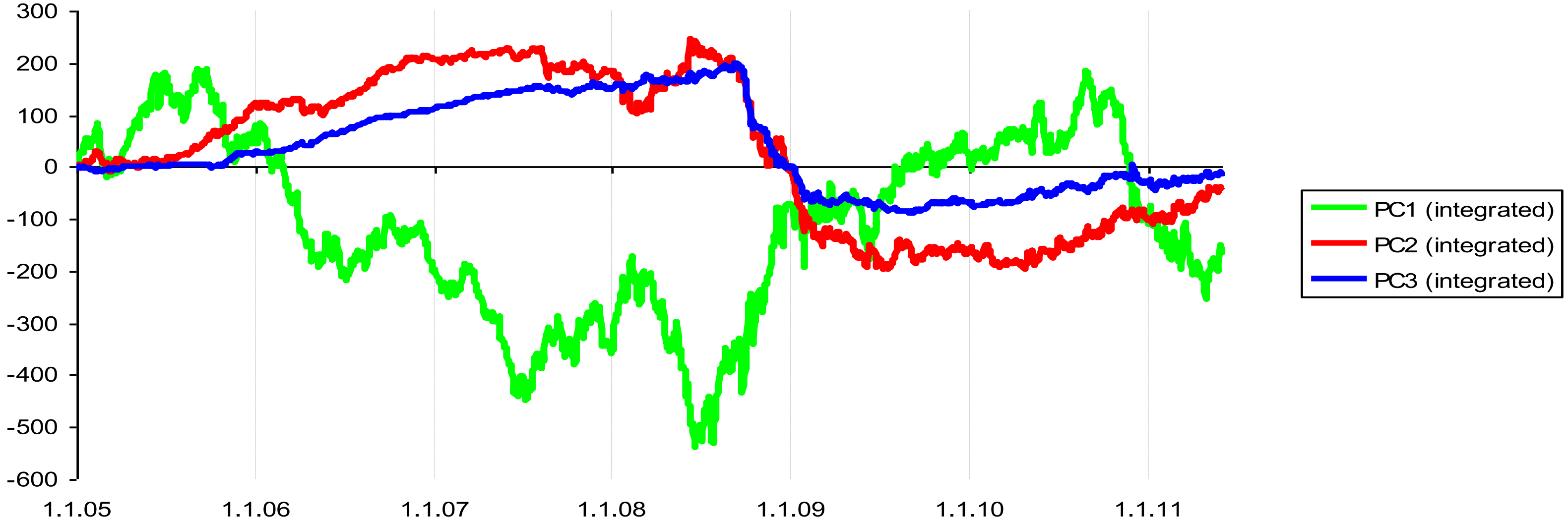
- Time series: daily absolute changes (first differences) of the interest rates
- Component loadings – first three PCs (97,7%)



Results

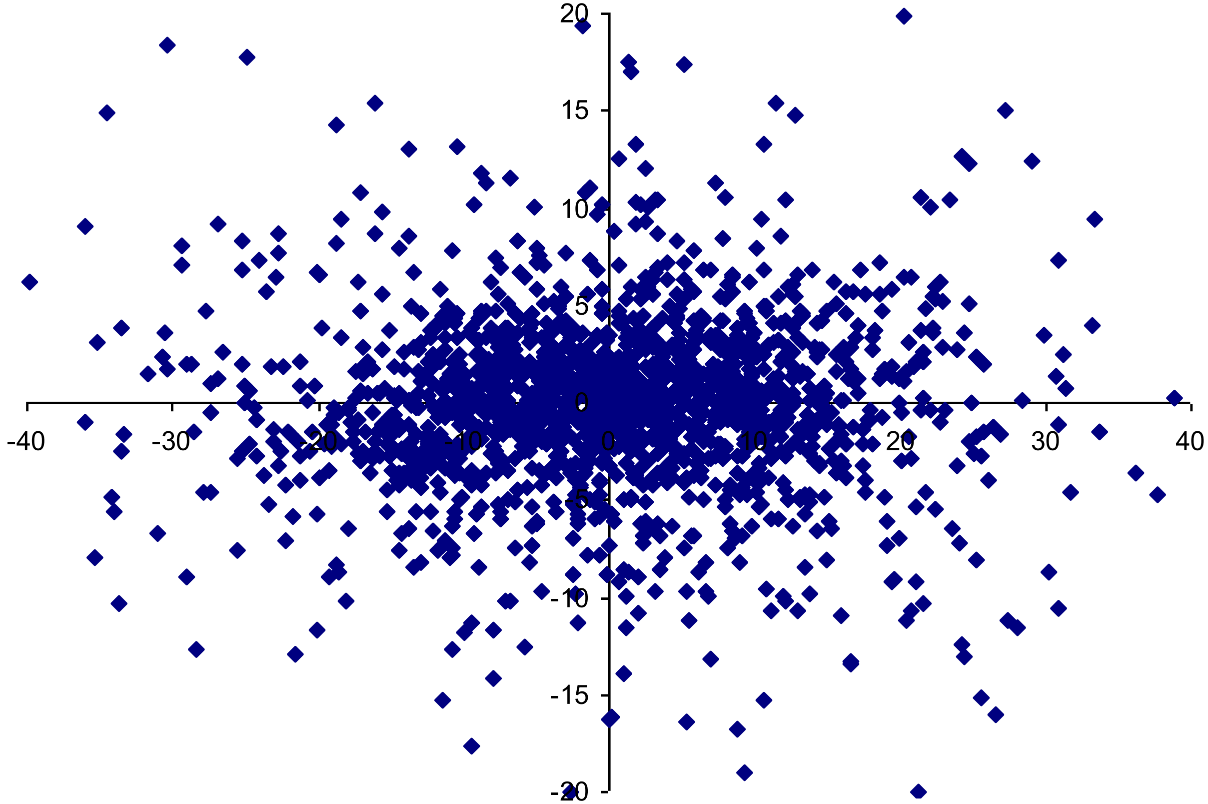
- Component scores





Results

- How to identify the type of the changes which have been prevailing in a given period?



- Biplot - Scatter plot in the (PC1, PC2) space

The scenario size

- Now we know the qualitative description of the „most representative“ change in the yield curve

- How the size of the scenarios should be determined?

The scenario size

The case of the first PC

- Conditional normal distribution for the given day: $N(\mu_{HK}, \sigma^2_{HK})$

- We choose $y = (\mu_{HK} - u_\alpha \sigma_{HK}, 0, ..., 0)^T$

- Backward transformation of the change in the PC to the yield curve

$$x = Py + \mu$$

- We use that
  - $P^T = P^{-1}$, where columns of P are the eigenvectors of the matrix Σ
  - PCs are pairwise linearly independent

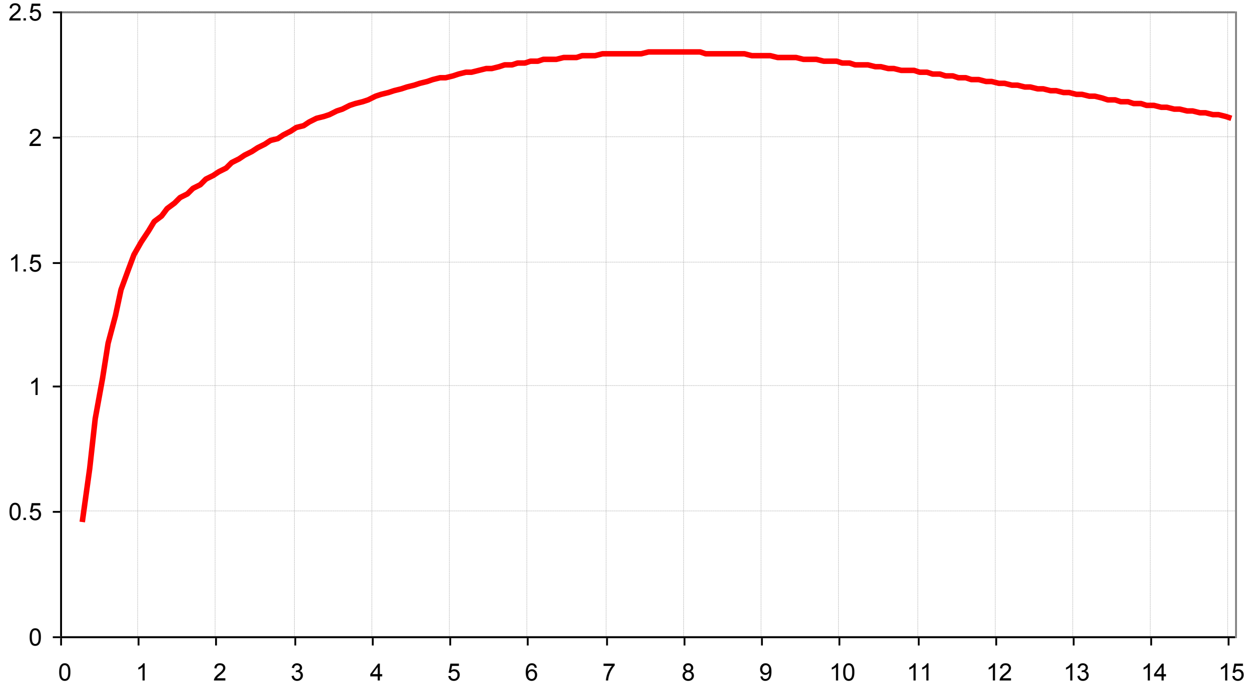
- After simplification:

$$x = P_1 (\mu_{HK} - u_\alpha \sigma_{HK}) + \mu,$$

where P_1 is the eigenvector corresponding to the largest eigenvalue

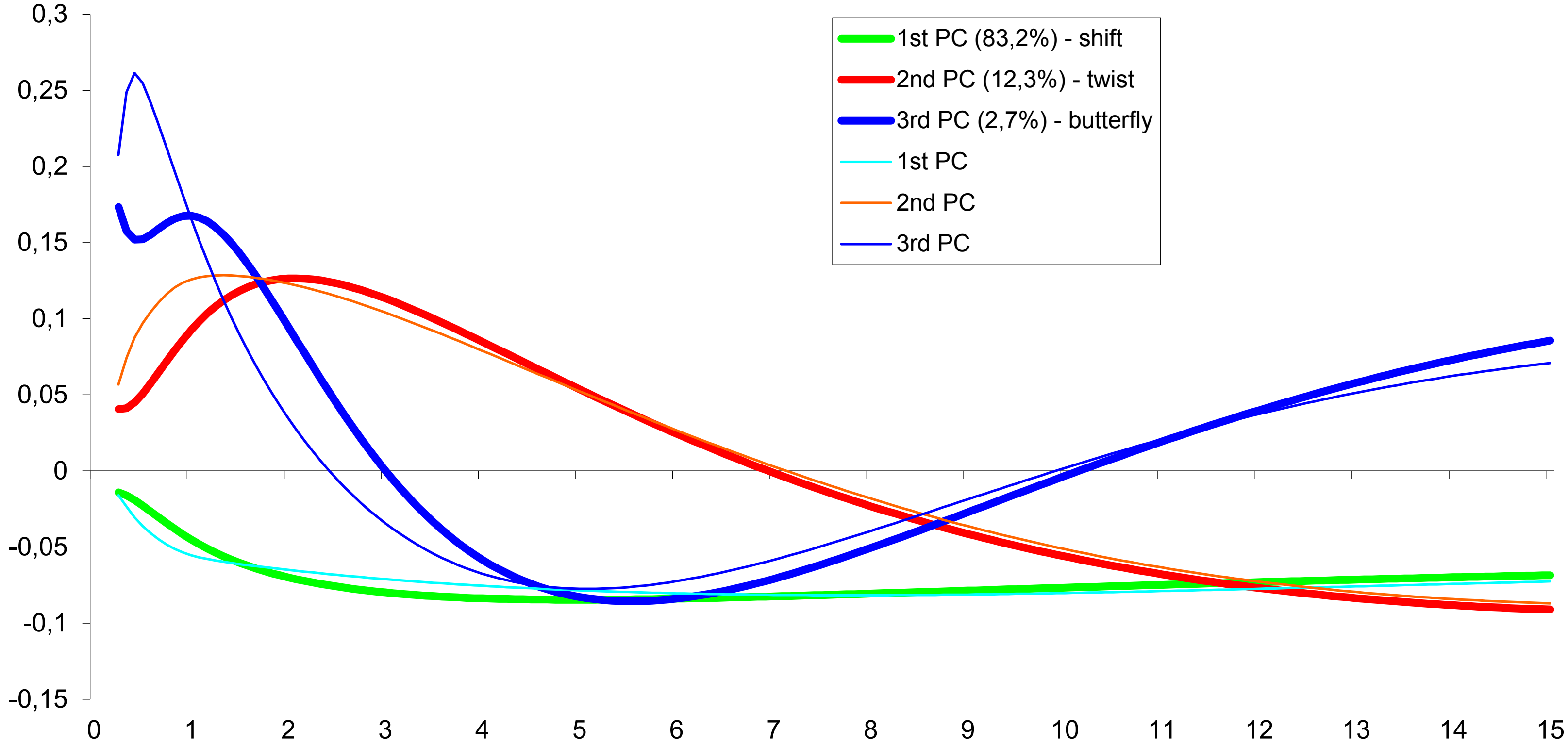
The scenario size

- The result for the „most typical“ scenario on the probability level of 99%:



Using of covariance matrix

- Component loadings – first three PCs (98,3%)



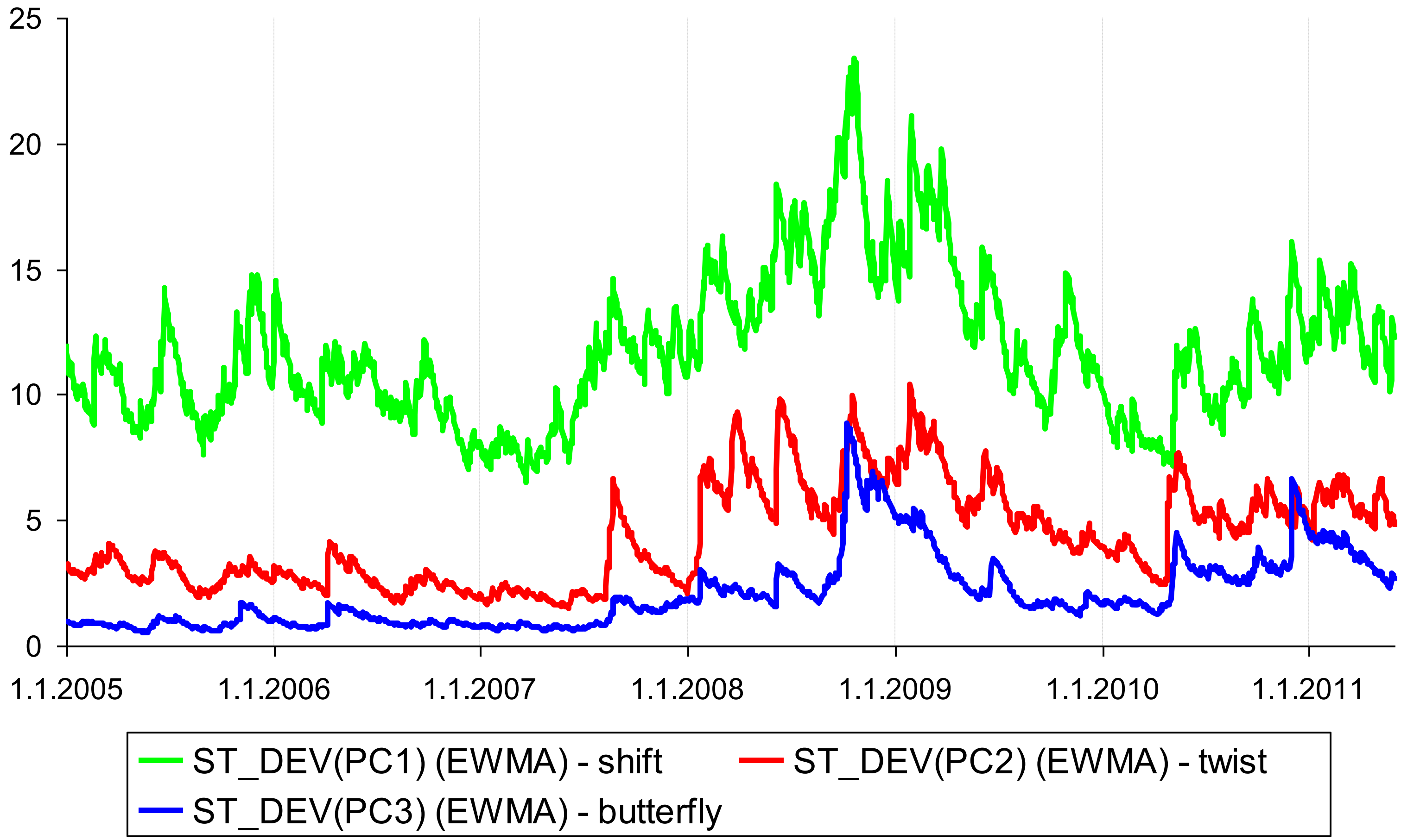
- Thin lines ilustrate the results based on the correlation matrix

Critics

- The interpretation of the principal components is often difficult
- The PCA only addresses the linear dependencies (i.e. correlations)
  - The pattern of an increasing dependence in market turmoils is not captured
- The PCA does not allow changing weights when constructing the PCs

Critics

- The dependence between (non-correlated) PCs in terms of volatilities



Possible extensions

1. Usage of a conditional correlation / covariance matrix
2. Orthogonal GARCH
3. Mapping of the portfolio to the set of risk factors
4. Factor analysis – principal component rotation can increase the interpretation