ASYMPTOTIC BEHAVIOUR OF SOME MARKOV OPERATORS APPEARING IN MATHEMATICAL MODELS OF BIOLOGY

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Abstract. A class of Markov operators satisfies the Foguel alternative if its members are either sweeping or have stationary densities. We show that this alternative holds for some integral Markov operators appearing in mathematical models of biology.

1. Introduction

Let $K : L_1(X) \to L_1(X)$ be an integral Markov operator of the form:

$$Kf(x) = \int_X K(x,y)f(y)dy \ , \tag{1.1.}$$

where $K(x,y)$ defined on XxX is a kernel. Such operators were intensively studied. In [1], [4], [6], [7] some sufficient conditions for sweeping (see def. 3.1.) and asymptotical stability were given. It was proved in [4] that,under the assumption of having subinvariant locally integrable function, the alternative of sweeping or having stationary density holds. The condition without the assumption of the existence of a subinvariant locally integrable function for operators satisfying some property (P) was given in [3]. The main result of this paper is the proof of the Foguel alternative for operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} (-\frac{\partial}{\partial x}(H(Q(\lambda(x)) - Q(y))f(y)dy \ , \tag{1.2.}$$

where $Q, \ \lambda \ , -H$ are nonnegative, nondecreasing, absolutely continuous functions on $\mathbb{R}^+$ satisfying:

$$H(0) = 1, \ \lim_{x\to\infty} H(x) = 0$$

$$Q(0) = \lambda(0) = 0, \ \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty \ .$$

Operators of this type need not satisfy the property (P). The asyptotic behaviour of operators of the form (1.2.) has many practical applications in biology.

In Section 2, some necessary results of [2] are presented. In Section 3, the main result (Theorem 3.2.) is proved.

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2. Some properties of Markov processes and integral Markov operators

Theorems 2.1 - 2.4. are proved in [2].

Definition 2.1. A Markov process is defined to be a quadruple (X, Σ, m, P) , where (X, Σ, m) is a $\sigma-$ finite measure space with positive measure and where P is an operator on $L_1(X)$ satisfying
(i) P is a contraction : $\|P\| \leq 1$
(ii) P is positive : if $0 \leq u \ \in L_1(X)$ then $Pu \geq 0$

Definition 2.2. If u is an arbitrary non-negative function, set $Pu := \lim_{k\to\infty} Pu_k$ for $0 \leq u_k \in L_1(X), u_k \nearrow u$, where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere. The sequence Pu_k is increasing so that $\lim_k Pu_k$ exists (it may be infinite). By [2] the definition of Pu is independent of the particular sequence u_k.

Definition 2.3. Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\}, D = X \setminus C$$

By [2] this definition is independent of the choice of u_0.

Theorem 2.1. *If* $0 \leq u \ \in L_1(X)$ *then*

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \textit{ for } x \in D, \ \sum_{k=0}^{\infty} P^k u(x) = 0 \textit{ or } \infty \textit{ for } x \in C.$$

Definition 2.4. A function $K(x,y) \geq 0$ defined on XxX which is jointly measurable with respect to its variables is called a kernel. If $\int_X K(x,y)dx = 1$, then K is called a stochastic kernel. Stochastic kernel defines an operator on $L_1(X)$:

$$Kf(x) = \int_X K(x,y)f(y)dy$$

with $\|K\| = 1$. So (X, Σ, m, K) is a Markov process.

Definition 2.5. Let P be an integral Markov operator, then (X, Σ, m, P) is said to be a Harris process if $X = C$.

Theorem 2.2. *Let K be an integral Markov operator and a Harris process. Then there exists $0 < u < \infty$ such that $Ku = u$ (a σ-finite invariant measure).*

Theorem 2.3. *Let P be a Markov process with $X = D$. Then there exists $0 < g < \infty$ such that $Pg \leq g$.*

Proof: Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$.

Definition 2.6. Let P be a Markov process. Define operators P_C, P_D:

$$P_C : L_1(C) \to L_1(C), P_C f = (P\tilde{f}) \restriction C,$$

where the symbol $\restriction$ denotes the restriction to the set C, $\tilde{f}$ is the function f extended by 0 on D,

$$P_D : L_1(D) \to L_1(D), P_D f = (P\tilde{f}) \restriction D,$$

where $\tilde{f}$ is the function f extended by 0 on C.

Theorem 2.4. *Let P be a Markov process. If supp $f \subseteq C$, then supp $Pf \subseteq C$. (supp $f = \{x : f(x) \neq 0\}$)*

Corollary 2.1. *Let K be an integral Markov operator. Then*

$$(C, \Sigma \upharpoonright C, m \upharpoonright C, K_C)$$

is a Harris process. ($\Sigma \upharpoonright C$ denotes the σ-algebra restricted to the space C, $m \upharpoonright C$ denotes the measure m restricted to the space $\Sigma \upharpoonright C$).

Proof: By Theorem 2.4. supp $f \subseteq C$ implies supp $Kf \subseteq C$. By Theorem 2.1. for $u > 0$ on C,$u = 0$ on D:

$$\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \upharpoonright C)(x)$$

for every $x \in C$.

Corollary 2.2. *Let P be a Markov process on $L_1(X)$. Then*

$$P_D(f \upharpoonright D) = (Pf) \upharpoonright D.$$

Proof: $f = f_D + f_C$, where $f_C = f.1_C, f_D = f.1_D$. By Theorem 2.4. $(Pf_C) \upharpoonright D = 0$, hence

$$(Pf) \upharpoonright D = (Pf_D) \upharpoonright D = P_D(f \upharpoonright D).$$

Corollary 2.3. $P_D^n(f \upharpoonright D) = (P^n f) \upharpoonright D$

Corollary 2.4. *Let P be a Markov process on X, let $u > 0$ on D. Then*

$$\sum_{n=0}^{\infty} P_D^n u < \infty.$$

Proof: Let $\tilde{u}$ be a function on X such that $\tilde{u} \upharpoonright C = 0, \tilde{u} \upharpoonright D = u$. By Corollary 2.3.

$$\sum_{n=0}^{\infty} P_D^n u = (\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D.$$

By Theorem 2.1. $(\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D < \infty$.

3. The Foguel alternative for integral Markov operators of the form (1.2.)

Definition 3.1. Let a family $\mathcal{A} \subset \Sigma$ be given. A Markov process is called sweeping with respect to $\mathcal{A}$, if

$$\lim_{n\to\infty} \int_A P^n f dm = 0$$

for $A \in \mathcal{A}$ and $f \in D$ $(D = \{f \in L_1(X), \|f\| = 1, f \geq 0\})$

In the sequel we shall assume that $\mathcal{A}$ satisfies the following properties:

(i) $0 < m(A) < \infty$ for $A \in \mathcal{A}$

(ii) $A_1, A_2 \in \mathcal{A}$ implies $A_1 \cup A_2 \in \mathcal{A}$

(iii) There exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\cup A_n = X$.

A family satisfying $(i) - (iii)$ will be called admissible.

Definition 3.2. Let (X, Σ, m) and an admissible family $\mathcal{A} \subseteq \Sigma$ be given. A measurable function $f : X \to \mathbb{R}$ is called locally integrable, if

$$\int_A |f| dm < \infty \text{ for } A \in \mathcal{A}.$$

The following theorem is proved in [4].

Theorem 3.1. *Let a measure space (X, Σ, m), an admissible family $\mathcal{A}$ and an integral Markov operator K be given. If K has no invariant density but there exists a positive locally integrable function f_* subinvariant with respect to K, then K is sweeping.*

Remark 3.1. Theorem 3.1. was proved in [4] for stochastic kernel operators ($\int_X K(x,y)dx = 1$). But the proof is completely same for integral Markov operators.

Let K be an integral Markov operator. Recall the definition of K_C and K_D (see def. 2.6.). By Corollary 2.1. K_C is a Harris process and by Corollary 2.4. K_D is dissipative ($X = D$). By Theorem 2.2. and Theorem 2.3. there exist g_C, g_D such that $K_C g_C = g_C$ and $K_D g_D \le g_D$. The following two lemmas (3.1. and 3.2.) claim that g_C, resp. g_D are locally integrable in all points $y \in C$, (resp. $y \in D$) such that

$$\int_C K_C(x,y)dm(x) > 0 \text{ (resp. } \int_D K_D(x,y)dm(x) > 0 \text{).}$$

Denote by $\mathbb{R}^+$ the set $[0, \infty)$ and by $\mathcal{T}$ the Euclidian metric topology on $\mathbb{R}^+$.

Lemma 3.1. *Let K be an integral Markov operator of the form (1.2.), let $y \in \mathbb{R}^+$. Let $0 < g < \infty$ and $K_C g \le g$. Let*

$$\int_C K_C(x,y)dm(x) > 0 \ .$$

Then there exists an open neighbourhood U_0 of y such that

$$\int_{U_0 \cap C} g(z)dz < \infty \ .$$

Proof: Let

$$\int_{U_y \cap C} g(z)dz = \infty \quad \forall U_y \in \mathcal{T} \text{ such that } y \in U_y \ .$$

Let $B = \{x \in C : K(x,y) > 0\}$. Let $E \subseteq B$ and $m(E) > 0$. Then

$$\begin{aligned} \int_E g(x)dx &\ge \int_E \int_{U_y \cap C} g(z)K(x,z)dzdx = \\ &= \int_{U_y \cap C} g(z) \int_E K(x,z)dxdz \ . \end{aligned} \tag{3.1.}$$

Since

$$K(x,y) = q(\lambda(x)).\lambda'(x)h(Q(\lambda(x)) - Q(y))$$

and $Q(y)$ is absolutely continuous,

$$\int_E K(x,z)dx = \int_{Q(\lambda(E))} h(t - Q(z))dt$$

is continuous with respect to z. By the assumption there exists $\varepsilon > 0$ such that

$$\int_E K(x,y)dx > \varepsilon > 0 \ .$$

Since $\int_E K(x,z)dx$ is continuous with respect to z, there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

$$\int_E K(x,z)dx > \varepsilon \quad \forall z \in U_y \ .$$

Now (3.1.) and $\int_{U_y \cap C} g(z)dz = \infty$ imply that

$$\int_E g(x)dx = \infty \ .$$

$E \subseteq B$ was arbitrary, so $g(x) = \infty$ on the set B. But by the assumption $0 < g < \infty$. $\square$

Lemma 3.2. *Let K be an integral Markov operator of the form (1.2.), let $y \in \mathbb{R}^+$. Let $0 < g < \infty$ and $K_D g \leq g$. Let*

$$\int_D K_D(x,y)dm(x) > 0 \ .$$

Then there exists an open neighbourhood U_0 of y such that

$$\int_{U_0 \cap D} g(z)dz < \infty \ .$$

The proof of Lemma 3.2. is the same as the proof of Lemma 3.1..

Theorem 3.2.. *Let K be an integral Markov operator of the form (1.2.). Let $\mathcal{A}$ be the family of compact subsets of $\mathbb{R}^+$ (with respect to the Euclidian metric topology). If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

Proof: Denote

$$\tilde{K}_C f = (Kf).1_C \ , \ \tilde{K}_D f = (Kf).1_D$$
$$f_C = f.1_C \ , \ f_D = f.1_D \ .$$

Now

$$||\tilde{K}_D^l f_D|| = ||K\tilde{K}_D^l f_D|| = ||\tilde{K}_C\tilde{K}_D^l f_D|| + ||\tilde{K}_D^{l+1} f_D|| \ ,$$

hence

$$||\tilde{K}_C\tilde{K}_D^l f_D|| = ||\tilde{K}_D^l f_D|| - ||\tilde{K}_D^{l+1} f_D|| \ ,$$

(3.2.) $$\sum_{l=k}^{n} ||\tilde{K}_C\tilde{K}_D^l f_D|| = ||\tilde{K}_D^k f_D|| - ||\tilde{K}_D^{n+1} f_D||$$

Lemma 1. *Let $y \in \mathbb{R}^+$. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap D} K_D^n f dm = 0$$

for every $f \in L_1(D)$.

Proof (of Lemma 1.): By Corollary 2.4.

$$0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty$$

for $u > 0$, hence the process K_D is dissipative. By Theorem 2.3. there exists a σ -finite subinvariant measure λ equivalent to $m \restriction D$.

Let $\mathcal{A}_\lambda$ be the family of all sets of finite measure (with respect to m) such that

$$\int_A \frac{d\lambda}{dm} dm < \infty \quad \forall A \in \mathcal{A}_\lambda .$$

Since $\frac{d\lambda}{dm} < \infty$, the family $\mathcal{A}_\lambda$ is admissible. K_D is dissipative, hence by Theorem 3.1. K_D is sweeping with respect to $\mathcal{A}_\lambda$. Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $D \cap U$ has positive measure. Denote $g = \frac{d\lambda}{dm}$. Let

$$\int_D K(x,y)dx > 0 \ .$$

By Lemma 3.2. there exists $U_y \in \mathcal{T}$ such that

$$\int_{U_y \cap D} g(x)dx < \infty \ ,$$

hence

$$U_y \cap D \in \mathcal{A}_\lambda, \ \lim_{n\to\infty} \int_{U_y \cap D} K_D^n f dm = 0 \ .$$

Let $\int_D K(x,y)dx = 0$. Let

$$\lim_{n\to\infty} \int_{U_y \cap D} K_D^n (f \restriction D) \neq 0$$

for all $U_y \in \mathcal{T}$ such that $y \in U_y$ and some $f \in L_1(\mathbb{R}^+)$. Now $\int_C K(x,y)dx = 1$. Since $\int_C K(x,y)dx$ is continuous with respect to y (see the proof of Lemma 3.1.), there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

$$\int_C K(x,z)dx > \varepsilon > 0 \ \ \forall z \in U_y \ .$$

By the assumption there exists $\delta > 0$ such that

$$\int_{U_y \cap D} K_D^n (f \restriction D) > \delta$$

for infinitely many n. By Corollary 2.3.

$$K_D^n(f \restriction D) = (\tilde{K}_D^n f_D) \restriction D .$$

Then

$$\begin{aligned}\int_C \tilde{K}_C \tilde{K}_D^n f_D(x)dx &\geq \int_C \int_{U_y \cap D} K(x,z)\tilde{K}_D^n f_D(z)dzdx = \\ &= \int_{U_y \cap D} \tilde{K}_D^n f_D(z) \int_C K(x,z)dxdz \geq \\ &\geq \varepsilon \int_{U_y \cap D} \tilde{K}_D^n f_D(z)dz \geq \varepsilon.\delta\end{aligned}$$

for infinitely many n. Hence

$$\sum_{n=0}^{\infty} \|\tilde{K}_C \tilde{K}_D^n f_D\| \geq \sum_{n=0}^{\infty} \int_C \tilde{K}_C \tilde{K}_D^n f_D(x)dx = \infty$$

which contradicts (3.2.). $\square$

Lemma 2. *Let $y \in \mathbb{R}^+$, let K_C has no stationary density. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap C} K_C^n f dm = 0$$

for every $f \in L_1(C)$.

Proof (of Lemma 2.): By Corollary 2.1. and Theorem 2.2. K_C is Harris and there exists a function g, $0 < g < \infty$ such that $K_C g = g$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $C \cap U$ has a positive measure. Since $\int_{\mathbb{R}^+} K(x,y)dx = 1$ and by Corollary 2.2 $K(x,y) = 0$ for $x \in D$, $y \in C$,

$$\int_C K(x,y)dx = 1 .$$

By Lemma 3.1. there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

$$\int_{U_y \cap C} g(x)dx < \infty . \tag{3.3.}$$

Let $\mathcal{A}_g$ be the family of all sets of finite measure such that

$$\int_A g dm < \infty \quad \forall A \in \mathcal{A}_g .$$

Since $g < \infty$, the family $\mathcal{A}_g$ is admissible. By (3.3.) $U_y \cap C \in \mathcal{A}_g$ and by Theorem 3.1.

$$\int_{U_y \cap C} K_C^n f dm \to 0 \quad \forall f \in L_1(C) . \quad \square$$

Lemma 3. *Let K_C has no stationary density, let $A \in \mathcal{A}$. Then*

$$\lim_{n\to\infty}\int_{A\cap C} K_C^n f_1 dm = 0\ ,\quad \lim_{n\to\infty}\int_{A\cap D} K_D^n f_2 dm = 0 \tag{3.4.}$$

for every $f_1 \in L_1(C)$, $f_2 \in L_1(D)$.

Proof (of Lemma 3.): Let $y \in \mathbb{R}^+$. By Lemma 1. there exists $U_1 \in \mathcal{T}$ such that $y \in U_1$ and

$$\lim_{n\to\infty}\int_{U_1\cap D} K_D^n f_2 dm = 0 \quad \forall f_2 \in L_1(D).$$

By Lemma 2. there exists $U_2 \in \mathcal{T}$ such that $y \in U_2$ and

$$\lim_{n\to\infty}\int_{U_2\cap C} K_C^n f_1 dm = 0 \quad \forall f_1 \in L_1(C).$$

Set $U_y = U_1 \cap U_2$. Then

$$\lim_{n\to\infty}\int_{U_y\cap C} K_C^n f_1 dm = 0\ ,\quad \lim_{n\to\infty}\int_{U_y\cap D} K_D^n f_2 dm = 0 \tag{3.5.}$$

Thus we have proved that for every $y \in \mathbb{R}^+$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and (3.5.) holds. Finally (3.4.) follows from compactness of A. □

Proof (of Theorem 3.2.): By Lemma 3. K_D is sweeping, K_C is sweeping or has a stationary density.

Let K_C have a stationary density $\tilde{f}$. Let f_* be a function on $\mathbb{R}^+$ such that $f_* \restriction C = \tilde{f}$, $f_* \restriction D = 0$. Then

$$(Kf_*) \restriction C = (K(f_*.1_C)) \restriction C + (K(f_*.1_D)) \restriction C = K_C\tilde{f} = \tilde{f}.$$

By Corollary 2.2. $(Kf_*) \restriction D = K_D(f_* \restriction D) = 0$, hence $Kf_* = f_*$. Let K_C be sweeping. We shall prove that K is sweeping.

Let $f \in L_1(\mathbb{R}^+)$, then $f = f_C + f_D$, where $f_C = f.1_C$, $f_D = f.1_D$. By Corollary 2.3.

$$(K^n f_C) \restriction D = 0,\ (K^n f) \restriction D = K_D^n(f \restriction D).$$

By Lemma 3.

$$\int_{A\cap D} K^n f dm \to 0 \text{ for every } A \in \mathcal{A}.$$

Now it is enough to prove that

$$\int_{A\cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A}.$$

Clearly

$$\begin{aligned}
\tilde{K}_C f &= \tilde{K}_C(f_C + f_D),\ Kf = \tilde{K}_C f + \tilde{K}_D f,\\
\tilde{K}_C(Kf) &= \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C\tilde{K}_D f_D\\
\tilde{K}_C(K^2 f) &= \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2\tilde{K}_D f_D + \tilde{K}_C\tilde{K}_D^2 f_D\\
&\dots\\
&\dots\\
K^n f.1_C &= \tilde{K}_C(K^{n-1} f) =\\
&= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1}\tilde{K}_D f_D + \dots\\
&+ \tilde{K}_C^{n-k}\tilde{K}_D^k f_D + \dots + \tilde{K}_C\tilde{K}_D^{n-1} f_D.
\end{aligned}$$

Take $1 < k < n$ and define:

$$M_{k,n}f = \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1}\tilde{K}_D f_D + \cdots + \tilde{K}_C^{n-k+1}\tilde{K}_D^{k-1} f_D$$
$$R_{k,n}f_D = \tilde{K}_C^{n-k}\tilde{K}_D^k f_D + \cdots + \tilde{K}_C \tilde{K}_D^{n-1} f_D.$$

$\tilde{K}_C$ is contraction, hence

$$\|R_{k,n}f_D\| \le \|\tilde{K}_C^{n-k}\tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C\tilde{K}_D^{n-1}f_D\| \le$$
$$\le \|\tilde{K}_C\tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C\tilde{K}_D^{n-1}f_D\|.$$

By (3.2.)

$$\|R_{k,n}f_D\| \le \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\|.$$

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| < \frac{\varepsilon}{2} \text{ for } n,k \ge n_0(\varepsilon),\ n \ge k.$$

Now fix $k \ge n_0(\varepsilon)$, $A \in \mathcal{A}$. $\tilde{K}_C$ be sweeping implies

$$\int_{A\cap C} M_{k,n} f dm < \frac{\varepsilon}{2}$$

for n sufficiently large, hence

$$\int_{A\cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A}. \quad \square$$

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