Systems that have a mixture of deterministic and probabilistic dynamics often appear in biological and physical sciences. If these systems are described by densities, then the evolution of densities is given often by the equation

$$f_{n+1}(x) = \int_X K(x,y) f_n(y) dm(y) \ , \tag{0.1.}$$

where $K(x,y)$ is a stochastic kernel, $f_n(x)$ is the density describing the n-th generation. If continuous time systems are involved an analogous situation occurs. Then the corresponding evolution of densities is given by the equation

$$f_t(x) = \int_X K_t(x,y) f_0(y) dm(y) \ , \tag{0.2.}$$

where $K_t(x,y)$ is a stochastic kernel. An integral Markov operator of the form

$$Kf(x) = \int_X K(x,y) f(y) dm(y) \tag{0.3.}$$

is defined by equation (0.1). A semigroup of integral Markov operators $\{P_t\}_{t \geq 0}$ of the form

$$P_t f(x) = \int_X K_t(x,y) f(y) dm(y) \ , \quad P_0 f = f \tag{0.4.}$$

is defined by the equation (0.2). The main aim of this dissertation is to describe the asymptotic behaviour of integral Markov operators of the form (0.3) with special attention to the operators which have practical applications in biology and theoretical physics. Only few results concerning the asymptotic behaviour of continuous time systems are given. These results are immediate consequences of results concerning asymptotic behaviour of discrete time systems. In the Chapter 1 some models possessed by a mixture of deterministic and probabilistic dynamics are described. To give an example of continuous time system, a continuous time system in the presence of noise is presented.

Chapter 2 contains the contemporary state of the study concerning asymptotic behaviour of integral Markov operators. There are several ways to deal with this problem.

The first way is based on the results of Foguel [2]. The most important of them are those giving conditions for the existence of a subinvariant measure and decomposition theorems. These results are summarized in Section 2.1. Section 2.2 contains results of Baron - Lasota ([1]) and Komorowski - Tyrcha ([7]) based on the results of Foguel.

The second way employs Lower Bound Function Theorem proved in [8]. The condition for the asymptotical stability, based on this theorem is presented in Section 2.3. Section 2.3 contains also some conditions for convergence of ergodic averages (also in [8]).

Another way is the study of asymptotical periodicity under the conditions of constrictivness of Markov operators. These results were summarized by J. Komornik in [4]. Some important results of [4] and applications are given in Section 2.4.

The last way I know is based on the conditions dealing with Lyapunov and Bielecki functions. Section 2.5 contains this method and its applications.

Section 2.6 shows the connection between the asymptotic behaviour of discrete time semigroups of Markov operators and continuous time semigroups of Markov operators.

The results of the dissertation are summarized in Chapter 3. The dissertation also contains 2 papers: **"The Foguel Alternative for Integral Markov Operators"** and **"Asymptotic Behaviour of Some Markov Operators Appearing in Mathematical Models of Biology"**.

1. Examples of deterministic systems with stochastic perturbations leading to integral Markov operators

1.1. Generalized Lasota-Mackey Model (GLM)

This model is described in [19]. Assume that the cell cycle consists of two phases: A-phase and B-phase. A-phase begins at birth and lasts until the occurrence of a critical event which is necessary for mitosis and cell division. Then the cell enters B-phase and is delayed for a time T_B until division occurs. The probability that the critical event occurs in the interval $[t, t+\triangle t)$, provided it does not occur up to the moment t, is equal $\varphi(s)\triangle t + 0(\triangle t)$, where s is the size of the cell at time t. Assume that the cell grows according to the equation

$$\frac{ds}{dt} = g(s) \ , \tag{1.1.1.}$$

where $g(s) > 0$ for $s > 0$, $g(0) = 0$. Denote by f_n the density function of the initial size of cells in the n-th generation. Let us denote by R_0 the initial size of randomly chosen cell from the first generation ($n = 0$) and by R_1 the initial size of its daughters ($n = 1$). For the moment it is convenient to assume that $R_0 = r$ is constant and only R_1 is a random variable. Denote by $s(t, r)$ the solution of the growth equation (1.1.1.) with the initial condition $s(0) = r$. Then for a given $x \geq 0$ we may calculate the probability $R_1 \geq x$:

$$\begin{aligned} \mathrm{Prob}(R_1 \geq x | R_0 = r) &= \mathrm{Prob}(\frac{1}{2} s(T_A + T_B, r) \geq x) \\ &= \mathrm{Prob}(T_A \geq T_r(2x)) \ , \end{aligned} \tag{1.1.2.}$$

where $T_r(z)$ is the solution of the equation $s(T + T_B, r) = z$ with respect to T. Denote

$$P(t) = \mathrm{Prob}(T_A \geq t) \ .$$

By the assumption

$$\frac{P(t) - P(t+\triangle t)}{P(t)} = \varphi(s(t,r))\triangle t + 0(\triangle t) \ ,$$

hence

$$\frac{dP(t)}{dt} = -\varphi(s(t,r))P(t)$$

and

$$P(t) = \exp\left\{ -\int_0^t \varphi(s(u,r))du \right\} \ .$$

Therefore

$$\mathrm{Prob}(T_A \geq T_r(2x)) = \exp\left\{ -\int_0^{T_r(2x)} \varphi(s(u,r))du \right\} \ . \tag{1.1.3.}$$

Substituting $s(u, r) = y$ into (1.1.3.) we obtain

$$dy = (\partial s(u,r)/\partial u)du = g(s(u,r))du = g(y)du$$

and consequently

$$\text{(1.1.4.)} \qquad \mathrm{Prob}(T_A \geq T_r(2x)) = \exp\left\{-\int_r^{\lambda(x)} q(y)dy\right\} ,$$

where

$$\text{(1.1.5.)} \qquad q(y) = \varphi(y)/g(y) \text{ and } \lambda(x) = s(T_r(2x), r) .$$

Observe that $\lambda(x)$ does not depend on r. In fact, since (1.1.1.) describes a dynamical system, we have

$$\lambda(x) = s(T_r(2x), r) = s(-T_B, s(T_B + T_r(2x), r))$$

which according to the definition of T_r gives

$$\text{(1.1.6.)} \qquad \lambda(x) = s(-T_B, 2x) .$$

Formulas (1.1.4.) and (1.1.5.) are valid if $T_r(2x) \geq 0$. In the remaining case we have

$$\text{(1.1.7.)} \qquad \mathrm{Prob}(R_1 \geq x | R_0 = r) = \mathrm{Prob}(T_A \geq T_r(2x)) = 1 .$$

From the definition of $\lambda(x)$ given by the second equality in (1.1.5.) it follows that the condition $T_r(2x) \geq 0$ ($T_r(2x) < 0$) is equivalent to $\lambda(x) \geq r$ ($\lambda(x) < r$). Thus combining (1.1.2), (1.1.4.) and (1.1.7.) we obtain

$$\mathrm{Prob}(R_1 \geq x | R_0 = r) = \exp\left\{-\int_r^{\lambda(x)} q(y)dy\right\} \text{ for } \lambda(x) \geq r$$

$$\mathrm{Prob}(R_1 \geq x | R_0 = r) = 1 \text{ for } \lambda(x) < r .$$

In the general case, when R_0 is a random variable with density function f_0 we have

$$\mathrm{Prob}(R_1 \geq x) = \int_0^\infty \mathrm{Prob}(R_1 \geq x | R_0 = r) f_0(r) dr$$

and consequently

$$f_1(x) = -\frac{d}{dx}\mathrm{Prob}(R_1 \geq x) = \int_0^{\lambda(x)} \left\{-\frac{d}{dx}\exp\left(-\int_r^{\lambda(x)} q(y)dy\right)\right\} f_0(r)dr .$$

If we denote

$$K(x,r) = -\frac{d}{dx}\exp\left(-\int_r^{\lambda(x)} q(y)dy\right) \text{ for } \lambda(x) \geq r$$

$$K(x,r) = 0 \text{ for } \lambda(x) < r ,$$

we have

$$f_1(x) = \int_0^{\lambda(x)} K(x,r) f_0(r) dr \ .$$

It is easy to verify, that

$$\int_0^{\infty} K(x,r) dx = 1 \text{ for } r \geq 0$$

and

$$K(x,r) \geq 0 \text{ for } x \geq 0 \ , \ r \geq 0 \ .$$

So if we define the operator $K : L_1(X) \to L_1(X)$ by

(1.1.8.) $$Kf(x) = \int_0^{\lambda(x)} K(x,y) f(y) dy \ ,$$

then K is an integral Markov operator (see def 2.1.4.). The sequence of densities $\{f_n\}$ describing the evolution of the initial size in GLM model is given by $f_n = K^n f_0$ where K is an integral Markov operator given by formula (1.1.8.).

1.2. General Model of Biological System Producing Events

This model is dealed in [11]. Consider a (biological) system which produces events. In addition to the usual laboratory time the system is also assumed to have an internal or physiological time. We denote this internal time by τ to distinguish it from the laboratory (or clock) time t. When an event appears the physiological time resets from the value $\tau = \tau_{max}$ to $\tau = 0$. We assume that the rate of maturation $d\tau/dt$ depends on the amount of an activator (or maturation factor) which we denote by a. Thus we have

(1.2.1.) $$\frac{d\tau}{dt} = \varphi(a), \quad \varphi \geq 0 \ .$$

We further assume that the activator is produced by a dynamics described by the solution of the differential equation

(1.2.2.) $$\frac{da}{dt} = g(a), \quad g \geq 0 \ .$$

The solution of (1.2.2.) satisfying the initial condition $a(0) = r$ will be denoted by

$$a(t) = \Pi(t,r) \ ,$$

and we assume it is defined for all $t \geq 0$. When an event is produced at a time $\tau = \tau_{max}$ and activator level a_{max}, then a portion $\varrho = \varrho(a_{max})$ of a_{max} is consumed in the production of the event. Thus after the event the activator resets to the level

(1.2.3.) $$a = a_{max} - \varrho(a_{max}) \ .$$

We call the function $y - \varrho(y)$ the **reset function** , and assume it is invertible. The inverse of $y - \varrho(y)$ is denoted by λ.

Our main assumption is related to the physiological time. Namely we assume that the survival function of τ_{max} is independent of the initial value of the activator. We denote this survival function by H. Thus we may write

$$\text{(1.2.4.)} \qquad \text{Prob}(\tau_{max} \geq x | a(\tau = 0) = r) = H(x)$$

for every $r > 0$. In the terminology of population dynamics we could say that the lifespan of an organism will be shorter when its rate of maturation is increased.

With these assumptions, we will derive a recurrence relation for the values of activator at the times when events occur. Assume that the events appear at the times

$$t_0 < t_1 < t_2 < \ldots .$$

Let a_n be the amount of the activator at the beginning of the time interval (t_n, t_{n+1}). According to Equation (1.2.2.), this amount at time $t \in (t_n, t_{n+1})$ is given by

$$a = \Pi(t - t_n, a_n) .$$

Now using (1.2.1.) we may calculate the physiological time τ corresponding to t. Namely

$$\text{(1.2.5.)} \qquad \tau = \int_{t_n}^{t} \varphi(\Pi(s - t_n, a_n))ds .$$

Substitute $z = \Pi(s - t_n, a_n))$, $dz = g(\Pi(s - t_n, a_n))ds$ and observe that $z = a_n$ for $s = t_n$ and $z = a$ for $s = t$. Then (1.2.5.) becomes

$$\text{(1.2.6.)} \qquad \tau = \int_{a_n}^{a} q(z)dz = Q(a) - Q(a_n) ,$$

where

$$\text{(1.2.7.)} \qquad q(z) = \frac{\varphi(z)}{g(z)} \text{ and } Q(z) = \int_0^z q(y)dy .$$

The function q has a simple biological interpretation, since it gives the rate of change of the physiological time relative to the activator.

When t approaches t_{n+1}, the physiological time τ and the amount of the activator a take their maximal values which we denote by τ_n and $a_{max,n}$ respectively. In this case equation (1.2.6.) gives

$$\text{(1.2.8.)} \qquad \tau_n = Q(a_{max,n}) - Q(a_n) .$$

Further, from the definition of the reset function we have $a_{n+1} = \lambda^{-1}(a_{max,n})$, and consequently

$$\text{(1.2.9.)} \qquad a_{n+1} = \lambda^{-1}(Q^{-1}(Q(a_n) + \tau_n)) \text{ for } n = 0,\ 1,\ \ldots .$$

This is the desired recurrence relation between successive activator levels at event occurrence. By assumption, the variables a_n and τ_n are independent, see Equation (1.2.4.), and thus we may consider (1.2.9.) as a discrete time dynamical system with stochastic perturbations by the τ_n.

The behaviour of this system from a statistical point of view may be described by the sequence of distributions

$$F_n(x) = \text{Prob}(a_n < x) \quad \text{for} \quad n = 0,\ 1,\ \dots .$$

Set $H_1 = 1 - H$ and denote by $h = H_1'$ the density function of the distribution of τ_n (assuming that this density exists). If a_n has a distribution F_n then $Q(a_n)$ has the distribution $G_n(x) = F_n(Q^{-1}(x))$. Further, since a_n and τ_n are independent, the variable $u_n = Q(a_n) + \tau_n$ has a distribution function given by the convolution

$$\int_0^x H_1(x-y)dG_n(y) = \int_0^{Q^{-1}(x)} H_1(x - Q(y))dF_n(y) \ . \tag{1.2.10.}$$

Finally, $\lambda^{-1}(Q^{-1}(u_n))$ has the distribution function

$$\int_0^{\lambda(x)} H_1(Q(\lambda(x)) - Q(y))dF_n(y) \ .$$

From this and the definition of the density, it follows that $a_{n+1} = \lambda^{-1}(Q^{-1}(u_n))$ has a density

$$f_{n+1}(x) = \lambda'(x)q(\lambda(x)) \int_0^{\lambda(x)} h(Q(\lambda(x)) - Q(y))f_n(y)dy \ . \tag{1.2.11.}$$

Introducing the operator K defined by

$$Kf(x) = \int_0^{\lambda(x)} \left[-\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) \right] f(y)dy \ , \tag{1.2.12.}$$

we may write these relations in the more abbreviated forms $f_{n+1} = Kf_n$ and $f_n = K^n f_0$. Under some simple regularity conditions concerning λ, Q and H, Equation (1.2.12.) defines an integral Markov operator on the space $L_1(\mathbb{R}^+)$ of all integrable functions defined on the half line $\mathbb{R}^+ = [0, \infty)$.These assumptions will be formulated in the section 2.2..

At this point it is worth noting the explicit use of the inverse function $Q^{-1}(x)$ in the derivation of Eqs. (1.2.9.) and (1.2.11.). In some applications it may happen that the functions $\varphi(x)$ and $q(x)$ vanish on an interval $[0, x_0]$ and are only positive for $x > x_0$. In this case it is clear that $Q(x)$ as given by (1.2.7.) also vanishes for $0 \le x \le x_0$ and is thus not invertible. In [11] is shown, that if we denote by Q^{-1} the inverse of Q restricted to $[x_0, \infty)$, then (1.2.9.) and (1.2.11.) are still valid.

If the densities f_1 are given then it is easy to find the density of the distribution of the interevent intervals, i.e., the time intervals $\triangle t_n = t_{n+1} - t_n$ between the n^{th} and $(n+1)^{st}$ events. In fact, Equation (1.2.5.) with $t = t_{n+1}$ gives

$$\tau_n = \int_{t_n}^{t^{n+1}} \varphi(\Pi(s - t_n, a_n))ds = \int_0^{\triangle t_n} \varphi(\Pi(s, a_n))ds \ .$$

Therefore

$$\begin{aligned} \text{Prob}(\triangle t_n \ge x) &= \text{Prob}\left(\tau_n \ge \int_0^x \varphi(\Pi(s, a_n))ds\right) \\ &= \int_0^\infty \text{Prob}\left(\tau_n \ge \int_0^x \varphi(\Pi(s,r))ds | a_n = r\right) f_n(r)dr \ . \end{aligned}$$

From this and (1.2.4.) it follows immediately that

$$\mathrm{Prob}(\triangle t_n \ge x) = \int_0^\infty H\left(\int_0^x \varphi(\Pi(s,r))ds\right) f_n(r)dr \ .$$

By differentiation we can find the density distribution of $\triangle t_n$ which we denote by $\alpha_n(x)$ Namely, the density of the interevent intervals is

$$\alpha_n(x) = \int_0^\infty h\left(\int_0^x \varphi(\Pi(s,r))ds\right) \varphi(\Pi(x,r)) f_n(r)dr \ . \tag{1.2.13.}$$

In the particular case when $f_n = f_*$, $(n = 0,\ 1,\ \ldots)$ is a time independent stationary sequence, α_n has the same property.

1.3. Discrete Time System with Constantly Applied Stochastic Perturbations

Let the process be defined by

$$x_{n+1} = S(x_n) + \xi_n \ , \tag{1.3.1.}$$

where $S : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable transformation and $\xi_0,\ \xi_1,\ \ldots$ are independent random vectors each having the same density g. Denote by f_n the density of x_n. In [8] is found the relation between f_n and f_{n+1}.

By (1.3.1.), x_{n+1} is the sum of two independent random vectors: $S(x_n)$ and ξ_n. Note that $S(x_n)$ and ξ_n are clearly independent since, in calculating $x_1,\ \ldots,\ x_n$, we only need $\xi_0,\ \ldots, \xi_{n-1}$. Let $h : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary, bounded, measurable function. It is easy to find the mathematical expectation of $h(x_{n+1})$, since

$$E(h(x_{n+1})) = \int_{\mathbb{R}^d} h(x) f_{n+1}(x)dx \ . \tag{1.3.2.}$$

Furthermore, by (1.3.1.) and the fact that the joint density of $(x_n,\ \xi_n)$ is just $f_n(y)g(z)$, we also have

$$\begin{aligned} E(h(x_{n+1})) &= E(h(S(x_n) + \xi_n)) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(S(y)+z) f_n(y) g(z) dy dz \ . \end{aligned}$$

By a change of variables, this can be rewritten as

$$E(h(x_{n+1})) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x) f_n(y) g(x - S(y)) dy dx \ . \tag{1.3.3.}$$

Equating (1.3.2.) and (1.3.3.), and using the fact that h was an arbitrary, bounded, measurable function, we immediately obtain

$$f_{n+1}(x) = \int_{\mathbb{R}^d} f_n(y) g(x - S(y)) dy \ .$$

So if we define an integral Markov operator $K : L_1(\mathbb{R}^d) \to L_1(\mathbb{R}^d)$ by

$$Kf(x) = \int_{\mathbb{R}^d} g(x - S(y)) f(y) dy \ ,$$

then K is an integral Markov operator with kernel

$$K(x,y) = g(x - S(y)) \ .$$

It is clear that

$$K(x,y) \ge 0 \text{ and } \int_{\mathbb{R}^d} K(x,y)dx = 1 \quad \forall y \in \mathbb{R}^d \ ,$$

so $K(x,y)$ is a stochastic kernel.

1.4. Discrete Time System with Multiplicative Perturbations

In this section we turn our attention to a discrete time system perturbated in a multiplicative way. Specifically, we examine a process

$$x_{n+1} = \xi_n S(x_n) \ , \tag{1.4.1.}$$

where $S : (0, \infty) \to (0, \infty)$ is continuous and positive a.e. and the ξ_n are independent random variables, each distributed with the same density g. Denote by f_n the density of x_n. In [8] is derived the relation between f_n and f_{n+1}.

Using exactly the same approach employed in Section 1.3., let $h : (0, \infty) \to (0, \infty)$ be an arbitrary bounded and Borel measurable function. The expectation of $h(x_{n+1})$ is given by

$$E(h(x_{n+1})) = \int_0^\infty h(x) f_{n+1}(x) dx \ . \tag{1.4.2.}$$

Using (1.4.1.) we also have

$$\begin{aligned} E(h(x_{n+1})) &= E(\xi_n S(x_n)) \\ &= \int_0^\infty \int_0^\infty h(zS(y)) f_n(y) g(z) dy dz \\ &= \int_0^\infty \int_0^\infty h(x) f_n(y) g\left(\frac{x}{S(y)}\right) \frac{1}{S(y)} dy dx \ , \end{aligned} \tag{1.4.3.}$$

where we used a change of variables $z = x/S(y)$ in passing the second to third lines of (1.4.3.). Equating (1.4.2.) and (1.4.3.), and using the fact that h was arbitrary by assumption, we arrive at

$$f_{n+1}(x) = \int_0^\infty f_n(y) g\left(\frac{x}{S(y)}\right) \frac{1}{S(y)} dy \ . \tag{1.4.4.}$$

From (1.4.4.) we may also write $f_{n+1} = Kf_n$, where the operator K, given by

$$Kf(x) = \int_0^\infty f(y) g\left(\frac{x}{S(y)}\right) \frac{1}{S(y)} dy \ ,$$

is an integral Markov operator with a stochastic kernel

$$K(x, y) = g\left(\frac{x}{S(y)}\right) \frac{1}{S(y)} \ .$$

1.5. The Fokker-Planck Equation

In this section continuous time systems in the presence of noise are presented. This leads to the Fokker-Planck equation, describing the evolution of densities for these systems. We are specifically interested in the behaviour of the system

$$\frac{dx}{dt} = b(x) + \sigma(x)\xi \tag{1.5.1.}$$

with initial condition

$$x(0) = x^0 \ , \tag{1.5.2.}$$

where

$$b(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_d(x) \end{pmatrix} \text{ and } \sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \dots & \sigma_{1d}(x) \\ \vdots & \ddots & \vdots \\ \sigma_{d1}(x) & \dots & \sigma_{dd}(x) \end{pmatrix}$$

are given functions of x and

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_d(t) \end{pmatrix}$$

is the unknown. The function $\sigma(x)$ is the amplitude of the perturbation $\xi = \frac{dw}{dt}$, where w is a d-dimensional Wiener process. The system (1.5.1.)-(1.5.2) is the continuous time analog of the discrete time system with a constantly applied perturbation considered in the Section 1.3. and is in detail discussed in [8]. As in discrete time systems we shall examine the evolution of densities. Denote by $u(t,x)$ the density of $x(t)$ (the solution of (1.5.1.) with (1.5.2.)). Then $u(t,x)$ satisfies

$$\mathrm{Prob}(x(t) \in B) = \int_B u(t,z)dz \ . \tag{1.5.3.}$$

By Theorem 1.5.1. $u(t,x)$ can be found without any knowledge concerning the solution $x(t)$ of the stochastic differential equations (1.5.1) with (1.5.2.) and is the solution of a partial differential equation, known as the Fokker-Planck (or Kolmogorov forward) equation.

Now set

$$a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x) \ . \tag{1.5.4.}$$

From (1.5.4.) it is clear that $a_{ij} = a_{ji}$ and, thus, the quadratic form,

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \ , \tag{1.5.5.}$$

is symmetric. Further, since

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j = \sum_{k=1}^{d}\left(\sum_{i=1}^{d}\sigma_{ik}(x)\lambda_i\right)^2 \ ,$$

(1.5.5.) is nonnegative. Theorem 1.5.1. is proved in [8].

Theorem 1.5.1. *If the functions σ_{ij}, $\partial\sigma_{ij}/\partial x_k$, $\partial^2\sigma_{ij}/\partial x_k\partial x_l$, b_i, $\partial b_i/\partial x_j$, $\partial u/\partial t$, $\partial u/\partial x_i$, and $\partial^2 u/\partial x_i\partial x_j$ are continuous for $t > 0$ and $x \in \mathbb{R}^d$, and if b_i, σ_{ij} and their first derivatives are bounded, then $u(t,x)$ satisfies the equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i\partial x_j}(a_{ij}u) - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}(b_i u) \ , \quad t > 0,\ x \in \mathbb{R}^d \ . \tag{1.5.6.}$$

Moreover, if the initial condition $x(0) = x^0$, which is a random variable, has a density f then

$$u(0,x) = f(x) \ , \quad x \in \mathbb{R}^d \tag{1.5.7.}$$

Equation (1.5.6.) is called the **Fokker-Planck equation** or **Kolmogorov forward equation.**Equation (1.5.6.) is of second order and may be rewritten in the form

$$\begin{aligned}\frac{\partial u}{\partial t} =& \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{d}\tilde{b}_i(x)\frac{\partial u}{\partial x_i} \\ &+ \tilde{c}(x)u \ , \quad t > 0,\ x \in \mathbb{R}^d \ ,\end{aligned} \tag{1.5.8.}$$

where

$$\tilde{b}(x) = -b_i(x) + \sum_{j=1}^{d}\frac{\partial a_{ij}(x)}{\partial x_j}$$

and

$$\tilde{c}(x) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 a_{i,j}(x)}{\partial x_i\partial x_j} - \sum_{i=1}^{d}\frac{\partial b_i}{\partial x_i} \ . \tag{1.5.9.}$$

We have shown that the quadratic form

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j$$

is always nonnegative. We will assume that stronger inequality,

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \geq \varrho\sum_{i=1}^{d}\lambda_i^2 \ , \tag{1.5.10.}$$

where ϱ is a positive constant, holds. This is called the **uniform parabolicity condition.**

It is known that, if the coefficients a_{ij}, $\tilde{b}_i$, and $\tilde{c}_i$ are smooth and satisfy the growth conditions

$$|a_{ij}(x)| \leq M,\ |\tilde{b}_i(x)| \leq M(1+|x|),\ |\tilde{c}(x)| \leq M(1+|x|^2) \ , \tag{1.5.11.}$$

then the classical solution of the Cauchy problem, equations (1.5.8.) with (1.5.7.), is unique and given by the integral formula

$$u(t,x) = \int_{\mathbb{R}^d}\Gamma(t,x,y)f(y)dy \ , \tag{1.5.12.}$$

where the kernel Γ, called the **fundamental solution,** is independent of the initial density f.

Definition 1.5.1. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function. A function $u(t, x)$, $t > 0$, $x \in \mathbb{R}^d$ is called a **classical solution** of Equation (1.5.6.) with initial condition (1.5.7.) if it satisfies the following conditions:

(a) For every $T > 0$ there exist $c > 0$, $\alpha > 0$ such that

$$|u(t,x)| \leq ce^{\alpha|x|^2} \quad \text{for } 0 < t \leq T,\ x \in \mathbb{R}^d \ .$$

(b) $u(t, x)$ has continuous derivatives u_t, u_x , $u_{x_i x_j}$ and satisfies Equation (1.5.8.) for every $t > 0$, $x \in \mathbb{R}^d$; and

(c) $\lim_{t\to 0} u(t, x) = f(x)$.

Remark. Condition (a) is necessary because for functions which grow faster than $e^{\alpha|x|^2}$, the Cauchy problem, even for the heat equation $u_t = \frac{1}{2}\sigma^2 u_{xx}$, is not uniquely determined.

Definition 1.5.2. We say that the coefficients a_{ij} and b_i of Equation (1.5.6.) are **regular for the Cauchy problem** if they are C^4 functions such that the corresponding coefficients a_{ij}, $\tilde{b}_i$, and $\tilde{c}$ of Equation (1.5.8.) satisfy the uniform parabolicity condition (1.5.10.) and growth conditions (1.5.11.).

The following theorem, that ensures the existence and uniqueness of classical solutions, is stated in [8].

Theorem 1.5.2. *Assume that the coefficients a_{ij} and b_i are regular for the Cauchy problem and that f is a continuous function satisfying the inequality $|f(x)| \leq ce^{\alpha|x|^2}$ with constants $c > 0$ and $\alpha > 0$. Then there is a unique classical solution of (1.5.6.)-(1.5.7.) which is given by (1.5.12.). The kernel $\Gamma(t, x, y)$, defined for $t > 0$, x, $y \in \mathbb{R}^d$, is continuous and differentiable with respect to t, is twice differentiable with respect to x_i, and satisfies (1.5.8.) as a function of (t, x) for every fixed y. Further, in every strip $0 < t \leq T$, $x \in \mathbb{R}$, $|y| \leq r$, Γ satisfies the inequalities*

$$0 < \Gamma(t,x,y) \leq \Phi(t, x-y), \qquad \left|\frac{\partial \Gamma}{\partial t}\right| \leq \Phi(t, x-y) \ ,$$

$$\left|\frac{\partial \Gamma}{\partial x_i}\right| \leq \Phi(t, x-y), \qquad \left|\frac{\partial^2 \Gamma}{\partial x_i \partial x_j}\right| \leq \Phi(t, x-y) \ ,$$

where

$$\Phi(t, x-y) = kt^{-(n+2)/2} \exp[-\delta(x-y)^2/t]$$

and the constants δ and k depend on T and r.

Remark. The explicit construction of the fundamental solution Γ for general coefficients a_{ij}, $\tilde{b}_i$ and $\tilde{c}$ is usually impossible. It is easy only for some special cases, such as the heat equation,

$$u_t = (\sigma^2/2)u_{xx} \ .$$

In this case, Γ is the familiar kernel

$$\Gamma(t,x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp[-(x-y)^2/2\sigma^2 t] \ .$$

2.1. Some Properties of Markov Processes and Integral Markov Operators.

In this section some necessary results of [2] are presented. These results are applied in [1], [7].

Definition 2.1.1. A Markov process is defined to be a quadruple (X, Σ, m, P) , where (X, Σ, m) is a $\sigma-$ finite measure space with positive measure and where P is an operator on $L_1(X)$ satisfying
 (i) P is a contraction : $\|P\| \leq 1$
 (ii) P is positive : if $0 \leq u \ \in L_1(X)$ then $Pu \geq 0$

Definition 2.1.2. If u is an arbitrary non-negative function, set

$$Pu := \lim_{k\to\infty} Pu_k \text{ for } 0 \leq u_k \in L_1(X),\ u_k \nearrow u \ ,$$

where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere.

Remark. The sequence Pu_k is increasing so that $\lim_k Pu_k$ exists (it may be infinite). To see that definition is independent of the particular sequence let $v_k \in L_1$ with $v_k \nearrow u$ also. Set $w_{k,n} = \min(u_n, v_k)$. For fixed k, $w_{k,n} \nearrow v_k$, and by Fatou's Lemma

$$\int w_{k,n} dm \nearrow \int v_k dm \ .$$

Thus $w_{k,n} \to v_k$ in L_1 norm and $Pw_{k,n} \nearrow Pv_k$. Since $Pu_n \geq Pw_{k,n}$ for each n, we have

$$\lim_{n\to\infty} Pu_n \geq Pv_k \ .$$

The result follows from symmetry.

Definition 2.1.3. Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\}\ ,\ D = X \setminus C$$

By Theorem 2.1.1. this definition is independent of the choice of u_0.

Lemma 2.1.1. (Hopf Maximal Ergodic Lemma). *Let (X, Σ, m, P) be a Markov process. Let $u \in L_1$ and define*

$$E = \{x : \sup_n (u(x) + Pu(x) + \cdots + P^n u(x)) > 0\} \ .$$

Then $\int_E u(x) dm(x) \geq 0$.

Proof: Note that u may take also negative values. For k a nonnegative integer define operators S_k by $S_0 u(x) \equiv 0$,

$$S_k u(x) = u(x) + Pu(x) + \cdots + P^{k-1} u(x) \ .$$

Set

$$S_n^+(x) = \max_{0\leq k\leq n} S_k u(x) \ .$$

Note that $S_n^+ u(x) \geq 0$ for all n.

Set $E_n = \{x : S_n^+ u(x) > 0\}$. Since $S_n^+ u$ is an increasing sequence, the sets E_n are increasing, and in fact $E_n \nearrow E$. Thus

$$\int_E udm = \lim_{n\to\infty} \int_{E_n} udm \ ,$$

and it suffices to show that $\int_{E_n} udm \geq 0$ for all n.

For $0 \leq k \leq n$, $S_n^+ u \geq S_k u$. Thus

$$PS_n^+ u(x) \geq PS_k u(x) \ ,$$

and

$$u(x) + PS_n^+ u(x) \geq u(x) + PS_k u(x) = S_{k+1} u(x) \ .$$

If $x \in E_n$,

$$S_n^+ u(x) = \max_{1\leq k \leq n} S_k u(x) \ ,$$

since $S_0 u(x) = 0 < S_n^+ u(x)$. The previous inequalities, for $1 \leq k+1 \leq n$, give

$$u(x) + PS_n^+ u(x) \geq S_n^+ u(x)$$

on E_n, or

$$u(x) \geq S_n^+ udm - PS_n^+ u(x)$$

for $x \in E_n$. Thus

$$\int_{E_n} udm \geq \int_{E_n} S_n^+ udm - \int_{E_n} PS_n^+ udm \ .$$

But $\int_{E_n} S_n^+ udm = \int_X S_n^+ udm$. Thus

$$\int_{E_n} udm \geq \int_X S_n^+ udm - \int_X PS_n^+ udm \geq 0 \ ,$$

since $\|P\| \leq 1$. $\square$

Lemma 2.1.2. *Let (X, Σ, m, P) be a Markov process. Let $u,\ v \in L_1$ be nonnegative functions. If*

$$\sum_{k=0}^{\infty} P^k u(x) = \infty \ ,$$

then

$$\textit{either} \ \sum_{k=0}^{\infty} P^k v(x) = \infty \quad \textit{or} \quad \sum_{k=0}^{\infty} P^k v(x) = 0 \ .$$

Proof: Set

$$A_{u,v} = \left\{ x : \sum_{k=0}^{\infty} P^k u(x) = \infty,\ \sum_{k=0}^{\infty} P^k v(x) < \infty \right\} \ .$$

So if $x \in A_{u,v}$ then

$$\sum_{k=0}^{\infty} P^k(u - av)(x) = \infty$$

for every $a > 0$. In particular, if $a > 0$

$$A_{u,v} \subseteq \{x : \sup_n (I + P + \cdots + P^n)(u - av)(x) > 0\} = B_a \ .$$

By Lemma 2.1.1.

$$0 \leq \int_{B_a} (u - av)dm \leq \int_X udm - a \int_{A_{u,v}} vdm \ .$$

Since this holds as $a \to \infty$ it follows that

$$\int_{A_{u,v}} vdm = 0 \ .$$

The same argument applies to the function $P^n v$; since $A_{u,v} = A_{u,P^n v}$, the conclusion is that

$$\int_{A_{u,v}} P^n vdm = 0$$

for every $n \geq 0$. $\square$

Theorem 2.1.1. *If* $0 \leq u \in L_1(X)$ *then*

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \text{ for } x \in D \ , \ \sum_{k=0}^{\infty} P^k u(x) = 0 \text{ or } \infty \text{ for } x \in C \ .$$

Proof: Let $0 < u_0 \in L_1$ be the same as in def. 2.1.3.. Whenever $\sum_{k=0}^{\infty} P^k u(x) = \infty$ then by Lemma 2.1.2. also $\sum_{k=0}^{\infty} P^k u_0(x) = \infty$. Thus $\sum_{k=0}^{\infty} P^k u(x) < \infty$ on D. If $x \in C$ then $\sum_{k=0}^{\infty} P^k u_0(x) = \infty$ and by Lemma 2.1.2. $\sum_{k=0}^{\infty} P^k u(x)$ is either zero or infinity. $\square$

Definition 2.1.4. A function $K(x, y) \geq 0$ defined on XxX which is jointly measurable with respect to its variables is called a kernel. Let $\int_X K(x, y)dx \leq 1$. Define an operator K on $L_1(X)$:

$$Kf(x) = \int_X K(x, y)f(y)dy \ .$$

Then $\|K\| \leq 1$ and K is called an integral Markov operator.

Definition 2.1.5. Let K be an integral Markov operator, then (X, Σ, m, K) is said to be a Harris process if $X = C$.

Theorem 2.1.2. *Let K be an integral Markov operator and a Harris process. Then there exists $0 < u < \infty$ such that $Ku = u$ (a σ-finite invariant measure).*

The proof of Theorem 2.1.2. is complicated. This theorem is one of the main results of [2] (see [2] - Chapter VI.).

Theorem 2.1.3. *Let P be a Markov process with $X = D$. Then there exists $0 < g < \infty$ such that $Pg \leq g$.*

Proof: Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$. $\square$

Definition 2.1.6. Let P be a Markov process. Define operators P_C, P_D:

$$P_C : L_1(C) \to L_1(C) \ , \ P_C f = (P\tilde{f}) \restriction C \ ,$$

where the symbol $\restriction$ denotes the restriction to the set C, $\tilde{f}$ is the function f extended by 0 on D,

$$P_D : L_1(D) \to L_1(D) \ , \ P_D f = (P\tilde{f}) \restriction D \ ,$$

where $\tilde{f}$ is the function f extended by 0 on C.

Lemma 2.1.3. *Let (X, Σ, m, P) be a Markov process. If $0 \leq f \in L_\infty$, then*

$$\sum_{k=0}^{\infty} P^{*k} f(x) = 0 \quad \textit{or} \quad \infty$$

for $x \in C$ (P^* denotes the operator adjoint to P).

Proof: Fix $0 \leq f \in L_\infty$ and suppose there is a set $A \subseteq C$ with $m(A) > 0$ on which

$$\sum_{k=0}^{\infty} P^{*k} f(x) \leq M < \infty \ .$$

Take $0 \leq u \in L_1$ with $u = 0$ outside of A and $u > 0$ in A. Then

$$\langle \sum_{k=0}^{\infty} P^k u, P^n f \rangle = \langle u, \sum_{k=n}^{\infty} P^{*k} f \rangle \leq M \|u\|_1 < \infty \ .$$

Since $A \subseteq C$, $\sum_{k=0}^{\infty} P^k u$ must be either 0 or ∞. Since $u > 0$ on A the former cannot occur. Thus $P^{*n} f$ must be 0 on A. $\square$

Theorem 2.1.4. *Let (X, Σ, m, P) be a Markov process. If supp $f \subseteq C$, then supp $Pf \subseteq C$. (supp $f = \{x : f(x) \neq 0\}$)*

Proof: Take $0 < u \in L_1$. Since $\sum_{k=0}^{\infty} P^k u < \infty$ on D, it is not difficult to find a sequence $B_n \nearrow D$ such that

$$\infty > \langle \sum_{k=0}^{\infty} P^k u, 1_{B_n} \rangle = \langle u, \sum_{k=0}^{\infty} P^{*k} 1_{B_n} \rangle \ .$$

Since $u > 0$,

$$\sum_{k=0}^{\infty} P^{*k} P^* 1_{B_n}(x) \leq \sum_{k=0}^{\infty} P^{*k} 1_{B_n}(x) < \infty \ .$$

By Lemma 2.1.3. $P^* 1_{B_n}(x) = 0$ if $x \in C$. Thus $P^* 1_D(x) = 0$ if $x \in C$, or $P^* 1_D \leq 1_D$.

Let $0 \leq f \in L_1$, supp $f \subseteq C$. Then

$$\langle Pf, 1_D \rangle = \langle f, P^* 1_D \rangle \leq \langle f, 1_D \rangle = 0 \ . \quad \square$$

Corollary 2.1.1. *Let K be an integral Markov operator. Then*

$$(C, \Sigma \restriction C, m \restriction C, K_C)$$

is a Harris process. ($\Sigma \restriction C$ denotes the σ-algebra restricted to the space C, $m \restriction C$ denotes the measure m restricted to the space $\Sigma \restriction C$).

Proof: By Theorem 2.1.4. supp $f \subseteq C$ implies supp $Kf \subseteq C$. By Theorem 2.1.1. for $u > 0$ on C,$u = 0$ on D:

$$\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \restriction C)(x)$$

for every $x \in C$. $\square$

Corollary 2.1.2. *Let P be a Markov process on $L_1(X)$. Then*

$$P_D(f \restriction D) = (Pf) \restriction D \ .$$

Proof: $f = f_D + f_C$, where $f_C = f.1_C, f_D = f.1_D$. By Theorem 2.1.4. $(Pf_C) \restriction D = 0$, hence

$$(Pf) \restriction D = (Pf_D) \restriction D = P_D(f \restriction D) \ . \quad \square$$

Corollary 2.1.3. $P_D^n(f \restriction D) = (P^n f) \restriction D$.

Corollary 2.1.4. *Let P be a Markov process on X, let $u > 0$ on D. Then*

$$\sum_{n=0}^{\infty} P_D^n u < \infty \ .$$

Proof: Let $\tilde{u}$ be a function on X such that $\tilde{u} \restriction C = 0, \tilde{u} \restriction D = u$. By Corollary 2.1.3.

$$\sum_{n=0}^{\infty} P_D^n u = (\sum_{n=0}^{\infty} P^n \tilde{u}) \restriction D \ .$$

By Theorem 2.1.1. $(\sum_{n=0}^{\infty} P^n \tilde{u}) \restriction D < \infty$. $\square$

The following two theorems deal an existence of a finite invariant measure. These theorems are not used in the dissertation, so we shall not prove Theorem 2.1.5.. Theorem 2.1.6. is an immediate consequence of Theorem 2.1.5.. The proof of Theorem 2.1.5. can be found in [2] (Chapter IV.).

Theorem 2.1.5. *Let (X, Σ, m, P) be a Markov process. Let P^* be the operator adjoint to P. Then X may be decomposed uniquely into disjoint union $X = A_0 \cup A_1$, where*

(i) $A_0 = \bigcup A_n$ *for sets* A_n *with* $A_n \subseteq A_{n+1}$ *and*

$$1/k \sum_{j=0}^{k-1} P^{*j} 1_{A_n} \to 0$$

uniformly off a set of measure 0;

(ii) There exists a finite measure μ with $P\mu = \mu$, μ is equivalent to the restriction of m to A_1, and every finite invariant measure is weaker than μ.

Theorem 2.1.6. *Let (X, Σ, m, P) be a Markov process. Let P^* be the operator adjoint to P. Then the following are equivalent:*

(a) *there exists a finite invariant measure equivalent to m*

(b) *if $0 \leq f \in L_\infty$, $f \not\equiv 0$, then $\liminf_n \langle P^n 1, f\rangle > 0$*

(c) *if $0 \leq f \in L_\infty$, $f \not\equiv 0$, then $\liminf_N 1/N \sum_{n=0}^{N-1} \langle P^n 1, f\rangle > 0$*

(d) *if $0 \leq f \in L_\infty$, $f \not\equiv 0$, then $\limsup_N 1/N \sum_{n=0}^{N-1} \langle P^n 1, f\rangle > 0$*

(e) *there is no set A of positive measure for which*

$$\lim_N 1/N \sum_{n=0}^{N-1} P^{*n} 1_A(x) = 0 \ \textit{(a.e.)}$$

(f) *there is no set A of positive measure for which*

$$\lim_N 1/N \sum_{n=0}^{N-1} P^{*n} 1_A(x) = 0 \ \textit{uniformly (a.e.)}$$

Proof Clearly $(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \Longrightarrow (f)$. $(f) \Longrightarrow (a)$ is an immediate consequence of Theorem 2.1.5.. $\square$

For the remainder of this section we shall assume that (X, Σ, λ, P) is a fixed Markov process and λ a σ -finite subinvariant measure $(P1 \leq 1)$.

Theorem 2.1.7. *P and P^* are both positive contractions on each L_p , $1 \leq p \leq \infty$.*

Proof: Since (X, Σ, λ, P) is a Markov process, P^* is a positive contraction on L_∞. Let $g = \sum c_i 1_{A_i}$, where A_i are disjoint measurable sets. Functions of this type are dense in L_1. Then $P^* g$ satisfies

$$\begin{aligned} \|P^* g\|_1 &= \int |P^*(\sum c_i 1_{A_i})| d\lambda \leq \sum |c_i| \int P^* 1_{A_i} d\lambda \\ &= \sum |c_i| (P\lambda)(A_i) \leq \sum |c_i| \lambda(A_i) = \|g\|_1 \ . \end{aligned}$$

Extending the operator P^* to all of L_1 we have that P^* is a contraction on L_1. The Riesz-Thorin Theorem now shows that P^* is also a positive contraction on each space L_p with $1 \leq p \leq \infty$.

The operator P^* has an adjoint acting on L_∞ which we call P^{**}. It is clear that P^{**} is a contraction on L_∞. If $f,\ g$ belong to $L_1 \cap L_\infty$ then

$$\langle f, P^* g\rangle = \langle P^{**} f, g\rangle \ .$$

But $\langle f, P^* g\rangle = \langle Pf, g\rangle$ and hence $P^{**} f = Pf$ for $f \in L_1 \cap L_\infty$. Extending P^{**} to all of L_1 we have $P^{**} f = Pf \quad \forall f \in L_1$. $\square$

Denote

$$K = \{f \in L_2(X) \ : \ \|P^n f\|_{L_2} = \|P^{*n} f\|_{L_2} = \|f\|_{L_2} \ , \ n = 1, 2, \ldots\}$$

Theorem 2.1.8.

(*i*) *K is an invariant subspace P and P^**

(*ii*) *on K, $PP^* = P^*P = I$*

(*iii*) *if $f \perp K$ then weak* $\lim P^n f =$ *weak* $\lim P^{*n} f = 0$.

Proof: If $f \in K$ then

$$\|f\|^2 = \langle P^n f, P^n f\rangle = \langle P^{*n}P^n f, f\rangle \le \|P^{*n}P^n f\|\|f\| = \|f\|^2 .$$

Equality in the Cauchy-Schwarz inequality shows that $P^{*n}P^n f = f$ and similarly $P^n P^{*n} f = f$. Conversely, if $f = P^{*n}P^n f = P^n P^{*n} f$ then

$$\|f\|^2 = \langle P^{*n}P^n f, f\rangle = \|P^n f\|^2$$

and $\|f\|^2 = \|P^{*n} f\|^2$. Thus

$$K = \{f \in L_2 : P^{*n}P^n f = f = P^n P^{*n} f, \quad n = 1, 2, \dots\} ,$$

which proves (*ii*). This characterization also shows, that K is a subspace.

By symmetry it will suffice to prove only half of (*iii*): we show that $P^n g$ converges weakly to 0 for all $g \in K^\perp$. Suppose that $P^n g$ does not converge weakly to 0. Then there exists $\varepsilon > 0$, $f \in L_2$ and a subsequence $\{n_i\}$ such that $|\langle P^{n_i} g, f\rangle| \ge \varepsilon$. Since P is a contraction and since the unit ball of L_2 is weakly sequentially compact, some subsequence of $\{P^{n_i} g\}$ must converge weakly. But we shall show that every such weakly convergent subsequence must converge weakly to 0, a contradiction.

Fix k, and let $h \in L_2$. Then

$$\begin{aligned}\|P^{*k}P^kP^n h - P^n h\|^2 &= \|P^{*k}P^{k+n}h\|^2 - 2\langle P^{*k}P^{k+n}h, P^n h\rangle + \|P^n h\|^2 \\ &\le \|P^{k+n}h\|^2 - 2\|P^{k+n}h\|^2 + \|P^n h\|^2 \\ &= \|P^n h\|^2 - \|P^{k+n}h\|^2 .\end{aligned}$$

Since $\|P^n h\|$ is a monotone decreasing sequence the last expression tends to 0 as $n \to \infty$ by the Cauchy criterion. Thus $\|P^{*k}P^kP^n h - P^n h\| \to 0$ as $n \to \infty$.

Suppose now that $P^{n_i} g \to \tilde f$ weakly. We show that $\tilde f = 0$. Now $P^{*k}P^kP^{n_i} g \to P^{*k}P^k \tilde f$ weakly. But by the above results

$$\text{weak lim } P^{*k}P^kP^{n_i} g = \text{ weak lim } P^{n_i} g = \tilde f .$$

Thus $\tilde f = P^{*k}P^k \tilde f$ for all $k \ge 1$ and similarly $\tilde f = P^kP^{*k}\tilde f$. Hence $\tilde f \in K$. But $g \in K^\perp$, so that $\tilde f =$ weak lim $P^{n_i} g \in K^\perp$ also. Thus $\tilde f \in K \cap K^\perp = \{0\}$. $\square$

Denote

$$\Sigma_1 = \{A \in \Sigma \ : 1_A \in K\}$$

$$X_1 = \bigcup \Sigma_1 \text{ (precisely } 1_{X_1} = \text{ess sup } \{1_A \ : A \in \Sigma_1\} \text{)}$$

$$X_2 = X \setminus X_1$$

Theorem 2.1.9.

(i) Σ_1 *is a field. If* $B = \bigcup A_n$ *for* $A_n \in \Sigma_1$ *and* $\lambda(B) < \infty$ *then* $B \in \Sigma_1$.

(ii) P *and* P^* *are automorphisms of* Σ_1.

(iii) Σ_1 *generates* K *:* $0 \leq f \in K$ *if and only if* $\{f > a\} \in \Sigma_1$ *for all* $a > 0$.

Proof: We prove (iii) and (i) in several stages.

(a) If $f \in K$ then $|f| \in K$:

$$\|f\| = \|P^n f\| \leq \|P^n |f|\| \leq \||f|\| = \|f\| ,$$

and similarly for P^*.

(b) If $f, g \in K$, then $\max(f, g)$ and $\min(f, g) \in K$:

$$\max(f, g) = \frac{1}{2}(|f - g| + f + g) \quad \text{and} \quad \min(f, g) = \frac{1}{2}(f + g - |f - g|) .$$

(c) If $A, B \in \Sigma_1$, then $A \cup B$, $A \cap B$ and $A \setminus B \in \Sigma_1$:

$$1_{A \cup B} = \max(1_A, 1_B),\ 1_{A \cap B} = \min(1_A, 1_B),\ 1_{A \setminus B} = 1_A - 1_{A \cap B} .$$

Thus Σ_1 is a field. Let A_n and B are as in (i). We may assume, that $A_{n+1} \supseteq A_n$. Then

$$P^k P^{*k} 1_{A_n} \nearrow 1_B .$$

Since the convergence is in L_2 norm $P^k P^{*k} 1_B = 1_B$, and $B \in \Sigma_1$.

(d) If $0 \leq f \in K$ and a is a positive constant, then $g = \min(f, a) \in K$: $Pg \leq Pf$ and $Pg \leq a$. Thus $Pg \leq \min(Pf, a)$. Similar results hold for P^*, P^n and P^{*n}. Thus

$$P^n P^{*n} g \leq P^n \min(P^{*n} f, a) \leq \min(P^n P^{*n} f, a) = g ;$$

similarly $P^{*n} P^n g \leq g$. Consequently

$$P^n P^{*n} (f - g) \geq f - g \quad \text{and} \quad P^{*n} P^n (f - g) \geq f - g .$$

Since P and P^* are contractions, equality must hold.

(e) If $f \in K$ then $\{x : f(x) > a > 0\} \in \Sigma_1$: we may assume $f \geq 0$ since $f^+ \in K$ by (b). By (d) $(f - a)^+ = f - \min(f, a) \in K$. Thus $\min(n(f - a)^+, 1) \in K$. As $n \to \infty$ this tends to $1_{\{f > a\}}$ in L_2 norm.

To prove (ii) let $f = P1_A$ for $A \in \Sigma_1$. We show that $f = 1_B$ for $B \in \Sigma_1$. Clearly $0 \leq f \leq 1$. Fix $\varepsilon, \delta > 0$ and set

$$E = \{x : \varepsilon < f \leq 1 - \delta\} = \{x : \varepsilon < f\} \setminus \{x : 1 - \delta < f\} .$$

Since $f \in K$, $E \in \Sigma_1$. Set $g = (1 - f)1_E$. Then $0 \leq g \leq 1$, $0 \leq f + g \leq 1$, and $g \not\equiv 0$ if E has positive measure. Also $g = \min(1_E - f, 1_E) \in K$. Thus $P^*(g + f) \leq 1$. But $P^* f = P^* P 1_A = 1_A$, so that $P^* g \leq 1 - 1_A$, and $P^* g = 0$ on A. So

$$0 = (1_A, P^* g) = (P1_A, g) = (f, g) \geq \int_E f g d\lambda ,$$

so that $\lambda(E) = 0$. Let ε and δ tend to 0: it follows that $f = 1_B$. It is clear that $B \in \Sigma_1$. The demonstration that P^* maps Σ_1 into Σ_1 is similar. Now P and P^* must map Σ_1 onto Σ_1: if A is given then $P(P^* 1_A) = 1_A$, and $P^* 1_A = 1_B$ for $B \in \Sigma_1$. $\square$

Remark. Let Σ_1 be atomic. Let W be an atom of Σ_1. Then $P^n 1_W = 1_{A_n}$ for $A_n \in \Sigma_1$; Since P^n is invertible, A_n must be an atom. Two possibilities arise: either all the atoms $P^n 1_W$ are distinct, or there is a smallest index k with $P^k 1_W = 1_W$. In the former case W is called wandering: then $W \subseteq D$, for $0 < \sum P^n 1_W < \infty$. In the latter case $P^j 1_W$ are distinct for $0 \leq j \leq k-1$, and W is called cyclic of order k.

Lemma 2.1.4. *Let (X, Σ, m) be a measure space. Let $K : L_1(X) \to L_1(X)$ be an integral Markov operator. Then Σ_1 is atomic.*

Proof: Let K be an integral Markov operator with kernel $K(x,y)$ on XxX. Suppose, that Σ_1 is not atomic. Then there exists $A \in \Sigma_1$, such that for every $B \in \Sigma_1,\ m(B) > 0,\ B \subseteq A$ there exists $C \in \Sigma_1,\ C \subseteq B$ with $0 < m(C) < m(B)$. Denote

$$\tilde{A}_\delta = \{y \in A : m\{x : K(x,y) > 0\} > \delta\} .$$

If $m(\tilde{A}_\delta) = 0$ for every $\delta > 0$, then $K(x,y) = 0$ on XxA, which contradicts $\|K 1_A\| = \|1_A\|$. So fix $\delta > 0$ such that $m(\tilde{A}_\delta) > 0$.

Now we prove that there exist $M, N \in \Sigma_1,\ M \cap N = \varnothing$ such that $M \cap \tilde{A}_\delta,\ N \cap \tilde{A}_\delta$ have positive measures. Suppose, that such sets M, N do not exist. Then

$$(\forall M \subseteq A,\ m \in \Sigma_1)(m(M \cap \tilde{A}_\delta) > 0 \Longrightarrow M \supseteq \tilde{A}_\delta) .$$

Let

$$\mathcal{F} = \{M \in \Sigma_1 : M \subseteq A \ \&\ M \supseteq \tilde{A}_\delta\} ,$$

$$c = \inf\{m(M); M \in \mathcal{F}\} . \tag{2.1.1.}$$

Clearly $c \geq m(\tilde{A}_\delta) > 0$. Since Σ_1 is a field, it is easy to construct a sequence of sets $\{M_i\}$, such that $M_{i+1} \subseteq M_i$, and $m(M_i) \to c$. Using (i) of Theorem 2.1.9. we have

$$\bigcap M_i = M \in \Sigma_1 .$$

Since $M \supseteq \tilde{A}_\delta,\ m(M) > 0$. Since no subset of A is atom of Σ_1, we have

$$M = P \cup Q ,$$

where $P, Q \in \Sigma_1,\ m(P), m(Q) > 0$. Then $P \supseteq \tilde{A}_\delta$ or $Q \supseteq \tilde{A}_\delta$. Let $P \supseteq \tilde{A}_\delta$. Then $m(P) < m(M) = c$, which contradicts (2.1.1.). Thus we have proved, that there exist $M, N \in \Sigma_1,\ M \cap N = \varnothing$ with positive measures such that $m(\tilde{A}_\delta \cap M) > 0$, $m(\tilde{A}_\delta \cap N) > 0$ and $A = M \cup N$. Then $m(M) \leq \frac{1}{2} m(A)$ or $m(N) \leq \frac{1}{2} m(A)$. Let $m(M) \leq \frac{1}{2} m(A)$. Put $A_1 = M$. By the same way construct the sequence of sets $A_i \in \Sigma_1$ such that

$$A \supseteq A_1 \supseteq A_2 \supseteq \dots , \quad m(A_{i+1}) \leq \frac{1}{2} m(A_i) ,$$

and $m(A_i \cap \tilde{A}_\delta) > 0\ \forall i \in \mathbb{N}$. Take n such that $m(A_n) < \delta/2$. Since $A_n \in \Sigma_1$, we have $K 1_{A_n} = 1_B$, where $m(B) = m(A_n) < \delta/2$. Now

$$\begin{aligned} 0 = \int_{X \setminus B} 1_B(x) dx &= \int_{X \setminus B} \int_X K(x,y) 1_{A_n}(y) dy dx = \\ &= \int_X \int_{X \setminus B} K(x,y) dx 1_{A_n}(y) dy = \\ &= \int_X \alpha(y) 1_{A_n}(y) dy . \end{aligned}$$

But $\alpha(y) = \int_{X \setminus B} K(x,y)dx > 0$ on the set $A_n \cap \tilde{A}_\delta$, which is in contradiction with $0 = \int_{X \setminus B} 1_B(x)dx$. Thus A must contain some atom. $\square$

Theorem 2.1.10. *Let Σ_1 be atomic. Let A be a set with $\lambda(A) < \infty$ and $f \in L_1(X)$. Then*

(i) if $A \subseteq X_2$ then

$$\lim_{n\to\infty} \int_A P^n f d\lambda = 0 \ , \ \lim_{n\to\infty} \int_A P^{*n} f d\lambda = 0$$

(ii) if $A \subseteq W$ where W is wandering then

$$\lim_{n\to\infty} \int_A P^n f d\lambda = \lim_{n\to\infty} \int_A P^{*n} f d\lambda = 0$$

(iii) if $A \subseteq W$ is cyclic of order k then

$$\lim_{n\to\infty} \int_A P^{nk+d} f d\lambda = \lambda(A) \frac{\int_W P^d f d\lambda}{\lambda(W)}$$

$$\lim_{n\to\infty} \int_A P^{*nk+d} f d\lambda = \lambda(A) \frac{\int_W P^{*d} f d\lambda}{\lambda(W)}$$

Proof: (i) is an immediate consequence of Theorem 2.1.8. (iii) and Theorem 2.1.9. (iii). For (ii) observe that $\sum \int_{P^n W} f d\lambda < \infty$ since $f \in L_1$, so that

$$\sum \langle f, P^n 1_A \rangle \leq \sum \langle f, P^n 1_W \rangle < \infty \ .$$

To prove (iii) it suffices to let X be the single cycle $\cup_0^{k-1} P^j W$. Fix $A \in \Sigma$ with $A \subseteq W$, and consider the function $(\lambda(A))/(\lambda(W))1_W$. Now $1_A - (\lambda(A))/(\lambda(W))1_W$ is orthogonal to K: it is clearly orthogonal to $P^j 1_W$, for $j \not\equiv 0 \pmod k$, and

$$(1_W, 1_A - \frac{\lambda(A)}{\lambda(W)}.1_W) = \lambda(A) - \lambda(A) = 0 \ .$$

Set $g = 1_A - (\lambda(A))/(\lambda(W))1_W$. By Theorem 2.1.8. $P^n g$ and $P^{*n} g$ tend weakly to 0. If $f \in L_2$ then

$$\begin{aligned} \langle P^{nk+d} f, 1_A \rangle &= \langle f, P^{*nk+d} g \rangle + \langle f, P^{*nk+d} \frac{\lambda(A)}{\lambda(W)} 1_W \rangle \\ &\to \frac{\lambda(A)}{\lambda(W)} \int_{P^{*d} W} f d\lambda = \frac{\lambda(A)}{\lambda(W)} \int_W P^d f d\lambda \end{aligned}$$

as $n \to \infty$.

Let $h \in L_1$ and fix $\varepsilon > 0$. Choose $f \in L_2$ with $\|h - f\|_1 < \varepsilon$. Then

$$\begin{aligned}\langle P^{nk+d}h, 1_A\rangle - \frac{\lambda(A)}{\lambda(W)}\int_W P^d h d\lambda \\ &= \langle h - f, P^{*nk+d}1_A\rangle - \frac{\lambda(A)}{\lambda(W)}\int_W P^d(h-f)d\lambda \\ &+ \langle f, P^{*nk+d}1_A\rangle - \frac{\lambda(A)}{\lambda(W)}\int_W P^d f d\lambda \\ &\leq \|h-f\|_1 + \frac{\lambda(A)}{\lambda(W)}\|h-f\|_1 + \varepsilon < 3\varepsilon\ ,\end{aligned}$$

since

$$\langle P^{nk+d}f, 1_A\rangle - \frac{\lambda(A)}{\lambda(W)}\int_W P^d f d\lambda < \varepsilon$$

for sufficiently large n. The results for P^* follow by symmetry. $\square$

2.2. The Results Based on the Results of Foguel ([2]).

Definition 2.2.1. Let K be an integral Markov operator:

$$Kf(x) = \int_X K(x,y)f(y)dy\ .$$

If $\int_X K(x,y)dx = 1$, then K is called a stochastic integral Markov operator.

Definition 2.2.2. Let a family $\mathcal{A} \subset \Sigma$ be given. A Markov process is called sweeping with respect to $\mathcal{A}$, if

$$\lim_{n\to\infty}\int_A P^n f dm = 0$$

for $A \in \mathcal{A}$ and every density f.

In the sequel we shall assume that $\mathcal{A}$ satisfies the following properties:
(i) $0 < m(A) < \infty$ for $A \in \mathcal{A}$
(ii) $A_1, A_2 \in \mathcal{A}$ implies $A_1 \cup A_2 \in \mathcal{A}$
(iii) There exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\cup A_n = X$.
A family satisfying $(i) - (iii)$ will be called admissible.

Definition 2.2.3. Let (X, Σ, m) and an admissible family $\mathcal{A} \subseteq \Sigma$ be given. A measurable function $f : X \to \mathbb{R}$ is called locally integrable, if

$$\int_A |f|dm < \infty \text{ for } A \in \mathcal{A}\ .$$

Definition 2.2.4. Let a Markov process P be given. A function f is called density, if $\|f\| = 1$ and $f \geq 0$. A density f is called stationary, if $Pf = f$. The operator is called asymptotically stable, if there is a density f_* such that

$$\lim_{n\to\infty}\|P^n f - f_*\| = 0$$

for every density f.

The following theorem is proved in [7] and is a consequence of Theorem 2.1.10..

Theorem 2.2.1. *Let a measure space (X, Σ, m), an admissible family $\mathcal{A}$ and an integral Markov operator K be given. If K has no stationary density but there exists a positive locally integrable f_* subinvariant with respect to K, then K is sweeping.*

Proof: Consider a new measure space (X, Σ, m_*) where $dm_* = f_* dm$ and define an operator $\tilde{K}$ by the formula

$$\tilde{K}h(x) = \int_X \tilde{K}(x,y)h(y)dm_*(y), \quad \tilde{K}(x,y) = K(x,y)/f_*(x) \ .$$

It is easy to verify that $\tilde{K}$ is a Markov operator on $\tilde{L}_1 = L_1(X, \Sigma, m_*)$ and that $\tilde{K}$ has no invariant density; further $\tilde{K}1 = Kf_*/f_* \leq 1$. This inequality allows to use the classical decomposition technique [2]. Let $\Sigma_1,\ X_1,\ X_2$, defined for operator $\tilde{K}$ be the same as in section 2.1.. We have for every $A \subseteq X_2,\ m_*(A) < \infty$ and $h \in \tilde{L}_1$

$$\lim_{n\to\infty} \int_A \tilde{K}^n h dm_* = \lim_{n\to\infty} \int_A \tilde{U}^n h dm_* = 0 \ ,$$

where $\tilde{U}$ is adjoint to $\tilde{K}$. By Lemma 2.1.4. Σ_1 is atomic. Thus Σ_1 consists of sums (with finite measure) of atoms: $A_1,\ A_2,\ \ldots$. Since $\tilde{K}$ has no invariant density

$$\lim_{n\to\infty} \int_{A_k} \tilde{K}^n h dm_* = \lim_{n\to\infty} \int_{A_k} \tilde{U}^n h dm_* = 0 \quad \text{for } h \in \tilde{L}_1$$

holds for every atom $A_k \in \Sigma_1$.

Now we are in a position to verify that K is sweeping. Let $f \in L_1$ and $A \in \mathcal{A}$ be fixed. Since f_* is integrable, $m_*(A) < \infty$. Setting $h = f/f_*$ we obtain

$$\int_A K^n f dm = \int_A \tilde{K}^n h dm_* = \int_{A\cap X_2} \tilde{K}^n h dm_* + \int_{A\cap X_1} \tilde{K}^n h dm_* \ .$$

Since $A \cap X_2 \subseteq X_2$ the first integral on the right hand side converges to zero. In order to evaluate the second one, choose a $\mu > 0$. There exist an $M > 0$ and a sequence of atoms $A_1,\ A_2,\ \ldots,\ A_k$ such that

$$\int_{X_1} (h - M1_B)^+ dm_* \leq \mu \quad \text{where } B = \cup_k A_k \ .$$

Thus

$$\begin{aligned}\int_{X_1\cap A} \tilde{K}^n h dm_* &\leq \int_{X_1} \tilde{K}^n (h - M1_B)^+ dm_* + M \int_{X_1\cap A} \tilde{K}^n 1_B dm_* \\ &\leq \mu + M \int_B \tilde{U}^n 1_{X_1\cap A} dm_* \ .\end{aligned}$$

Since B is a finite sum of atoms, the last integral converges to zero and the proof is completed. $\square$

Theorems 2.2.2, 2.2.3. and Corollary 2.2.1. are proved in [1].

Definition 2.2.5. We say that a Markov process P overlaps supports if for every two densities f, g there is a positive integer $n_0 = n_0(f, g)$ such that

$$\mu(\text{supp } P^{n_0} f \cap \text{supp } P^{n_0} g) > 0 \ .$$

Observe that this condition implies that

$$\mu(\text{supp } P^n f \cap \text{supp } P^n g) > 0 \text{ for } n \geq n_0(f, g) \ .$$

In fact,

$$\text{supp } P^n f \cap \text{supp } P^n g \supseteq \ \text{supp } P^{n-n_0}(\min \{P^{n_0} f, P^{n_0} g\}) \ .$$

Theorem 2.2.2. *A stochastic integral Markov operator which overlaps supports and has a stationary density $f_* > 0$ a.e. is asymptotically stable.*

Proof As in the proof of Theorem 2.2.1. consider a new measure space (X, Σ, m_*) where $dm_* = f_* dm$ and define an operator $\tilde{K}$ by the formula

$$\tilde{K}h(x) = \int_X \tilde{K}(x, y) h(y) dm_*(y), \quad \tilde{K}(x, y) = K(x, y)/f_*(x) \ .$$

It is evident that $\tilde{K}$ is an integral Markov operator on the space $\tilde{L}_1 = L_1(X, \Sigma, m_*)$ and that

$$\tilde{K} 1_X = 1_X \ .$$

This equality allows to use the classical decomposition technique [2]. Let Σ_1, X_1, X_2 defined for operator $\tilde{K}$ be the same as in section 2.1..

Let $f = 1_{X_2}$, $g = 1_X$. Then by Theorem 2.1.10. (i)

$$\lim_{n\to\infty} \int_X g . \tilde{K}^n f dm_* = 0 \ .$$

Since $\tilde{K}$ preserves the integral with respect to m_*, this gives $m_*(X_2) = 0$. Assume that W_i is a wandering atom of Σ_1. Then

$$\text{supp } (\tilde{K}^n 1_{W_i}) \cap \text{supp } (\tilde{K}^n \tilde{K} 1_{W_i}) = \text{supp } (\tilde{K}^n 1_{W_i}) \cap \text{supp } (\tilde{K}^{n+1} 1_{W_i}) = \varnothing$$

for every n, which contradicts the fact that $\tilde{K}$ overlaps supports (since $\tilde{K}^n f = 1/f_* K^n(f.f_*)$, $\tilde{K}$ also overlaps supports). Thus there are no atoms of Σ_1. Assume now that W_i is a cyclic atom of Σ_1 with period $k \geq 2$. Then, as previously,

$$\text{supp } (\tilde{K}^n 1_{W_i}) \cap \text{supp } (\tilde{K}^n \tilde{K} 1_{W_i}) = \text{supp } (\tilde{K}^n 1_{W_i}) \cap \text{supp } (\tilde{K}^{n+1} 1_{W_i}) = \varnothing$$

for every n. Consequently each atom W_i of Σ_1 is cyclic with period $k = 1$. Assume that there are two such sets W_1 and W_2. Then

$$\text{supp } (\tilde{K}^n 1_{W_1}) \cap \text{supp } (\tilde{K}^n 1_{W_2}) = W_1 \cap W_2 = \varnothing$$

for every n which again contradicts overlaping supports. Thus there is exactly one cyclic set W_1 with length of cycle $k = 1$ and $X = W_1$. By Theorem 2.1.10 (iii) we have

$$\lim_{n\to\infty} \|\tilde{K}^n f - \left(\int_X f dm_*\right) 1_X\|_{\tilde{L}_1} = 0$$

for every $f \in \tilde{L}_1$. Let $f \in L_1$, $h = f/f_*$. Then

$$\lim_{n\to\infty} \|K^n f - f_*\|_{L_1} = \lim_{n\to\infty} \|\tilde{K}^n h - 1_X\|_{\tilde{L}_1} = 0 \ .$$

Thus K is symptotically stable. □

Sometimes we have a stationary density that is not positive on the whole space which is an important assumption in the Theorem 2.2.2.. This situation may be improved by studying P restricted to the support of the invariant density.

Let a Markov process $P : L_1(X, \Sigma, m) \to L_1(X, \Sigma, m)$ be given. It is well known that for every nonnegative f, $f_* \in L_1(X)$ the inclusion supp $f \subseteq$ supp f_* implies supp $Pf \subseteq$ supp Pf_*. In particular, if $f_* = Pf_*$ and supp $f_* = C$ then supp $f \subseteq C$ implies supp $Pf \subseteq C$. This property allows to consider P on the space $L_1(C)$ of all functions from $L_1(X)$ with supports contained in C. We will denote P restricted to $L_1(C)$ by P_C.

Theorem 2.2.3. *Let $P : L_1(X, \Sigma, m) \to L_1(X, \Sigma, m)$ be a Markov process with $\|P\| = 1$ having an invariant density f_*. Assume that the operator P_C with $C =$ supp f_* is asymptotically stable. Assume moreover that there is a $\delta > 0$ such that*

$$\sup_n \int_C P^n f dm \geq \delta \text{ for every density } f \ . \tag{2.2.1.}$$

Then the operator $P : L_1(X) \to L_1(X)$ is also asymptotically stable.

Proof: According to the Lower Bound Function Theorem (Theorem 2.3.3.) it is sufficient to find a nonegative $h \in L_1$, $\|h\| > 0$ such that

$$\lim_{n\to\infty} \|(P^n f - h)^-\|_{L_1} = 0$$

for every density f. Define $h = \frac{1}{2}\delta f_*$ and fix a density f. According to (2.2.1.) there is an integer m such that

$$\eta := \int_C P^m f dm \geq \frac{1}{2}\delta \ .$$

For $n \geq m$ we have

$$P^n f = P^{n-m}(1_{X\setminus C} P^m f) + P_C^{n-m}(1_C P^m f) \ .$$

Since P_C is asymptotically stable with the invariant density f_* we also have

$$\lim_{n\to\infty} \|P_C^{n-m}(1_C P^m f) - \eta f_*\| = 0 \ .$$

Since $h \leq \eta f_*$ we have

$$\lim_{n\to\infty} \|(P^n f - h)^-\| \leq \lim_{n\to\infty} \|P_C^{n-m}(1_C P^m f) - \eta f_*\| = 0 \ . \quad \square$$

Corollary 2.2.1. *Let $K : L_1(X, \Sigma, m) \to L_1(X, \Sigma, m)$ be a stochastic integral Markov operator which overlaps supports and has the invariant density f_*. Denote $C =$ supp f_*. If there is a $\delta > 0$ such that (2.2.1) is satisfied then K is asymptotically stable.*

Proof: According to Theorem 2.2.3. it is enough to prove that the operator K_C is asymptotically stable. Evidently

$$K_C f(x) = \int_C K(x,y) f(y) dy$$

for every $f \in L_1(C)$ and

$$0 = \int_C f_*(y)dm(y) - \int_C K_C f_*(x)dm(x) =$$
$$= \int_C (1 - \int_C K(x,y)dm(x)) f_*(y)dm(y) \ ,$$

hence

$$\int_C K(x,y)dm(x) = 1 \quad \text{for} \quad y \in C \quad \text{a.e.}$$

This shows that K_C is a stochastic integral Markov operator. Thus we can apply Theorem 2.2.2. to the operator K_C and its asymptotical stability follows. □

Theorem 2.2.3. has the assumption of the existence of an invariant density. Now we give some sufficient condition for the existence of a stationary density.

Definition 2.2.6. A Banach limit L is a linear functional defined on the space l_∞ of bounded sequences $(a_n) = (a_1, a_2, \dots)$ of real numbers which satisfies the following conditions:

(*i*) $L(a_n) \geq 0$ if $a_i \geq 0 \ \ (i = 1, 2, \dots)$

(*ii*) $L(a_1, a_2, \dots) = L(a_2, a_3, \dots)$

(*iii*) $L(1, 1, \dots) = 1$

If (a_n) is convergent then $L(a_n) = \lim_{n\to\infty} a_n$, and if $\limsup_n a_n \leq c$ then $L(a_n) \leq c$.

Theorem 2.2.4. *Let $P : L_1(X, \Sigma, m) \to L_1(X, \Sigma, m)$ be a Markov process with $\|P\| = 1$ and L a Banach limit. Assume there exists a set $A \in \mathcal{A}$, $m(A) < \infty$, a number $\delta > 0$ and a density f such that*

$$L(\int_{(X\setminus A)\cup E} P^n f dm) < 1 \ \textit{for} \ E \subseteq A \ \textit{and} \ m(E) < \delta \ . \tag{2.2.2.}$$

Then P admits a stationary density.

The proof of this result was given by Socala [18]. It should be noted, however, that in Socala statement a stronger form of condition (2.2.2.) was used. Namely functional L was replaced by $\limsup$. The above formulation was proposed by Komorowski and Tyrcha [7].

Theorem 2.2.4. and Corollary 2.2.1. were (in [1]) applied to some operators appearing in the mathematical theory of the cell cycle. We shall consider stochastic integral Markov operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)f(y)dy \ , \tag{2.2.3.}$$

where

$$K(x,y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y)) \ . \tag{2.2.4.}$$

In the remainder of this section we shall assume that
(K1) The function $H : [0, \infty) \to [0, \infty)$ is nonincreasing, absolutely continuous, and

$$H(0) = 1 \ , \ \lim_{x \to \infty} H(x) = 0 \ .$$

(K2) The functions $Q : [0, \infty) \to [0, \infty)$ and $\lambda : [0, \infty) \to [0, \infty)$ are nondecreasing, absolutely continuous, and

$$Q(0) = \lambda(0) = 0 \ , \ \lim_{x \to \infty} Q(x) = \lim_{x \to \infty} \lambda(x) = \infty \ .$$

Denote $h(x) = -H'(x)$. Theorems 2.2.5. and 2.2.6. are proved in [1].

Theorem 2.2.5. *If there exists an $\alpha \in (0, 1]$ such that*

$$\int_0^\infty x^\alpha h(x) dx < \liminf_{x \to \infty} ((Q(\lambda(x))^\alpha - Q(x)^\alpha) \ , \tag{2.2.5.}$$

then the operator K given by formulas (2.2.3.) and (2.2.4.) has a stationary density.

The proof of this theorem is based on the Theorem 2.2.4..

Theorem 2.2.6. *If there exists a positive number $\alpha \leq 1$ such that (2.2.5.) holds, and a nonnegative number c such that*

$$h(x) > 0 \ \textit{for} \ x \geq c \ \textit{a.e.},$$

then the operator given by (2.2.3.) and (2.2.4.) is asymptotically stable.

The proof of this theorem is based on the Theorem 2.2.4. and Corollary 2.2.1..

2.3. The Results of [8]: the Convergence of Ergodic Averages, the Lower Bound Function Theorem, the Lyapunov Function and the Condition for the Asymptotical Stability.

All theorems and corollaries from this section are proved in [8].
Let (X, Σ, m) be a measure space, $\mathcal{F}$ a set of functions in L_p.

Definition 2.3.1. The set $\mathcal{F}$ will be called strongly precompact if every sequence of functions $\{f_n\}$, $f_n \in \mathcal{F}$ contains a strongly convergent subsequence $\{f_{\alpha_n}\}$ that converges to $\bar{f} \in L_p$.

Definition 2.3.2. The set $\mathcal{F}$ will be called weakly precompact if every sequence of functions $\{f_n\}$, $f_n \in \mathcal{F}$ contains a weakly convergent subsequence $\{f_{\alpha_n}\}$ that converges to $\bar{f} \in L_p$.

Let $A_n f = \frac{1}{n} \sum_{k=0}^{n-1} P^k f$ for $f \in L_1$.

Theorem 2.3.1. *Let (X, Σ, m) be a measure space and $P : L_1(X) \to L_1(X)$ a Markov process with $\|P\| = 1$. If for given $f \in L_1(X)$ the sequence $\{A_n f\}$ is weakly precompact, then it converges strongly to some $f_* \in L_1(X)$ that is a fixed point of P. Furthermore, if f is a density, then f_* is a density.*

The proof of this theorem is based on the Hahn - Banach Theorem.

Theorem 2.3.2. *Let (X, Σ, m) be a measure space and $P : L_1(X) \to L_1(X)$ a Markov process with $\|P\| = 1$ with unique stationary density f_*. If $f_* > 0 \ \forall x \in X$, then*

$$\lim_{n\to\infty} A_n f = f_*$$

for every density f.

The following theorem gives the necessary and sufficient condition for the asymptotical stability.

Definition 2.3.3. A function $h \in L_1$ is a lower - bound function for a Markov process $P : L_1 \to L_1$ if

$$\lim_{n\to\infty} \|(P^n f - h)^-\| = 0$$

for every density f. A lower bound function is called nontrivial if $h \geq 0$ and $\|h\| > 0$.

Theorem 2.3.3. (The Lower Bound Function Theorem). *Let (X, Σ, m) be a measure space and $P : L_1(X) \to L_1(X)$ a Markov process with $\|P\| = 1$. P is asymptotically stable if and only if there exists a nontrivial lower bound function for P.*

Corollary 2.3.1. *Let (X, Σ, m) be a measure space, $K : XxX \to \mathbb{R}$ a stochastic kernel. Denote by $K_n(x, y)$ the kernel corresponding to the operator K^n. If for some m*

$$\int_X \inf_y K_m(x, y) dx > 0 \ ,$$

then K is asymptotically stable.

Proof: By the definition of K_n, for every density we have

$$K^n f(x) = \int_X K_n(x, y) f(y) fy \ .$$

Furthemore, from the associative property of the composition of operators,

$$K_{n+m}(x, y) = \int_X K_m(x, z) K_n(z, y) dz \ ,$$

so that

$$\begin{aligned} K^{n+m} f(x) &= \int_X K_{n+m}(x, y) f(y) dy = \\ &= \int_X \left\{ \int_X K_m(x, z) K_n(z, y) dz \right\} f(y) dy \ . \end{aligned}$$

If we set

$$h(x) = \inf_y K_m(x, y) \ ,$$

then

$$\begin{aligned} K^{n+m} f(x) &\geq h(x) \int_X \left\{ \int_X K_n(z, y) dz \right\} f(y) dy = \\ &= h(x) \int_X f(y) dy \end{aligned}$$

since K is a stochastic kernel. Furthemore, since f is a density,

$$K^{n+m}f(x) \geq h(x)$$

for $n \geq 1$ and every density f. Thus

$$(K^n - h)^- = 0 \quad \text{for} \quad n \geq m+1 \ ,$$

which completes the prooof. $\square$

The last theorem from this section gives a sufficient condition for the asymptotical stability and has an application in the mathematical theory of the cell cycle (see section 2.5.).

Definition 2.3.4. Let X be an unbounded measurable subset of a d - dimensional Euclidian space $\mathbb{R}^d$, $X \subseteq \mathbb{R}^d$, $K : X\text{x}X \to \mathbb{R}$ a measurable stochastic kernel. We will call any continuous nonnegative function $V : X \to \mathbb{R}$ satisfying $\lim\limits_{x\to\infty} V(x) = \infty$ a Lyapunov function.

Theorem 2.3.4. *If a stochastic kernel $K(x,y)$ satisfies*

$$\int_X \inf_{|y|\leq r} K(x,y)dx > 0 \quad \forall r > 0$$

and has a Lyapunov function $V : X \to \mathbb{R}$ such that

$$\int_X K(x,y)V(x)dx \leq \alpha V(y) + \beta \qquad 0 \leq \alpha < 1 \ , \ \beta \geq 0 \ ,$$

then K is asymptotically stable.

2.4. The Asymptotic Periodicity of Markov and Related Operators.

The paper [4] provides a unified exposition of some results in the theory of the asymptotical behaviour of Markov (and related) operators. In this section we summarize some results of this paper.

Throughout this section by a Markov operator we mean Markov process with norm 1. We shall deal with operators on $L_1(X)$, where (X, Σ, m) is a measure space, m a σ - finite measure on a σ - algebra Σ. Denote the set of nonnegative elements of L_1 by

$$L_1^+ = \{f \in L_1 : \ f(x) \geq 0 \ a.e.\}$$

and densities

$$D = \{f \in L_1^+ : \|f\| = 1\} \ .$$

Definition 2.4.1. An operator P is called

(i) power bounded if there exists an $M > 0$ such that

$$\|P^n\| \leq M \qquad \forall n \in \mathbb{N}$$

(ii) trivial if

$$\lim_{n\to\infty} \|P^n f\| = 0 \quad \forall f \in L_1$$

(*iii*) strictly nontrivial if

$$\liminf_{n\to\infty} \|P^n f\| > 0$$

for every density f.

(*iv*) weakly almost periodic if for every $f \in L_1$ the trajectory $\{P^n f\}$ is weakly precompact.

(*v*) constrictive if there exists a weakly compact subset $F \subset L_1$ such that

$$\lim_{n\to\infty} d(P^n f, F) = 0$$

for every density f, where $d(g, F)$ is the infimum of $\|g - h\|$, $h \in F$.

(*vi*) asymptotically periodic if it is either trivial or if there exist finitely many distinct functions $g_1, \ldots, g_a \in L_1^+$, a permutation α of the set $\{1, \ldots, a\}$ and positive continuous linear functionals $\lambda_1, \ldots, \lambda_a$ on L_1 such that

$$\lim_{n\to\infty} \|P^n(f - \sum_{i=1}^{a} \lambda_i(f) g_i)\| = 0 \text{ and } Pg_i = g_{\alpha(i)} \ , \ i = 1, \ldots, a \ . \tag{2.4.1.}$$

Theorem 2.4.1. *A constrictive operator is asymptotically periodic.*

Definition 2.4.2. A Markov operator P is called quasi constrictive if it has a constrictor $F \subset L_1^+$ satisfying: There exist $K \in \Sigma$, $m(K) < \infty$ and numbers $\kappa < 1$, $\delta > 0$ such that

$$\int_{B\cup(X\setminus K)} f dm \leq \kappa \text{ for } f \in F \ , \ m(B) < \delta \ .$$

Definition 2.4.3. A Markov operator is called almost constrictive if it satisfied: There exist $\kappa < 1$ such that for any decreasing sequence $\{B_m\}$ with empty intersection

$$\lim_{m\to\infty} \left(\limsup_{n} \int_{B_m} P^n f dm \right) < \kappa$$

for every density f holds.

Definition 2.4.4. An operator P is called

(*i*) quasi constrictive if there exist $K \in \Sigma$, $m(K) < \infty$, real numbers ε , $\delta > 0$ and a function n_0 such that

$$\int_{K\setminus B} P^n f dm \geq \varepsilon \text{ for } m(B) < \delta \ , \ f \in D \ , \ n \geq n_0(f)$$

(*ii*) almost constrictive if for any decreasing sequence $\{B_m\}$ with empty intersection

$$\lim_{m\to\infty} \left(\liminf_{n} \int_{X\setminus B_m} P^n f dm \right) > \varepsilon$$

for some $\varepsilon > 0$ and all densities f.

Remark. For Markov operators, both definitions are equivalent.

Theorem 2.4.2. *If P is quasi - constrictive operator then it is asymptotically periodic.*

A more detailed characterization of the asymptotic periodicity of operators is given by the following result.

Theorem 2.4.3. *Let P be an asymptotically periodic operator.*
(i) The functions $g_1, \dots, g_a \in L_1^+$ satisfying (2.4.1.) are linearly independent.
(ii) For every i , $j \in \{1, \dots a\}$, $i \neq j$ and $f \in L_1^+$

$$supp\ (f) \subseteq\ supp\ (g_i) \cap\ supp\ (g_j) \quad \Rightarrow \lim_{n \to \infty} \|P^n f\| = 0\ .$$

(iii) If P is strictly nontrivial operator, then the functions $g_1, \dots, g_a$ satisfying (2.4.1.) have disjoint supports.
(iv) If P is a Markov operator then there exist densities $g_1, \dots, g_a$ with disjoint supports satisfying (2.4.1.).

Definition 2.4.5. $B \in \Sigma$ with $m(B) > 0$ is called a lower set for an operator P if

$$\lim_{n \to \infty} m(B - \text{ supp } (P^n f)) = 0 \quad \forall f \in D\ .$$

Theorem 2.4.4. *If P is an asymptotically periodic strictly nontrivial operator having a lower set, then P is asymptotically stable.*

This result is a consequence of the following one.

Theorem 2.4.5. *If P is a weakly almost periodic strictly nontrivial operator having a lower set, then it is weakly asymptotically stable i.e. there exists a P - invariant density g and a positive linear functional λ on L_1 such that for every $f \in L_1$ the differences*

$$P^n f - \lambda(f).g$$

converge weakly to 0.

[4] contains an interesting criterion of existence of invariant density for a given nontrivial operator.

Theorem 2.4.6. *A strictly nontrivial operator P has an invariant density if and only if there exists $f \in D$ such that for every decreasing sequence $\{B_m\} \subset \Sigma$ with empty intersection the inequality*

$$\lim_{m \to \infty} \left(\liminf_{n} \int_{X \setminus B_m} A_n(f) dm \right) > 0$$

holds. ($A_n(f) = 1/n \sum_{i=0}^{n-1} P^i f$)

Theorem 2.4.7. *An almost constrictive operator is weakly almost periodic.*

We shall end summarizing the results of [4] with one result that might help in the studying the asymptotical stability of Markov operators. Under some assumptions the studying the asymptotical stability can be restricted to the support of invariant density.

Theorem 2.4.8. *Let P be an operator.*

(i) For a given $f \in L_1^+$ the trajectory $\{P^n f\}$ is weakly precompact if and only if the sequence

$$A_n(f) = 1/n \sum_{i=0}^{n-1} P^i f$$

converges in L_1.

(ii) P is weakly almost periodic if and only if there exists a $f_0 \in D$ such that $f_0 > 0$ a.e. and the sequence $\{A_n(f_0)\}$ converges in L_1 to a P - invariant density g. Moreover, the set $G =$ supp (g) and its characteristic function 1_G satisfy

$$\lim_{n\to\infty} \|P^n f - 1_G . P^n f\| = 0 \quad \forall f \in L_1 \ .$$

Theorem 2.4.2. was in [11] applied to the class of Markov operators appearing in the mathematical theory of the cell cycle. Let

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)dy \ , \tag{2.4.2.}$$

where $K(x,y) = -\frac{\partial}{\partial x} H(Q(\lambda(x)) - Q(y))$. We shall assume that Q, λ and H satisfy the following conditions:

(i) The functions $Q : \mathbb{R}^+ \to \mathbb{R}^+$ and $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ are non - decreasing and absolutely continuous on each subinterval $[0,c]$ of the half line $\mathbb{R}^+$. Moreover

$$Q(0) = \lambda(0) = 0 \text{ and } \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty \ .$$

(ii) The function $H : \mathbb{R}^+ \to \mathbb{R}^+$ is non - increasing, absolutely continuous on each interval $[0,c]$, and

$$H(0) = 1 \ , \ \lim_{x\to\infty} H(x) = 0 \ .$$

Denote $h(x) = -H'(x)$. The following theorem is proved in [11].

Theorem 2.4.9. *Assume that*

$$\int_0^\infty x^\varepsilon h(x)dx < \infty$$

for some $\varepsilon > 0$, and that

$$\liminf_{x\to\infty} \frac{Q(\lambda(x))}{Q(x)} > 0 \ .$$

Then P given by (2.4.2.) is asymptotically periodic. If there is a number $x_0 \geq 0$ such that

$$h(x) > 0 \qquad \text{for } x > x_0 \ ,$$

then P given by (2.4.2.) is asymptotically stable.

2.5. The Results Based on Lyapunov and Bielecki Functions.

Theorem 2.5.1. is proved in [3].

Theorem 2.5.1. *Let K be a stochastic integral Markov operator of the form*

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)f(y)dy \ ,$$

where

$$K(x,y) = -1_{[y,\infty]}(\lambda(x))\frac{d}{dx}\exp\left(-\int_y^{\lambda(x)} q(z)dz\right) \ .$$

($1_{[y,\infty]}$ is the characteristic function of the interval $[y,\infty]$). Let λ and q satisfy the following conditions:

(i) The function $\lambda :\ \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable. Moreover, $\lambda'(x) > 0$ for $x \geq 0$, $\lambda(0) = 0$, and $\lim_{x\to\infty}\lambda(x) = \infty$.

(ii) The function $q :\ \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and

$$\int_0^\infty q(x)dx = \infty \ .$$

Denote $H(x) = Q(\lambda(x)) - Q(x)$, where $Q(x) = \int_0^x q(y)dy$. If $\liminf\limits_{x\to\infty} H(x) > 1$ then K is asymptotically stable.

The proof of this theorem is based on the Theorem 2.3.4..

Definition 2.5.1. A function $V :\ \mathbb{R}^+ \to \mathbb{R}^+$ is called a Bielecki function if it is measurable, locally bounded, and

$$\inf_{0\leq x\leq c} V(x) > 0 \qquad \forall c > 0 \ .$$

Theorem 2.5.2. *Let P be a Markov process with $\|P\| = 1$. If there exists a Bielecki function V and a nonnegative constant $\gamma < 1$ such that*

$$\int_0^\infty V(x)Pf(x)dx \leq \gamma\int_0^\infty V(x)f(x)dx$$

for every density f, then the operator P is sweeping with respect to the compact sets of $[0,\infty)$.

This theorem is proved in [3]. The following theorem is proved in [1].

Theorem 2.5.3. *Let K be a stochastic integral Markov operator of the form*

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)f(y)dy \ ,$$

where

$$K(x,y) = -\frac{\partial}{\partial x}H(Q(\lambda(x)) - Q(y)) \ .$$

Let Q, λ, $-H$ are nonnegative, nondecreasing, absolutely continuous functions satisfying:

$$H(0) = 1 \ , \ \lim_{x\to\infty} H(x) = 0 \ ,$$

$$Q(0) = \lambda(0) = 0 \ , \ \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty \ .$$

Denote $h(x) = -H'(x)$. Assume that

$$\sup_{x\geq x_0} ((Q(\lambda(x)))^\beta - Q(x)^\beta) < \int_0^\infty x^\beta h(x)dx < \infty$$

for an $x_0 \geq 0$ and $\beta \geq 1$ and that

$$\int_{Q(\lambda(x_0))}^\infty h(x)dx > 0 \ .$$

Then K is sweeping with respect to the compact sets of $[0,\infty)$.

The proof of this theorem is based on the Theorem 2.5.2..

2.6. Asymptotical Stability, Sweeping and Stationary Densities for Stochastic Semigroups of Operators

In this section we give some consequences of the results given in the sections 2.1.-2.5. for stochastic semigroups of operators.

Definition 2.6.1. Let (X, Σ, m) be a measure space. A family of operators $P_t : L_1(X) \to L_1(X)$, $t \geq 0$, satisfying

(*i*) $P_t(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 P_t f_1 + \lambda_2 P_t f_2 \ \ \forall f_1, \ f_2 \in L_1, \ \lambda_1, \ \lambda_2 \in \mathbb{R}$;
(*ii*) $P_t f \geq 0$, if $f \geq 0$;
(*iii*) $\int_X P_t f(x)dm(x) = \int_X f(x)dm(x), \ \ \forall f \in L_1$;
(*iv*) $P_{t+t'} f = P_t(P_{t'} f) \ \ \forall f \in L_1, \ t, \ t' \geq 0$;
(*v*) $P_0 f = f \ \ \forall f \in L_1$,

is called a **stochastic semigroup**. Further, if, for every $f \in L_1$ and $t_0 \geq 0$,

$$\lim_{t\to t_0} \|P_t f - P_{t_0} f\| = 0 \ ,$$

then this semigroup is called **continuous**.

Definition 2.6.2. A stochastic semigroup of operators $\{P_t\}_{t\geq 0}$ is called asymptotically stable, if there exists a unique density f_* such that $P_t f_* = f_*$ for all $t \geq 0$ (unique stationary density), and

$$\lim_{t\to\infty} P_t f = f_*$$

for every density f.

Definition 2.6.3. Let (X, Σ, m) be a measure space, $\mathcal{A}$ an admissible family of measurable sets. A stochastic semigroup $P_t : L_1(X) \to L_1(X)$ is called sweeping with respect to $\mathcal{A}$ if

$$\lim_{t\to\infty} \int_A P_t f(x)dm(x) = 0$$

for every density f and $A \in \mathcal{A}$.

Theorems 2.6.1.-2.6.3. are proved in [8] and give the connection between asymptotical behaviour of discrete time semigroups of operators and continuous time systems of operators.

Theorem 2.6.1. *If $P_t : L_1(X) \to L_1(X)$ is a continuous stochastic semigroup and if $P_{t_0} f_0 = f_0$ for some $t_0 > 0$ and some density f_0, then*

$$f_*(x) = \frac{1}{t_0} \int_0^{t_0} P_t f_0(x) dt$$

is a density and satisfies $P_t f_* = f_*$ for all $t \geq 0$.

Theorem 2.6.2. *Let (X, Σ, m) be a measure space, $\mathcal{A}$ an admissible family of measurable sets, and $P_t : L_1(X) \to L_1(X)$ a continuous stochastic semigroup. If for some $t_0 > 0$ the sequence $\{P_{t_0}^n\}$ is sweeping with respect to $\mathcal{A}$, then the semigroup $\{P_t\}_{t \geq 0}$ is also sweeping with respect to $\mathcal{A}$.*

Theorem 2.6.3. *Let $P_t : L_1(X) \to L_1(X)$ be a continuous stochastic semigroup. If for some $t_0 > 0$ the sequence $\{P_{t_0}^n\}$ is asymptotically stable, then the semigroup $\{P_t\}_{t \geq 0}$ is also asymptotically stable.*

Now applying theorems 2.6.1.-2.6.3. and using results that we have for discrete time semigroups, we may describe the asymptotic behaviour of continuous time semigroups. Theorems 2.6.4.-2.6.6. are proved in [8].

Theorem 2.6.4. *Let $\{P_t\}_{t \geq 0}$ be a semigroup of Markov operators, not necessarily continuous. Assume that there is an $h \in L_1$, $h \geq 0$, $\|h\| > 0$ such that*

$$\lim_{t \to \infty} \|(P_t f - h)^-\| = 0 \tag{2.6.1.}$$

for every density f. Then there is a unique density f_* such that $P_t f_* = f_*$ for all $t \geq 0$. Furthermore,

$$\lim_{t \to \infty} P_t f = f_* \tag{2.6.2.}$$

for every density f.

Proof: Take any $t_0 > 0$ and define $P = P_{t_0}$ so that $P_{nt_0} = P^n$. Then from (2.6.1.)

$$\lim_{n \to \infty} \|(P^n f - h)^-\| = 0$$

for every density f. Using Theorem 2.6.3. and Theorem 2.3.3. (The Lower Bound Function Theorem) we have (2.6.2.). □

Theorem 2.6.5. *Let $X, \Sigma, m)$ be a measure space, and $\mathcal{A}$ a given admissible family of measurable sets. Furthermore, let $P_t : L_1(X) \to L_1(X)$ be a continuous stochastic semigroup for which there exists a Bielecki function $V : X \to \mathbb{R}$, a constant $\gamma < 1$, and a point $t_0 > 0$ such that*

$$\int_X V(x) P_{t_0} f(x) dm(x) \leq \gamma \int_X V(x) f(x) dm(x)$$

for every density f. Then the semigroup $\{P_{t_0}\}_{t \geq 0}$ is sweeping.

Proof: Since the operator P_{t_0} satisfies the conditions of Theorem 2.5.2., the sequence $\{P_{t_0}\}$ is sweeping. Theorem 2.6.2. completes the proof. □

Theorem 2.6.6. *Let $X, \Sigma, m)$ be a measure space, and $\mathcal{A}$ a given admissible family of measurable sets. Furthermore, let $P_t : L_1(X) \to L_1(X)$ be a continuous stochastic semigroup such that for some $t_0 > 0$ the operator P_{t_0} satisfies the following conditions:*

(i) P_{t_0} is an integral operator given by a stochastic kernel; and

(ii) There is a locally integrable function f_* such that

$$P_{t_0} f_* \le f_* \quad \text{and } f_* > 0 \quad a.e.$$

Under these conditions, the semigroup $\{P_{t_0}\}_{t\ge 0}$ either has an invariant density, or it is sweeping. If a positive invariant density exists and, in addition, P_{t_0} overlaps supports, then the semigroup is asymptotically stable.

Proof: Assume first that $\{P_{t_0}\}_{t\ge 0}$ is not sweeping so by Theorem 2.6.2. the sequence $\{P_{t_0}^n\}$ is also not sweeping. In this case, by Theorem 2.2.1. the operator P_{t_0} has an invariant density. Theorem 2.6.1. then implies that $\{P_{t_0}\}_{t\ge 0}$ must have an invariant density $\tilde{f}$. In the particular case that $\tilde{f} > 0$ and P_{t_0} overlaps supports, it follows from Theorem 2.2.2. that $\{P_{t_0}^n\}$ is asymptotically stable. Finally Theorem 2.6.3. implies that $\{P_{t_0}\}_{t\ge 0}$ is also asymptotically stable. $\square$

Definition 3.1. We say that an integral Markov operator $K : L_1(X) \to L_1(X)$ satisfies the property (P) with respect to a topology $\mathcal{T}$ on X, if

$$(\forall y \in X)(\exists B \in \Sigma \text{ with } m(B) > 0 \text{ such that } ((\forall x \in B)(\exists U_y^x \in \mathcal{T},$$

$$\varepsilon_x > 0 \text{ such that } y \in U_y^x, \text{ and } \forall z \in U_y^x : \; K(x,z) > \varepsilon_x)))$$

Theorem 3.1. *Let K be an integral Markov operator satisfying the property (P) with respect to a topology $\mathcal{T}$. Let the measure m be locally finite (with respect to $\mathcal{T}$). Let the sets of $\mathcal{A}$ be compact. If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

The proof of this theorem is in the paper **"The Foguel Alternative for Integral Markov Operators"** .

Example 3.1. In the mathematical theory of the cell cycle an important role is played by the class of integral Markov operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} K(x,y) f(y) dy \ ,$$

where

$$K(x,y) = -\frac{\partial}{\partial x} \exp\{-\int_y^{\lambda(x)} q(z) dz\} \ .$$

Assume the following conditions:
(i) $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable. Moreover, $\lambda'(x) > 0$ for $x \geq 0$, $\lambda(0) = 0$, and $\lim_{x \to \infty} \lambda(x) = \infty$.
(ii) The function $q : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and $\int_0^\infty q(x) dx = \infty$.
Let $\mathcal{T}$ be the Euclidian metric topology, $\mathcal{A}$ the family of compact subsets of $[0, \infty)$. In the paper **"The Foguel Alternative for Integral Markov Operators"** is proved, that K satisfies the property (P) and hence, by Theorem 3.1., is sweeping or has a stationary density. Using Theorem 2.2.3. is proved, that even the alternative of sweeping or asymptotical stability holds.

Theorem 3.2. *Let (X, Σ, m) be a Banach measure space with metric topology. Let the measure m be locally finite. Let K be an integral Markov operator with kernel $K(x,y)$ satisfying:*

$$\int_X \inf_{\|y\| < r} K(x,y) dx > 0 \quad \forall r > 0 \ .$$

Then K is sweeping with respect to the compact sets or has a stationary density.

Proof: By Theorem 3.1. it is enough to prove, that K satisfies the property (P). Take $y \in X$ and put $r = \|y\| + 1$. Denote by $B(r)$ the set $\{x : \|x\| < r\}$. Put

$$B = \{x \in X : \inf_{\|z\| < r} K(x,z) > 0\} \ ,$$

$$U_y^x = B(r) \ , \ \varepsilon_x = \inf_{\|z\| < r} K(x,z) \text{ for } x \in B$$

(see def. 3.1.). $\square$

Theorem 3.3. *Let (X, Σ, m) be a measure space with a topology $\mathcal{T}$. Let the measure m be locally finite (with respect to $\mathcal{T}$) and nonempty open sets have positive measure. Let $K : L_1(X) \to L_1(X)$ be an integral Markov operator with continuous kernel satisfying:*

$$\int_X K(x,y)dx = 1$$

for every $y \in X$. Then K is sweeping with respect to the compact sets or has a stationary density.

Proof: It is enough to prove (by Theorem 3.1.), that K satisfies the property (P). Take $y \in X$. Since

$$\int_X K(x,y)dx = 1 \ ,$$

there exists $x_0 \in X$ such that $K(x_0, y) > 0$. $K(x,y)$ is continuous, so there exist $\varepsilon > 0$, U_y - a neighbourhood of y, U_{x_0} - a neighborhood of x_0 such that $K(x,y) > \varepsilon$ on the set U_{x_0}xU_y. Now put $B = U_{x_0}$, $\varepsilon_x = \varepsilon$, $U_y^x = U_y$ for $x \in B$ (see def. 3.1.). □

Remark. For integral Markov operators the condition

$$\int_X K(x,y)dx = 1$$

must be satisfied for almost every y. So the condition

$$\int_X K(x,y)dx = 1$$

for every $y \in X$ is not trivial.

Theorem 3.4. *Let $K : L_1([0,\infty)) \to L_1([0,\infty))$ be an integral Markov operator of the form*

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)dy \ ,$$

where λ is a continuous, nondecreasing function with $\lambda(0) = 0$ and $\lim_{x\to\infty} \lambda(x) = \infty$. Let the kernel $K(x,y)$ be continuous on the set

$$\{(x,y) : 0 \le y < \lambda(x)\} \ ,$$

and

$$\int_0^{\infty} K(x,y)dx = 1$$

for every $y \ge 0$. Then K is sweeping (with respect to the compact sets of the Euclidian metric topology on $[0,\infty)$) or has a stationary density.

Proof: We show that K satisfies the property (P). Take $y \in [0,\infty)$. Since

$$\int_{\mathbb{R}+} K(x,y)dx = 1 \ ,$$

there exists $x_0 \in (0,\infty)$ such that $\lambda(x_0) > y$ and $K(x_0,y) > 0$. $K(x,y)$ is continuous, so there exist $\varepsilon, \delta_1, \delta_2 > 0$ such that $K(x,y) > \varepsilon$ on the set $(x_0 - \delta_1, x_0 + \delta_1)x(y - \delta_2, y + \delta_2)$. Now put

$$B = (x_0 - \delta_1, x_0 + \delta_1) \ ,$$

$$\varepsilon_x = \varepsilon \ , \ U_y^x = (y - \delta_2, y + \delta_2)$$

for $x \in B$ (see def. 3.1.). □

Example 3.2. (Discrete time system with constantly applied stochastic perturbations.) In the Chapter 1 is dealed the process

$$x_{n+1} = S(x_n) + \xi_n \ ,$$

where $S : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable transformation and $\xi_0, \ \xi_1, \ \ldots$ are independent random vectors, each having the same density g. If we denote by f_n the density of x_n, then

$$f_{n+1}(x) = Kf_n(x) \ ,$$

where

$$Kf(x) = \int_{\mathbb{R}^d} f(y)g(x - S(y))dy \ .$$

If the kernel $K(x,y) = g(x - S(y))$ is continuous, then by Theorem 3.3. K is sweeping or has a stationary density.

Example 3.3. (Discrete time system with multiplicative perturbations.) Let $X = (0,\infty)$. Define the process on X by

$$x_{n+1} = \xi_n S(x_n) \ ,$$

where $S : (0,\infty) \to (0,\infty)$ is continuous and positive and ξ_n are independent random variables, each having the same density g. In the Chapter 1 is shown that, if we denote by f_n the density of x_n, then

$$f_{n+1}(x) = Kf_n(x) \ ,$$

where

$$Kf(x) = \int_0^\infty f(y)g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}dy \ .$$

If the kernel $K(x,y) = g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}$ is continuous, then by Theorem 3.3. K is sweeping (with respect to the compact subsets of $(0,\infty)$) or has a stationary density. Note that if the density g is continuous, then the kernel $K(x,y)$ is continuous.

Example 3.4. (The linear Boltzmann equation.) Consider the linear Boltzmann equation

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = Ku \ , \quad u(0,x) = f \ ,$$

where $u : \mathbb{R}^+ \mathrm{x} X \to \mathbb{R}$ is a function, $f \in L_1(X)$, $K : L_1(X) \to L_1(X)$ is an integral Markov operator. Consider the solution $u(t,x)$ as a function of positive real numbers $\mathbb{R}^+$ into L_1

$$u : \mathbb{R}^+ \to L_1(X) \ .$$

Thus we may write the equation in the form

$$\frac{du}{dt} = (K - I)u \ , \quad u(0) = f \ .$$

In [8] is shown, that this equation generates a unique continuous semigroup of Markov operators $\{K_t\}_{t\geq 0}$ given by

(3.1.) $$u(t) = K_t f = e^{t(K-I)}f \ .$$

From (3.1.) easily follows (using $e^{t(K-I)}f = \sum_{n=0}^{\infty} \frac{t^n}{n!}(K-I)^n f$), that if $Kf_* = f_*$ for some density f_*, then also $K_t f_* = f_*$. Now we prove, that if K is sweeping with respect to some admissible family $\mathcal{A}$, then also $\{K_t\}_{t\geq 0}$ is sweeping with respect to $\mathcal{A}$. Take a density f, $A \in \mathcal{A}$, $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\int_A K^n f dm < \varepsilon/2 \quad \forall n > n_0 \ .$$

Now

$$K_t f = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} K^n f = e^{-t} \sum_{n=0}^{n_0} \frac{t^n}{n!} K^n f + e^{-t} \sum_{n=n_0+1}^{\infty} \frac{t^n}{n!} K^n f \ ,$$

and

$$\int_A K_t f \leq e^{-t} \sum_{n=0}^{n_0} \frac{t^n}{n!} \int_A K^n f + \varepsilon/2 \ .$$

Since $\lim_{t\to\infty} e^{-t} t^n = 0$, there exists $t_0 > 0$ such that

$$e^{-t} \sum_{n=0}^{n_0} \frac{t^n}{n!} \int_A K^n f < \varepsilon/2 \text{ for } t > t_0 \ ,$$

hence

$$\int_A K_t f < \varepsilon \text{ for } t > t_0 \ .$$

Thus we have proved, that the semigroup $\{K_t\}_{t\geq 0}$ has a stationary density, if the operator K has a stationary density and is sweeping, if the operator K is sweeping. Now we may study the veracity of the Foguel alternative for the semigroup $\{K_t\}_{t\geq 0}$ using Theorems 3.1.-3.4. on the operator K.

The following corollaries (3.1.-3.3.) are immediate consequences of Theorems 3.1.-3.3. and Theorems 2.6.1.-2.6.2. for continuous stochastic semigroups of the type

$$K_t f(x) = \int_X K(t,x,y) f(y) dy \ . \tag{3.2.}$$

Corollary 3.1. *Let a measure space (X, Σ, m) with a topology $\mathcal{T}$ and a continuous stochastic semigroup $\{K_t\}_{t\geq 0}$ of the type (3.2.) be given. Let the measure m be locally finite. Let $\mathcal{A}$ be the family of compact sets. Let for some $t_0 > 0$ the property (P) is satisfied for the kernel $K(t_0, x, y)$. Then $\{K_t\}_{t\geq 0}$ is sweeping with respect to $\mathcal{A}$ or has a stationary density.*

Corollary 3.2. *Let (X, Σ, m) be a Banach measure space with metric topology. Let the measure m be locally finite. Let $\{K_t\}_{t\geq 0}$ be a continuous stochastic semigroup of the form (3.2.). Let for some $t_0 > 0$ the kernel $K(t_0, x, y)$ satisfy*

$$\int_X \inf_{\|y\|<r} K(t_0, x, y) dx > 0 \quad \forall r > 0 \ .$$

Then $\{K_t\}_{t\geq 0}$ is sweeping with respect to the compact sets or has a stationary density.

Corollary 3.3. *Let (X, Σ, m) be a measure space with topology $\mathcal{T}$. Let the measure m be locally finite (with respect to $\mathcal{T}$) and nonempty sets have positive measure. Let $\{K_t\}_{t\geq 0}$ be a continuous stochastic semigroup of the form (3.2.). Let for some $t_0 > 0$ the kernel $K(t_0, x, y)$ be continuous and*

$$\int_X K(t_0, x, y)dx = 1$$

for every $y \in X$. Then $\{K_t\}_{t\geq 0}$ is sweeping with respect to the compact sets or has a stationary density.

Example 3.5. In the Chapter 1 is dealed the Fokker-Planck Equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}[a_{ij}(x)u] - \sum_{i=1}^{d}[b_i(x)u] \ , \quad t > 0 \ , \ x \in \mathbb{R}^d \ . \tag{3.3.}$$

If the coefficients a_{ij}, b_i and the function f satisfy the properties of the Theorem 1.5.2., then the classical solution of (3.3.) is unique and given by the integral formula

$$u(t, x) = \int_{\mathbb{R}^d} \Gamma(t, x, y) f(y) dy \ ,$$

where the kernel $\Gamma(t, x, y)$ is continuous, positive and independent of the initial density f. By Corollary 3.3. the semigroup $\{K_t\}_{t\geq 0}$

$$K_t f(x) = u(t, x) = \int_{\mathbb{R}^d} \Gamma(t, x, y) f(y) dy$$

is sweeping or has a stationary density. Fix $t_0 > 0$. Then the operator

$$K_{t_0} f(x) = \int_{\mathbb{R}^d} \Gamma(t_0, x, y) f(y) dy$$

has positive continuous kernel and hence, by Theorem 3.3. is sweeping or has a stationary density. Let the operator K_{t_0} has a stationary density. Since $\Gamma(t_0, x, y) > 0$, supp $f_* = \mathbb{R}^d$. It is clear, that K_{t_0} overlaps supports. Hence by Theorem 2.2.2. K_{t_0} must be asymptotically stable. By Theorem 2.6.3. $\{K_t\}_{t\geq 0}$ is asymptotically stable. Thus we have proved,that $\{K_t\}_{t\geq 0}$ is sweeping or asymptotically stable.

Remark. The fact, that integral Markov operator with continuous positive kernel is sweeping or asymptotically stable is known and might be proved without using Theorem 3.1.. The Foguel alternative for the Fokker-Planck equation is proved in [8].

The following theorem claims the veracity of the Foguel alternative for integral Markov operators of the type

$$Kf(x) = \int_0^{\lambda(x)} (-\frac{\partial}{\partial x}(H(Q(\lambda(x)) - Q(y))f(y)dy \ , \tag{3.4.}$$

where Q, λ , $-H$ are nonnegative, nondecreasing, absolutely continuous functions on $\mathbb{R}^+$ satisfying:

$$H(0) = 1, \ \lim_{x\to\infty} H(x) = 0$$
$$Q(0) = \lambda(0) = 0, \ \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty \ .$$

Operators of this type need no satisfy the property (P). Applications of the operators of the form (3.4.) are discussed in the Chapter 1.

Theorem 3.5. *Let K be an integral Markov operator of the form (3.4.). Let $\mathcal{A}$ be the family of compact subsets of $[0,\infty)$ (with respect to the Euclidian metric topology). If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

This theorem is proved in the paper **"Asymptotic Behaviour of Some Markov Operators Appearing in Mathematical Models of Biology"** .

Theorem 3.6. claims, that the Foguel alternative holds for every integral Markov operator, if we do not specify the admissible family $\mathcal{A}$.

Theorem 3.6. *Let K be an integral Markov operator. If K has no stationary density, then K is sweeping with respect to some admissible family $\mathcal{A}$.*

Proof: By Corollary 2.1.4.

$$0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty$$

for $u > 0$, hence the process K_D is dissipative. By Theorem 2.1.3. there exists a σ -finite subinvariant measure λ equivalent to $m \upharpoonright D$. Let $g = \frac{d\lambda}{dm}$. Let $\mathcal{A}_D$ be the family of sets of finite measure (with respect to m) such that

$$\int_A g dm < \infty \quad \forall A \in \mathcal{A}_D \ .$$

Since $g < \infty$, the family $\mathcal{A}_D$ is admissible. K_D is dissipative, hence by Theorem 2.2.1. K_D is sweeping with respect to $\mathcal{A}_D$.

Let K_C have a stationary density $\tilde{f}$. Let f_* be a function on X such that $f_* \upharpoonright C = \tilde{f}$, $f_* \upharpoonright D = 0$. Then

$$(Kf_*) \upharpoonright C = (K(f_*.1_C)) \upharpoonright C + (K(f_*.1_D)) \upharpoonright C = K_C\tilde{f} = \tilde{f} \ .$$

By Corollary 2.1.2. $(Kf_*) \upharpoonright D = K_D(f_* \upharpoonright D) = 0$, hence $Kf_* = f_*$.

Let K_C have no stationary density. By Theorem 2.1.2. and Corollary 2.1.1. there exists $0 < u < \infty$ such that $Ku = u$ (a σ - finite invariant measure). Let $\mathcal{A}_C$ be the family of all sets of finite measure (with respect to m) such that

$$\int_A u dm < \infty \quad \forall A \in \mathcal{A}_C \ .$$

Since $u < \infty$, the family $\mathcal{A}_C$ is admissible. By Theorem 2.2.1. K_C is sweeping with respect to $\mathcal{A}_C$. Now we shall prove that K is sweeping with respect to $\mathcal{A}_D \cup \mathcal{A}_C$.

Let $A \in \mathcal{A}_D$. Then by Corollary 2.1.3.

$$\int_A K^n f dm = \int_A K_D^n(f \upharpoonright D) \ .$$

Since K_D is sweeping with respect to $\mathcal{A}_D$:

$$\lim_{n\to\infty} \int_A K^n f dm = 0 \quad \forall A \in \mathcal{A}_D \ .$$

Now it is enough to prove that

$$\int_A K^n f dm \to 0 \text{ for } A \in \mathcal{A}_C \ .$$

Denote

$$\tilde{K}_C f = (Kf).1_C \ , \ \tilde{K}_D f = (Kf).1_D$$
$$f_C = f.1_C \ , \ f_D = f.1_D \ .$$

Clearly

$$\begin{aligned}
\tilde{K}_C f &= \tilde{K}_C(f_C + f_D), \ Kf = \tilde{K}_C f + \tilde{K}_D f \ , \\
\tilde{K}_C(Kf) &= \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C \tilde{K}_D f_D \\
\tilde{K}_C(K^2 f) &= \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2 \tilde{K}_D f_D + \tilde{K}_C \tilde{K}_D^2 f_D \\
&\dots \\
&\dots \\
K^n f.1_C &= \tilde{K}_C(K^{n-1} f) = \\
&= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots \\
&\ \ + \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \ .
\end{aligned}$$

Take $1 < k < n$ and define:

$$\begin{aligned}
M_{k,n} f &= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots + \tilde{K}_C^{n-k+1} \tilde{K}_D^{k-1} f_D \\
R_{k,n} f_D &= \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \ .
\end{aligned}$$

$\tilde{K}_C$ is contraction, hence

$$\begin{aligned}
\|R_{k,n} f_D\| &\le \|\tilde{K}_C^{n-k} \tilde{K}_D^k f_D\| + \dots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \le \\
&\le \|\tilde{K}_C \tilde{K}_D^k f_D\| + \dots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \ .
\end{aligned}$$

Now

$$\|\tilde{K}_D^l f_D\| = \|K \tilde{K}_D^l f_D\| = \|\tilde{K}_C \tilde{K}_D^l f_D\| + \|\tilde{K}_D^{l+1} f_D\| \ ,$$

hence

$$\|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^l f_D\| - \|\tilde{K}_D^{l+1} f_D\| \ ,$$

$$\sum_{l=k}^{n} \|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^{n+1} f_D\| \ ,$$

$$\|R_{k,n} f_D\| \le \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| \ .$$

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| < \frac{\varepsilon}{2} \text{ for } n, k \ge n_0(\varepsilon), \ n \ge k \ .$$

Now fix $k \ge n_0(\varepsilon)$, $A \in \mathcal{A}_C$. $\tilde{K}_C$ be sweeping implies

$$\int_A M_{k,n} f dm < \frac{\varepsilon}{2}$$

for n sufficiently large, hence

$$\int_A K^n f dm \to 0 \text{ for } A \in \mathcal{A}_C \ . \quad \square$$

Corollary 3.4. is an immediate consequence of Theorem 3.6. for integral continuous stochastic semigroups.

Corollary 3.4. *Let $\{K_t\}_{t\geq 0}$ be a continuous stochastic semigroup of the form*

$$K_t f(x) = \int_X K(t,x,y) f(y) dy \ .$$

If $\{K_t\}_{t\geq 0}$ has no stationary density, then $\{K_t\}_{t\geq 0}$ is sweeping with respect to some admissible family $\mathcal{A}$.

Proof: It is an immediate consequence of Theorem 3.6. and Theorems 2.6.1.-2.6.2. $\square$

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THE FOGUEL ALTERNATIVE FOR INTEGRAL MARKOV OPERATORS

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ABSTRACT. A class of Markov operators satisfies the Foguel alternative if its members are either sweeping or have stationary densities. New sufficient condition for this property is given.

1. INTRODUCTION

We shall consider Markov operators $K : L_1(X) \longrightarrow L_1(X)$ of the form :

$$Kf(x) = \int_X K(x,y)f(y)dy \ ,$$

where $K(x,y)$ defined on XxX is a kernel. Such operators were intensively studied. In [1], [4], [6], [7] some sufficient conditions for sweeping (see def.3.1.) and asymptotical stability were given. It was proved in [4] that, under the assumption of having subinvariant locally integrable function, the alternative of sweeping or having stationary density holds. The main result of this paper is the proof of this alternative without the assumption of having subinvariant locally integrable function (Th.3.2.).

In the section 2., some necessary results of [2] are presented. In the section 3., the main result is proved. Section 4. contains an application of Theorem 3.2. to the class of Markov operators appearing in the mathematical theory of the cell cycle.

2. SOME PROPERTIES OF MARKOV PROCESSES AND INTEGRAL MARKOV OPERATORS

Theorems 2.1 - 2.4. are proved in [2].

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition 2.1. A Markov process is defined to be a quadruple (X, Σ, m, P) , where (X, Σ, m) is a $\sigma-$ finite measure space with positive measure and where P is an operator on $L_1(X)$ satisfying

(i) P is a contraction : $\|P\| \leq 1$

(ii) P is positive : if $0 \leq u \ \in L_1(X)$ then $Pu \geq 0$

Definition 2.2. If u is an arbitrary non-negative function, set $Pu := \lim_{k\to\infty} Pu_k$ for $0 \leq u_k \in L_1(X), u_k \nearrow u$, where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere. The sequence Pu_k is increasing so that $\lim_k Pu_k$ exists (it may be infinite). By [2] the definition of Pu is independent of the particular sequence u_k.

Definition 2.3. Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\} \ , \ D = X \setminus C$$

By [2] this definition is independent of the choice of u_0.

Theorem 2.1. *If* $0 \leq u \ \in L_1(X)$ *then*

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \ \textit{for}\ x \in D \ , \ \sum_{k=0}^{\infty} P^k u(x) = 0 \ \textit{or} \ \infty \ \textit{for}\ x \in C \ .$$

Definition 2.4. A function $K(x,y) \geq 0$ defined on $X\mathrm{x}X$ which is jointly measurable with respect to its variables is called a kernel. Let $\int_X K(x,y)dx \leq 1$. Define an operator K on $L_1(X)$:

$$Kf(x) = \int_X K(x,y)f(y)dy \ .$$

Then $\|K\| \leq 1$ and K is called an integral Markov operator.

Definition 2.5. Let P be an integral Markov operator, then (X, Σ, m, P) is said to be a Harris process if $X = C$.

Theorem 2.2. *Let* K *be an integral Markov operator and a Harris process. Then there exists* $0 < u < \infty$ *such that* $Ku = u$ *(a* σ*-finite invariant measure).*

Theorem 2.3. *Let* P *be a Markov process with* $X = D$*. Then there exists* $0 < g < \infty$ *such that* $Pg \leq g$*.*

Proof: Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$. $\square$

Definition 2.6. Let P be a Markov process. Define operators P_C, P_D:

$$P_C : L_1(C) \to L_1(C) \ , \ P_C f = (P\tilde{f}) \restriction C \ ,$$

where the symbol $\restriction$ denotes the restriction to the set C, $\tilde{f}$ is the function f extended by 0 on D,

$$P_D : L_1(D) \to L_1(D) \ , \ P_D f = (P\tilde{f}) \restriction D \ ,$$

where $\tilde{f}$ is the function f extended by 0 on C.

Theorem 2.4. *Let P be a Markov process. If supp $f \subseteq C$, then supp $Pf \subseteq C$. (supp $f = \{x : f(x) \neq 0\}$)*

Corollary 2.1. *Let K be an integral Markov operator. Then*

$$(C, \Sigma \upharpoonright C, m \upharpoonright C, K_C)$$

is a Harris process. ($\Sigma \upharpoonright C$ denotes the σ-algebra restricted to the space C, $m \upharpoonright C$ denotes the measure m restricted to the space $\Sigma \upharpoonright C$).

Proof: By Theorem 2.4. supp $f \subseteq C$ implies supp $Kf \subseteq C$. By Theorem 2.1. for $u > 0$ on C,$u = 0$ on D:

$$\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \upharpoonright C)(x)$$

for every $x \in C$. $\square$

Corollary 2.2. *Let P be a Markov process on $L_1(X)$. Then*

$$P_D(f \upharpoonright D) = (Pf) \upharpoonright D .$$

Proof: $f = f_D + f_C$, where $f_C = f.1_C, f_D = f.1_D$. By Theorem 2.4. $(Pf_C) \upharpoonright D = 0$, hence

$$(Pf) \upharpoonright D = (Pf_D) \upharpoonright D = P_D(f \upharpoonright D) . \quad \square$$

Corollary 2.3. $P_D^n(f \upharpoonright D) = (P^n f) \upharpoonright D$.

Corollary 2.4. *Let P be a Markov process on X, let $u > 0$ on D. Then*

$$\sum_{n=0}^{\infty} P_D^n u < \infty .$$

Proof: Let $\tilde{u}$ be a function on X such that $\tilde{u} \upharpoonright C = 0, \tilde{u} \upharpoonright D = u$. By Corollary 2.3.

$$\sum_{n=0}^{\infty} P_D^n u = (\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D .$$

By Theorem 2.1. $(\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D < \infty$. $\square$

3. The Foguel alternative for integral Markov operators

Definition 3.1. Let a family $\mathcal{A} \subset \Sigma$ be given. A Markov process is called sweeping with respect to $\mathcal{A}$, if

$$\lim_{n\to\infty} \int_A P^n f dm = 0$$

for $A \in \mathcal{A}$ and $f \in D$ ($D = \{f \in L_1(X), \|f\| = 1, f \geq 0\}$)

In the sequel we shall assume that $\mathcal{A}$ satisfies the following properties:

(i) $0 < m(A) < \infty$ for $A \in \mathcal{A}$

(ii) $A_1, A_2 \in \mathcal{A}$ implies $A_1 \cup A_2 \in \mathcal{A}$

(iii) There exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\cup A_n = X$.

A family satisfying $(i) - (iii)$ will be called **admissible.**

Definition 3.2. Let (X, Σ, m) and an admissible family $\mathcal{A} \subseteq \Sigma$ be given. A measurable function $f : X \to \mathbb{R}$ is called locally integrable, if

$$\int_A |f| dm < \infty \text{ for } A \in \mathcal{A} \ .$$

Theorem 3.1. *Let a measure space (X, Σ, m), an admissible family $\mathcal{A}$ and an integral Markov operator K be given. If K has no invariant density but there exists a positive locally integrable function f_* subinvariant with respect to K, then K is sweeping.*

Remark 3.1. Theorem 3.1. was proved in [4] for stochastic kernel operators ($\int_X K(x, y)dx = 1$). But the proof is completely same for integral Markov operators.

We say that an integral Markov operator $K : L_1(X) \to L_1(X)$ satisfies a property (P) with respect to topology $\mathcal{T}$ on X, if

$$(\forall y \in X)(\exists B \in \Sigma \text{ with } m(B) > 0 \text{ such that } ((\forall x \in B)(\exists U_y^x \in \mathcal{T},$$

$$\varepsilon_x > 0 \text{ such that } y \in U_y^x, \text{ and } \forall z \in U_y^x : \ K(x, z) > \varepsilon_x)))$$

Theorem 3.2. *Let K be an integral Markov operator satisfying property (P) with respect to a topology $\mathcal{T}$. Let the measure m be locally finite (with respect to $\mathcal{T}$). Let the sets of $\mathcal{A}$ be compact. If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

Proof: Denote

$$\tilde{K}_C f = (Kf).1_C \ , \ \tilde{K}_D f = (Kf).1_D$$

$$f_C = f.1_C \ , \ f_D = f.1_D \ .$$

Now

$$\|\tilde{K}_D^l f_D\| = \|K\tilde{K}_D^l f_D\| = \|\tilde{K}_C \tilde{K}_D^l f_D\| + \|\tilde{K}_D^{l+1} f_D\| \ ,$$

hence

$$\|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^l f_D\| - \|\tilde{K}_D^{l+1} f_D\| \ ,$$

(3.1.)
$$\sum_{l=k}^{n} \|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^{n+1} f_D\|$$

Lemma 1. *Let $y \in X$. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n \to \infty} \int_{U_y \cap D} K_D^n f dm = 0$$

for every $f \in L_1(D)$.

Proof (of Lemma 1.): By Corollary 2.4.

$$0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty$$

for $u > 0$, hence the process K_D is dissipative. By Theorem 2.3. there exists a σ-finite subinvariant measure λ equivalent to $m \restriction D$.

Let $\mathcal{A}_\lambda$ be the family of all sets of finite measure (with respect to m) such that

$$\int_A \frac{d\lambda}{dm} dm < \infty \quad \forall A \in \mathcal{A}_\lambda .$$

Since $\frac{d\lambda}{dm} < \infty$, the family $\mathcal{A}_\lambda$ is admissible. K_D is dissipative, hence by Theorem 3.1. K_D is sweeping with respect to $\mathcal{A}_\lambda$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $D \cap U$ has positive measure. By the assumption there is a set B $(m(B) > 0)$ such that for every $x \in B$ there is $U_y^x \in \mathcal{T}$ such that $K(x,z) > \varepsilon_x$ on U_y^x. No loss of generality we may assume that the sets U_y^x have finite measure.

Let $m(B \cap D) > 0$. Then $\forall x \in B \cap D$

$$g(x) \geq \int_{U_y^x \cap D} K(x,z) g(z) dz \geq \int_{U_y^x \cap D} \varepsilon_x g(z) dz ,$$

hence

$$\int_{U_y^x \cap D} g(z) dz \leq \frac{1}{\varepsilon_x} g(x) < \infty$$

for every $x \in B \cap D$ and $U_y^x \cap D \in \mathcal{A}_\lambda$, $\lim_{n\to\infty} \int_{U_y^x \cap D} K_D^n f dm = 0$ for every $x \in B \cap D$.

Let $m(B \cap D) = 0$. Let $\lim_{n\to\infty} \int_{U_y^x \cap D} K_D^n (f \restriction D) \neq 0$ for some $f \in L_1(X)$ and every $x \in B$. By Corollary 2.3.

$$K_D^n (f \restriction D) = (\tilde{K}_D^n f_D) \restriction D .$$

Then

$$\int_{U_y^x \cap D} \tilde{K}_D^n f_D(x) > \delta_x > 0$$

for some δ_x and infinitely many n,

$$\forall x \in B \quad \tilde{K}_C \tilde{K}_D^n f_D(x) > \delta_x . \varepsilon_x$$

for infinitely many n. By the Lebesgue Monotone Convergence Theorem

$$\infty = \| \sum_{n=0}^{\infty} \tilde{K}_C \tilde{K}_D^n f_D \| = \sum_{n=0}^{\infty} \| \tilde{K}_C \tilde{K}_D^n f_D \| ,$$

which contradicts (3.1.). $\square$

Lemma 2. *Let $y \in X$, let K_C has no stationary density. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap C} K_C^n f dm = 0$$

for every $f \in L_1(C)$.

Proof (of Lemma 2.): By Corollary 2.1. and Theorem 2.2. K_C is Harris and there exists a function g, $0 < g < \infty$ such that $K_C g = g$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $C \cap U$ has a positive measure. By the assumption there is a set B such that for every $x \in B$ there is $U_y^x \in \mathcal{T}$ such that $K(x,z) > \varepsilon_x$ on U_y^x. By Corollary 2.2. $K(x,z) = 0$ for $x \in D,\ z \in C$, hence $B \subseteq C$. Now

$$g(x) \geq \int_{U_y^x \cap C} K(x,z)g(z)dz \geq \int_{U_y^x \cap C} \varepsilon_x g(z)dz \ ,$$

hence

$$\int_{U_y^x \cap C} g(z)dz < \frac{1}{\varepsilon_x} g(x) < \infty$$

for some $x \in B$. Let $\mathcal{A}_g$ be the family of all sets of finite measure such that

$$\int_A g dm < \infty \ \ \forall A \in \mathcal{A}_g \ .$$

Since $g < \infty$, the family $\mathcal{A}_g$ is admissible. Then $U_y^x \cap C \in \mathcal{A}_g$ and by Theorem 3.1.

$$\int_{U_y^x \cap C} K_C^n f dm \to 0 \ \ \forall f \in L_1(C) \ . \quad \square$$

Lemma 3. *Let K_C has no stationary density, let $A \in \mathcal{A}$. Then*

$$\lim_{n\to\infty} \int_{A\cap C} K_C^n f_1 dm = 0 \ , \quad \lim_{n\to\infty} \int_{A\cap D} K_D^n f_2 dm = 0 \tag{3.2.}$$

for every $f_1 \in L_1(C)$, $f_2 \in L_1(D)$.

Proof (of Lemma 3.): Let $y \in X$. By Lemma 1. there exists $U_1 \in \mathcal{T}$ such that $y \in U_1$ and

$$\lim_{n\to\infty} \int_{U_1\cap D} K_D^n f_2 dm = 0 \ \ \forall f_2 \in L_1(D) \ .$$

By Lemma 2. there exists $U_2 \in \mathcal{T}$ such that $y \in U_2$ and

$$\lim_{n\to\infty} \int_{U_2\cap C} K_C^n f_1 dm = 0 \ \ \forall f_1 \in L_1(C) \ .$$

Set $U_y = U_1 \cap U_2$. Then

$$\lim_{n\to\infty} \int_{U_y\cap C} K_C^n f_1 dm = 0 \ , \ \lim_{n\to\infty} \int_{U_y\cap D} K_D^n f_2 dm = 0 \tag{3.3.}$$

Thus we have proved that for every $y \in X$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and (3.3.) holds. Finally (3.2.) follows from compactness of A . $\square$

Proof (of Theorem 3.2.): By Lemma 3. K_D is sweeping, K_C is sweeping or has a stationary density.

Let K_C have a stationary density $\tilde{f}$. Let f_* be a function on X such that $f_* \restriction C = \tilde{f},\ \ f_* \restriction D = 0$. Then

$$(Kf_*) \restriction C = (K(f_*.1_C)) \restriction C + (K(f_*.1_D)) \restriction C = K_C \tilde{f} = \tilde{f} \ .$$

By Corollary 2.2. $(Kf_*) \restriction D = K_D(f_* \restriction D) = 0$, hence $Kf_* = f_*$. Let K_C be sweeping. We shall prove that K is sweeping.

Let $f \in L_1(X)$, then $f = f_C + f_D$, where $f_C = f.1_C$, $f_D = f.1_D$. By Corollary 2.3.

$$(K^n f_C) \restriction D = 0 \; , \; (K^n f) \restriction D = K_D^n (f \restriction D) \; .$$

By Lemma 3.

$$\int_{A \cap D} K^n f dm \to 0 \text{ for every } A \in \mathcal{A} \; .$$

Now it is enough to prove that

$$\int_{A \cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A} \; .$$

Clearly

$$\begin{aligned}
\tilde{K}_C f &= \tilde{K}_C(f_C + f_D), \; Kf = \tilde{K}_C f + \tilde{K}_D f \; , \\
\tilde{K}_C(Kf) &= \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C \tilde{K}_D f_D \\
\tilde{K}_C(K^2 f) &= \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2 \tilde{K}_D f_D + \tilde{K}_C \tilde{K}_D^2 f_D \\
&\dots \\
&\dots \\
K^n f.1_C &= \tilde{K}_C(K^{n-1} f) = \\
&= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots \\
&\quad + \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \; .
\end{aligned}$$

Take $1 < k < n$ and define:

$$\begin{aligned}
M_{k,n} f &= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots + \tilde{K}_C^{n-k+1} \tilde{K}_D^{k-1} f_D \\
R_{k,n} f_D &= \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \; .
\end{aligned}$$

$\tilde{K}_C$ is contraction, hence

$$\begin{aligned}
\|R_{k,n} f_D\| &\le \|\tilde{K}_C^{n-k} \tilde{K}_D^k f_D\| + \dots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \le \\
&\le \|\tilde{K}_C \tilde{K}_D^k f_D\| + \dots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \; .
\end{aligned}$$

By (3.1.)

$$\|R_{k,n} f_D\| \le \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| \; .$$

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| < \frac{\varepsilon}{2} \text{ for } n, k \ge n_0(\varepsilon), \; n \ge k \; .$$

Now fix $k \ge n_0(\varepsilon)$, $A \in \mathcal{A}$. $\tilde{K}_C$ be sweeping implies

$$\int_{A \cap C} M_{k,n} f dm < \frac{\varepsilon}{2}$$

for n sufficiently large, hence

$$\int_{A \cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A} \; . \quad \square$$

In the mathematical theory of the cell cycle an important role is played by the class of integral Markov operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} K(x,y)f(y)dy \ ,$$

where

$$K(x,y) = -\frac{\partial}{\partial x} \exp\{-\int_y^{\lambda(x)} q(z)dz\} \ .$$

Assume the following conditions:

(i) $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable. Moreover, $\lambda'(x) > 0$ for $x \geq 0$, $\lambda(0) = 0$, and $\lim_{x\to\infty} \lambda(x) = \infty$.

(ii) The function $q : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and $\int_0^\infty q(x)dx = \infty$.

Let $\mathcal{T}$ be the Euclidian metric topology, $\mathcal{A}$ the family of compact subsets of $\mathbb{R}^+$. Then it is quite easy to prove that K satisfies the property (P) with respect to $\mathcal{T}$:

$$K(x,y) = \lambda'(x)q(\lambda(x)) \exp\{-\int_y^{\lambda(x)} q(z)dz\} \ ,$$

$\lambda'(x) > 0$ for every x. Let $y_0 \in \mathbb{R}^+$, let

$$B = \{x : \ q(\lambda(x)) > 0 \ , \lambda(x) > y_0\} \ .$$

$m(B) > 0$ follows from $\int_0^\infty q(z)dz = \infty$. Further

$$\int_y^{\lambda(x)} q(z)dz \leq \int_0^{\lambda(x)} q(z)dz < \infty \ ,$$

hence

$$\exp\{-\int_y^{\lambda(x)} q(z)dz\} \geq \delta(x) > 0 \ ,$$

and

$$K(x,y) \geq \lambda'(x)q(\lambda(x))\delta(x) = \varepsilon(x) > 0 \tag{4.1.}$$

on the set $\{x : q(\lambda(x)) > 0\}$. Now set $U^x_{y_0} = [0, \lambda(x))$ and the property (P) is fulfilled. By Theorem 3.2. K is sweeping with respect to $\mathcal{A}$ or has a stationary density.

Let K has stationary density f_*. We show that K is asymptotically stable. Denote $C = \text{supp } f_*$. Lemma 4.1. was proved in [1].

Definition 4.1. We say that a Markov process P overlaps supports if for every two densities $f, \ g$ there is a positive integer $n_0 = n_0(f,g)$ such that

$$\mu(\text{supp } P^{n_0}f \cap \text{supp } P^{n_0}g) > 0 \ .$$

Lemma 4.1. *Let $K : L_1(X,\Sigma,m) \to L_1(X,\Sigma,m)$ be a stochastic integral Markov operator which overlaps supports and has the invariant density f_*. Denote $C =$ supp f_*. If there is a $\delta > 0$ such that*

$$\sup_n \int_C K^n f dm \geq \delta$$

for every density f, then K is asymptotically stable.

Since

$$K(x,y) = \lambda'(x).q(\lambda(x)).\exp\{-\int_y^{\lambda(x)} q(z)dz\} ,$$

$q(\lambda(x)) > 0$ on C follows from

$$f_*(x) = \int_0^{\lambda(x)} K(x,y)f_*(y)dy .$$

Let

$$m_0 = \inf\{z : m((0,z) \cap C) > 0\} ,$$
$$m_1 = \lambda^{-1}(m_0) ,$$
$$m_2 = \lambda^{-1}(m_1) .$$

Then the set

$$((m_1,\infty) \cap \{z : q(\lambda(z)) > 0\}) \setminus C$$

has measure zero, since by (4.1.)

$$Kf_*(x) = \int_0^{\lambda(x)} K(x,y)f_*(y)dy \geq \int_{m_0}^{\lambda(x)} \varepsilon(x)f_*(y)dy > 0$$

on the set $(m_1,\infty) \cap \{z : q(\lambda(z)) > 0\}$.

Now $(0,m_1) \cup C \supset$ supp Kf for every density f, and

$$\begin{aligned}\int_C K^2 f(x)dx &= \int_C \left(\int_{\mathbb{R}+} K(x,y)Kf(y)dy\right) dx = \\ &= \int_C \left(\int_{(0,m_1)\cup C} K(x,y)Kf(y)dy\right) dx = \\ &= \int_0^{m_1} \left(\int_C K(x,y)dx\right) Kf(y)dy \\ &+ \int_C \left(\int_{C\setminus(0,m_1)} K(x,y)Kf(y)dy\right) dx .\end{aligned}$$

By (4.1.)

$$\begin{aligned}\int_0^{m_1} \left(\int_C K(x,y)dx\right) Kf(y)dy &\geq \int_0^{m_1} \left(\int_{C\cap(m_2,\infty)} \varepsilon(x)dx\right) Kf(y)dy \geq \\ &\geq \delta \int_0^{m_1} Kf(y)dy . \qquad (4.2.)\end{aligned}$$

For Markov operators supp $f \subset C$ implies supp $Kf \subset C$, if C is support of invariant density, hence

$$\int_C \left(\int_{C\setminus(0,m_1)} K(x,y)Kf(y)dy \right) dx = \int_X \left(\int_{C\setminus(0,m_1)} K(x,y)Kf(y)dy \right) dx =$$

$$= \int_{C\setminus(0,m_1)} \int_X K(x,y)dxKf(y)dy =$$

$$= \int_{C\setminus(0,m_1)} Kf(y)dy \ . \tag{4.3.}$$

Finally (4.2.) and (4.3.) imply that

$$\int_C K^2 f(x)dx \geq \delta \int_{(0,m_1)\cup C} Kf(x)dx = \delta \int_X Kf(x)dx = \delta$$

for every density f. By Lemma 4.1. it is enough to prove that K overlaps supports.

By (4.1.)

$$K(x,y) \geq \varepsilon(x) > 0$$

on the set

$$S = \{x : q(\lambda(x)) > 0\} \ .$$

Since $\int_0^\infty q(z)dz = \infty$ and q is locally integrable, the set $(k,\infty)\cap S$ has positive measure for every $k > 0$. If f, g are arbitrary densities such that the sets

$$(0,k)\cap \text{ supp } f \quad \text{and} \quad (0,k)\cap \text{ supp } g$$

have positive measures, than on the set

$$(\lambda^{-1}(k),\infty)\cap S$$

$Kf > 0$ and $Kg > 0$, hence K overlaps supports. □

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ASYMPTOTIC BEHAVIOUR OF SOME MARKOV OPERATORS APPEARING IN MATHEMATICAL MODELS OF BIOLOGY

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Abstract. A class of Markov operators satisfies the Foguel alternative if its members are either sweeping or have stationary densities. We show that this alternative holds for some integral Markov operators appearing in mathematical models of biology.

1. Introduction

Let $K : L_1(X) \to L_1(X)$ be an integral Markov operator of the form:

$$Kf(x) = \int_X K(x,y)f(y)dy \ , \tag{1.1.}$$

where $K(x,y)$ defined on XxX is a kernel. Such operators were intensively studied. In [1], [4], [6], [7] some sufficient conditions for sweeping (see def. 3.1.) and asymptotical stability were given. It was proved in [4] that,under the assumption of having subinvariant locally integrable function, the alternative of sweeping or having stationary density holds. The condition without the assumption of the existence of a subinvariant locally integrable function for operators satisfying some property (P) was given in [3]. The main result of this paper is the proof of the Foguel alternative for operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} (-\frac{\partial}{\partial x}(H(Q(\lambda(x)) - Q(y))f(y)dy \ , \tag{1.2.}$$

where $Q, \ \lambda \ , -H$ are nonnegative, nondecreasing, absolutely continuous functions on $\mathbb{R}^+$ satisfying:

$$H(0) = 1, \ \lim_{x\to\infty} H(x) = 0$$

$$Q(0) = \lambda(0) = 0, \ \lim_{x\to\infty} Q(x) = \lim_{x\to\infty} \lambda(x) = \infty \ .$$

Operators of this type need not satisfy the property (P). The asyptotic behaviour of operators of the form (1.2.) has many practical applications in biology.

In Section 2, some necessary results of [2] are presented. In Section 3, the main result (Theorem 3.2.) is proved.

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2. Some properties of Markov processes and integral Markov operators

Theorems 2.1 - 2.4. are proved in [2].

Definition 2.1. A Markov process is defined to be a quadruple (X, Σ, m, P) , where (X, Σ, m) is a $\sigma-$ finite measure space with positive measure and where P is an operator on $L_1(X)$ satisfying

(*i*) P is a contraction : $\|P\| \leq 1$

(*ii*) P is positive : if $0 \leq u \ \in L_1(X)$ then $Pu \geq 0$

Definition 2.2. If u is an arbitrary non-negative function, set $Pu := \lim_{k\to\infty} Pu_k$ for $0 \leq u_k \in L_1(X), u_k \nearrow u$, where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere. The sequence Pu_k is increasing so that $\lim_k Pu_k$ exists (it may be infinite). By [2] the definition of Pu is independent of the particular sequence u_k.

Definition 2.3. Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\}, D = X \setminus C$$

By [2] this definition is independent of the choice of u_0.

Theorem 2.1. *If* $0 \leq u \ \in L_1(X)$ *then*

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \text{ for } x \in D, \ \sum_{k=0}^{\infty} P^k u(x) = 0 \text{ or } \infty \text{ for } x \in C.$$

Definition 2.4. A function $K(x, y) \geq 0$ defined on XxX which is jointly measurable with respect to its variables is called a kernel. If $\int_X K(x, y)dx = 1$, then K is called a stochastic kernel. Stochastic kernel defines an operator on $L_1(X)$:

$$Kf(x) = \int_X K(x, y)f(y)dy$$

with $\|K\| = 1$. So (X, Σ, m, K) is a Markov process.

Definition 2.5. Let P be an integral Markov operator, then (X, Σ, m, P) is said to be a Harris process if $X = C$.

Theorem 2.2. *Let K be an integral Markov operator and a Harris process. Then there exists* $0 < u < \infty$ *such that* $Ku = u$ *(a σ-finite invariant measure).*

Theorem 2.3. *Let P be a Markov process with $X = D$. Then there exists* $0 < g < \infty$ *such that* $Pg \leq g$.

Proof: Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$.

Definition 2.6. Let P be a Markov process. Define operators P_C, P_D:

$$P_C : L_1(C) \to L_1(C), P_C f = (P\tilde{f}) \restriction C,$$

where the symbol $\restriction$ denotes the restriction to the set C, $\tilde{f}$ is the function f extended by 0 on D,

$$P_D : L_1(D) \to L_1(D), P_D f = (P\tilde{f}) \restriction D,$$

where $\tilde{f}$ is the function f extended by 0 on C.

Theorem 2.4. *Let P be a Markov process. If supp $f \subseteq C$, then supp $Pf \subseteq C$. (supp $f = \{x : f(x) \neq 0\}$)*

Corollary 2.1. *Let K be an integral Markov operator. Then*

$$(C, \Sigma \upharpoonright C, m \upharpoonright C, K_C)$$

is a Harris process. ($\Sigma \upharpoonright C$ denotes the σ-algebra restricted to the space C, $m \upharpoonright C$ denotes the measure m restricted to the space $\Sigma \upharpoonright C$).

Proof: By Theorem 2.4. supp $f \subseteq C$ implies supp $Kf \subseteq C$. By Theorem 2.1. for $u > 0$ on C,$u = 0$ on D:

$$\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \upharpoonright C)(x)$$

for every $x \in C$.

Corollary 2.2. *Let P be a Markov process on $L_1(X)$. Then*

$$P_D(f \upharpoonright D) = (Pf) \upharpoonright D.$$

Proof: $f = f_D + f_C$, where $f_C = f.1_C, f_D = f.1_D$. By Theorem 2.4. $(Pf_C) \upharpoonright D = 0$, hence

$$(Pf) \upharpoonright D = (Pf_D) \upharpoonright D = P_D(f \upharpoonright D).$$

Corollary 2.3. $P_D^n(f \upharpoonright D) = (P^n f) \upharpoonright D$

Corollary 2.4. *Let P be a Markov process on X, let $u > 0$ on D. Then*

$$\sum_{n=0}^{\infty} P_D^n u < \infty.$$

Proof: Let $\tilde{u}$ be a function on X such that $\tilde{u} \upharpoonright C = 0, \tilde{u} \upharpoonright D = u$. By Corollary 2.3.

$$\sum_{n=0}^{\infty} P_D^n u = (\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D.$$

By Theorem 2.1. $(\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D < \infty$.

3. The Foguel alternative for integral Markov operators of the form (1.2.)

Definition 3.1. Let a family $\mathcal{A} \subset \Sigma$ be given. A Markov process is called sweeping with respect to $\mathcal{A}$, if

$$\lim_{n \to \infty} \int_A P^n f dm = 0$$

for $A \in \mathcal{A}$ and $f \in D$ $(D = \{f \in L_1(X), \|f\| = 1, f \geq 0\})$

In the sequel we shall assume that $\mathcal{A}$ satisfies the following properties:

(i) $0 < m(A) < \infty$ for $A \in \mathcal{A}$

(ii) $A_1, A_2 \in \mathcal{A}$ implies $A_1 \cup A_2 \in \mathcal{A}$

(iii) There exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\cup A_n = X$.

A family satisfying $(i) - (iii)$ will be called admissible.

Definition 3.2. Let (X, Σ, m) and an admissible family $\mathcal{A} \subseteq \Sigma$ be given. A measurable function $f : X \to \mathbb{R}$ is called locally integrable, if

$$\int_A |f| dm < \infty \text{ for } A \in \mathcal{A}.$$

The following theorem is proved in [4].

Theorem 3.1. *Let a measure space (X, Σ, m), an admissible family $\mathcal{A}$ and an integral Markov operator K be given. If K has no invariant density but there exists a positive locally integrable function f_* subinvariant with respect to K, then K is sweeping.*

Remark 3.1. Theorem 3.1. was proved in [4] for stochastic kernel operators ($\int_X K(x, y)dx = 1$). But the proof is completely same for integral Markov operators.

Let K be an integral Markov operator. Recall the definition of K_C and K_D (see def. 2.6.). By Corollary 2.1. K_C is a Harris process and by Corollary 2.4. K_D is dissipative ($X = D$). By Theorem 2.2. and Theorem 2.3. there exist g_C, g_D such that $K_C g_C = g_C$ and $K_D g_D \leq g_D$. The following two lemmas (3.1. and 3.2.) claim that g_C, resp. g_D are locally integrable in all points $y \in C$, (resp. $y \in D$) such that

$$\int_C K_C(x, y)dm(x) > 0 \text{ (resp. } \int_D K_D(x, y)dm(x) > 0 \text{).}$$

Denote by $\mathbb{R}^+$ the set $[0, \infty)$ and by $\mathcal{T}$ the Euclidian metric topology on $\mathbb{R}^+$.

Lemma 3.1. *Let K be an integral Markov operator of the form (1.2.), let $y \in \mathbb{R}^+$. Let $0 < g < \infty$ and $K_C g \leq g$. Let*

$$\int_C K_C(x, y)dm(x) > 0 \ .$$

Then there exists an open neighbourhood U_0 of y such that

$$\int_{U_0 \cap C} g(z)dz < \infty \ .$$

Proof: Let

$$\int_{U_y \cap C} g(z)dz = \infty \quad \forall U_y \in \mathcal{T} \text{ such that } y \in U_y \ .$$

Let $B = \{x \in C : K(x, y) > 0\}$. Let $E \subseteq B$ and $m(E) > 0$. Then

$$\int_E g(x)dx \geq \int_E \int_{U_y \cap C} g(z)K(x, z)dzdx =$$

$$= \int_{U_y \cap C} g(z) \int_E K(x, z)dxdz \ . \qquad (3.1.)$$

Since

$$K(x, y) = q(\lambda(x)).\lambda`(x)h(Q(\lambda(x)) - Q(y))$$

and $Q(y)$ is absolutely continuous,

$$\int_E K(x,z)dx = \int_{Q(\lambda(E))} h(t - Q(z))dt$$

is continuous with respect to z. By the assumption there exists $\varepsilon > 0$ such that

$$\int_E K(x,y)dx > \varepsilon > 0 \ .$$

Since $\int_E K(x,z)dx$ is continuous with respect to z, there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

$$\int_E K(x,z)dx > \varepsilon \quad \forall z \in U_y \ .$$

Now (3.1.) and $\int_{U_y \cap C} g(z)dz = \infty$ imply that

$$\int_E g(x)dx = \infty \ .$$

$E \subseteq B$ was arbitrary, so $g(x) = \infty$ on the set B. But by the assumption $0 < g < \infty$. $\square$

Lemma 3.2. *Let K be an integral Markov operator of the form (1.2.), let $y \in \mathbb{R}^+$. Let $0 < g < \infty$ and $K_D g \leq g$. Let*

$$\int_D K_D(x,y)dm(x) > 0 \ .$$

Then there exists an open neighbourhood U_0 of y such that

$$\int_{U_0 \cap D} g(z)dz < \infty \ .$$

The proof of Lemma 3.2. is the same as the proof of Lemma 3.1..

Theorem 3.2.. *Let K be an integral Markov operator of the form (1.2.). Let $\mathcal{A}$ be the family of compact subsets of $\mathbb{R}^+$ (with respect to the Euclidian metric topology). If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

Proof: Denote

$$\tilde{K}_C f = (Kf).1_C \ , \ \tilde{K}_D f = (Kf).1_D$$

$$f_C = f.1_C \ , \ f_D = f.1_D \ .$$

Now

$$\|\tilde{K}_D^l f_D\| = \|K\tilde{K}_D^l f_D\| = \|\tilde{K}_C \tilde{K}_D^l f_D\| + \|\tilde{K}_D^{l+1} f_D\| \ ,$$

hence

$$\|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^l f_D\| - \|\tilde{K}_D^{l+1} f_D\| \ ,$$

$$\sum_{l=k}^{n} \|\tilde{K}_C \tilde{K}_D^l f_D\| = \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^{n+1} f_D\| \tag{3.2.}$$

Lemma 1. *Let $y \in \mathbb{R}^+$. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap D} K_D^n f dm = 0$$

for every $f \in L_1(D)$.

Proof (of Lemma 1.): By Corollary 2.4.

$$0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty$$

for $u > 0$, hence the process K_D is dissipative. By Theorem 2.3. there exists a σ -finite subinvariant measure λ equivalent to $m \upharpoonright D$.

Let $\mathcal{A}_\lambda$ be the family of all sets of finite measure (with respect to m) such that

$$\int_A \frac{d\lambda}{dm} dm < \infty \quad \forall A \in \mathcal{A}_\lambda .$$

Since $\frac{d\lambda}{dm} < \infty$, the family $\mathcal{A}_\lambda$ is admissible. K_D is dissipative, hence by Theorem 3.1. K_D is sweeping with respect to $\mathcal{A}_\lambda$. Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $D \cap U$ has positive measure. Denote $g = \frac{d\lambda}{dm}$. Let

$$\int_D K(x,y)dx > 0 \ .$$

By Lemma 3.2. there exists $U_y \in \mathcal{T}$ such that

$$\int_{U_y \cap D} g(x)dx < \infty \ ,$$

hence

$$U_y \cap D \in \mathcal{A}_\lambda, \ \lim_{n\to\infty} \int_{U_y \cap D} K_D^n f dm = 0 \ .$$

Let $\int_D K(x,y)dx = 0$. Let

$$\lim_{n\to\infty} \int_{U_y \cap D} K_D^n (f \upharpoonright D) \neq 0$$

for all $U_y \in \mathcal{T}$ such that $y \in U_y$ and some $f \in L_1(\mathbb{R}^+)$. Now $\int_C K(x,y)dx = 1$. Since $\int_C K(x,y)dx$ is continuous with respect to y (see the proof of Lemma 3.1.), there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

$$\int_C K(x,z)dx > \varepsilon > 0 \quad \forall z \in U_y \ .$$

By the assumption there exists $\delta > 0$ such that

$$\int_{U_y \cap D} K_D^n (f \upharpoonright D) > \delta$$

for infinitely many n. By Corollary 2.3.

$$K_D^n(f \upharpoonright D) = (\tilde{K}_D^n f_D) \upharpoonright D \ .$$

Then

$$\begin{aligned}\int_C \tilde{K}_C \tilde{K}_D^n f_D(x)dx &\geq \int_C \int_{U_y \cap D} K(x,z)\tilde{K}_D^n f_D(z)dzdx = \\ &= \int_{U_y \cap D} \tilde{K}_D^n f_D(z) \int_C K(x,z)dxdz \geq \\ &\geq \varepsilon \int_{U_y \cap D} \tilde{K}_D^n f_D(z)dz \geq \varepsilon.\delta\end{aligned}$$

for infinitely many n. Hence

$$\sum_{n=0}^{\infty} \|\tilde{K}_C \tilde{K}_D^n f_D\| \geq \sum_{n=0}^{\infty} \int_C \tilde{K}_C \tilde{K}_D^n f_D(x)dx = \infty$$

which contradicts (3.2.). $\square$

Lemma 2. *Let $y \in \mathbb{R}^+$, let K_C has no stationary density. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n \to \infty} \int_{U_y \cap C} K_C^n f dm = 0$$

for every $f \in L_1(C)$.

Proof (of Lemma 2.): By Corollary 2.1. and Theorem 2.2. K_C is Harris and there exists a function g, $0 < g < \infty$ such that $K_C g = g$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $C \cap U$ has a positive measure. Since $\int_{\mathbb{R}^+} K(x,y)dx = 1$ and by Corollary 2.2 $K(x,y) = 0$ for $x \in D$, $y \in C$,

$$\int_C K(x,y)dx = 1 \ .$$

By Lemma 3.1. there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and

(3.3.) $$\int_{U_y \cap C} g(x)dx < \infty \ .$$

Let $\mathcal{A}_g$ be the family of all sets of finite measure such that

$$\int_A gdm < \infty \quad \forall A \in \mathcal{A}_g \ .$$

Since $g < \infty$, the family $\mathcal{A}_g$ is admissible. By (3.3.) $U_y \cap C \in \mathcal{A}_g$ and by Theorem 3.1.

$$\int_{U_y \cap C} K_C^n f dm \to 0 \quad \forall f \in L_1(C) \ . \quad \square$$

Lemma 3. *Let K_C has no stationary density, let $A \in \mathcal{A}$. Then*

$$\lim_{n\to\infty} \int_{A\cap C} K_C^n f_1 dm = 0 \ , \quad \lim_{n\to\infty} \int_{A\cap D} K_D^n f_2 dm = 0 \tag{3.4.}$$

for every $f_1 \in L_1(C)$, $f_2 \in L_1(D)$.

Proof (of Lemma 3.): Let $y \in \mathbb{R}^+$. By Lemma 1. there exists $U_1 \in \mathcal{T}$ such that $y \in U_1$ and

$$\lim_{n\to\infty} \int_{U_1\cap D} K_D^n f_2 dm = 0 \quad \forall f_2 \in L_1(D).$$

By Lemma 2. there exists $U_2 \in \mathcal{T}$ such that $y \in U_2$ and

$$\lim_{n\to\infty} \int_{U_2\cap C} K_C^n f_1 dm = 0 \quad \forall f_1 \in L_1(C).$$

Set $U_y = U_1 \cap U_2$. Then

$$\lim_{n\to\infty} \int_{U_y\cap C} K_C^n f_1 dm = 0 \ , \quad \lim_{n\to\infty} \int_{U_y\cap D} K_D^n f_2 dm = 0 \tag{3.5.}$$

Thus we have proved that for every $y \in \mathbb{R}^+$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and (3.5.) holds. Finally (3.4.) follows from compactness of A. $\square$

Proof (of Theorem 3.2.): By Lemma 3. K_D is sweeping, K_C is sweeping or has a stationary density.

Let K_C have a stationary density $\tilde{f}$. Let f_* be a function on $\mathbb{R}^+$ such that $f_* \restriction C = \tilde{f}$, $f_* \restriction D = 0$. Then

$$(Kf_*) \restriction C = (K(f_*.1_C)) \restriction C + (K(f_*.1_D)) \restriction C = K_C \tilde{f} = \tilde{f}.$$

By Corollary 2.2. $(Kf_*) \restriction D = K_D(f_* \restriction D) = 0$, hence $Kf_* = f_*$. Let K_C be sweeping. We shall prove that K is sweeping.

Let $f \in L_1(\mathbb{R}^+)$, then $f = f_C + f_D$, where $f_C = f.1_C$, $f_D = f.1_D$. By Corollary 2.3.

$$(K^n f_C) \restriction D = 0, \ (K^n f) \restriction D = K_D^n (f \restriction D).$$

By Lemma 3.

$$\int_{A\cap D} K^n f dm \to 0 \text{ for every } A \in \mathcal{A}.$$

Now it is enough to prove that

$$\int_{A\cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A}.$$

Clearly

$$\begin{aligned}
\tilde{K}_C f &= \tilde{K}_C(f_C + f_D), \ Kf = \tilde{K}_C f + \tilde{K}_D f, \\
\tilde{K}_C(Kf) &= \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C \tilde{K}_D f_D \\
\tilde{K}_C(K^2 f) &= \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2 \tilde{K}_D f_D + \tilde{K}_C \tilde{K}_D^2 f_D \\
&\dots \\
&\dots \\
K^n f.1_C &= \tilde{K}_C(K^{n-1} f) = \\
&= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots \\
&+ \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D.
\end{aligned}$$

Take $1 < k < n$ and define:

$$M_{k,n}f = \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1}\tilde{K}_D f_D + \cdots + \tilde{K}_C^{n-k+1}\tilde{K}_D^{k-1} f_D$$
$$R_{k,n}f_D = \tilde{K}_C^{n-k}\tilde{K}_D^k f_D + \cdots + \tilde{K}_C\tilde{K}_D^{n-1} f_D.$$

$\tilde{K}_C$ is contraction, hence

$$\|R_{k,n}f_D\| \leq \|\tilde{K}_C^{n-k}\tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C\tilde{K}_D^{n-1} f_D\| \leq$$
$$\leq \|\tilde{K}_C\tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C\tilde{K}_D^{n-1} f_D\|.$$

By (3.2.)

$$\|R_{k,n}f_D\| \leq \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\|.$$

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| < \frac{\varepsilon}{2} \text{ for } n, k \geq n_0(\varepsilon),\ n \geq k.$$

Now fix $k \geq n_0(\varepsilon)$, $A \in \mathcal{A}$. $\tilde{K}_C$ be sweeping implies

$$\int_{A\cap C} M_{k,n} f dm < \frac{\varepsilon}{2}$$

for n sufficiently large, hence

$$\int_{A\cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A}. \quad \square$$

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