THE FOGUEL ALTERNATIVE FOR INTEGRAL MARKOV OPERATORS

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ABSTRACT

A class of Markov operators satisfies the Foguel alternative if its members are either sweeping or have stationary densities. New sufficient condition for this property is given.

1. Introduction

We shall consider Markov operators $K : L_1(X) \longrightarrow L_1(X)$ of the form :

$$Kf(x) = \int_X K(x,y)f(y)dy \ ,$$

where $K(x,y)$ defined on XxX is a kernel. Such operators were intensively studied. In [1],[4],[6],[7] some sufficient conditions for sweeping (see def.3.1.) and asymptotical stability were given. It was proved in [4] that, under the assumption of having subinvariant locally integrable function, the alternative of sweeping or having stationary density holds. The main result of this paper is the proof of this alternative without the assumption of having subinvariant locally integrable function (Th.3.2.).

In the section 2., some necessary results of [2] are presented. In the section 3., the main result is proved. Section 4. contains an application of Theorem 3.2. to the class of Markov operators appearing in the mathematical theory of the cell cycle.

2. Some properties of Markov processes and integral Markov operators

Theorems 2.1 - 2.4. are proved in [2].

Definition 2.1. A Markov process is defined to be a quadruple (X, Σ, m, P) , where (X, Σ, m) is a $\sigma-$ finite measure space with positive measure and where P is an operator on $L_1(X)$ satisfying
 (i) P is a contraction : $\|P\| \leq 1$
 (ii) P is positive : if $0 \leq u \in L_1(X)$ then $Pu \geq 0$

Definition 2.2. If u is an arbitrary non-negative function, set $Pu := \lim_{k\to\infty} Pu_k$ for $0 \leq u_k \in L_1(X), u_k \nearrow u$, where the symbol $\nearrow$ denotes monotone pointwise convergence almost everywhere. The sequence Pu_k is increasing so that $\lim_k Pu_k$ exists (it may be infinite). By [2] the definition of Pu is independent of the particular sequence u_k.

Definition 2.3. Take $u_0 \in L_1(X)$ with $u_0 > 0$. Define

$$C = \{x : \sum_{k=0}^{\infty} P^k u_0(x) = \infty\} \ , \ D = X \setminus C$$

By [2] this definition is independent of the choice of u_0.

Theorem 2.1. *If* $0 \leq u \in L_1(X)$ *then*

$$\sum_{k=0}^{\infty} P^k u(x) < \infty \ \textit{for} \ x \in D \ , \ \sum_{k=0}^{\infty} P^k u(x) = 0 \ \textit{or} \ \infty \ \textit{for} \ x \in C \ .$$

Definition 2.4. A function $K(x, y) \geq 0$ defined on XxX which is jointly measurable with respect to its variables is called a kernel. Let $\int_X K(x, y)dx \leq 1$. Define an operator K on $L_1(X)$:

$$Kf(x) = \int_X K(x, y)f(y)dy \ .$$

Then $\|K\| \leq 1$ and K is called an integral Markov operator.

Definition 2.5. Let P be an integral Markov operator, then (X, Σ, m, P) is said to be a Harris process if $X = C$.

Theorem 2.2. *Let* K *be an integral Markov operator and a Harris process. Then there exists* $0 < u < \infty$ *such that* $Ku = u$ *(a* σ*-finite invariant measure).*

Theorem 2.3. *Let P be a Markov process with $X = D$. Then there exists $0 < g < \infty$ such that $Pg \leq g$.*

Proof: Let $0 < u_0 \in L_1(X)$. Set $g = \sum_{k=0}^{\infty} P^k u_0$. $\square$

Definition 2.6. Let P be a Markov process. Define operators P_C, P_D:

$$P_C : L_1(C) \to L_1(C) \ , \ P_C f = (P\tilde{f}) \restriction C \ ,$$

where the symbol $\restriction$ denotes the restriction to the set C, $\tilde{f}$ is the function f extended by 0 on D,

$$P_D : L_1(D) \to L_1(D) \ , \ P_D f = (P\tilde{f}) \restriction D \ ,$$

where $\tilde{f}$ is the function f extended by 0 on C.

Theorem 2.4. *Let P be a Markov process. If supp $f \subseteq C$, then supp $Pf \subseteq C$. (supp $f = \{x : f(x) \neq 0\}$)*

Corollary 2.1. *Let K be an integral Markov operator. Then*

$$(C, \Sigma \restriction C, m \restriction C, K_C)$$

is a Harris process. ($\Sigma \restriction C$ denotes the σ-algebra restricted to the space C, $m \restriction C$ denotes the measure m restricted to the space $\Sigma \restriction C$).

Proof: By Theorem 2.4. supp $f \subseteq C$ implies supp $Kf \subseteq C$. By Theorem 2.1. for $u > 0$ on C,$u = 0$ on D:

$$\infty = \sum_{k=0}^{\infty} K^k u(x) = \sum_{k=0}^{\infty} K_C^k (u \restriction C)(x)$$

for every $x \in C$. $\square$

Corollary 2.2. *Let P be a Markov process on $L_1(X)$. Then*

$$P_D(f \restriction D) = (Pf) \restriction D \ .$$

Proof: $f = f_D + f_C$, where $f_C = f.1_C, f_D = f.1_D$. By Theorem 2.4. $(Pf_C) \restriction D = 0$, hence

$$(Pf) \restriction D = (Pf_D) \restriction D = P_D(f \restriction D) \ . \quad \square$$

Corollary 2.3. $P_D^n(f \restriction D) = (P^n f) \restriction D$.

Corollary 2.4. *Let P be a Markov process on X, let $u > 0$ on D. Then*

$$\sum_{n=0}^{\infty} P_D^n u < \infty \ .$$

Proof: Let $\tilde{u}$ be a function on X such that $\tilde{u} \upharpoonright C = 0, \tilde{u} \upharpoonright D = u$. By Corollary 2.3.

$$\sum_{n=0}^{\infty} P_D^n u = (\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D \ .$$

By Theorem 2.1. $(\sum_{n=0}^{\infty} P^n \tilde{u}) \upharpoonright D < \infty$. $\square$

3. The Foguel alternative for integral Markov operators

Definition 3.1. Let a family $\mathcal{A} \subset \Sigma$ be given. A Markov process is called sweeping with respect to $\mathcal{A}$, if

$$\lim_{n \to \infty} \int_A P^n f dm = 0$$

for $A \in \mathcal{A}$ and $f \in D$ $(D = \{f \in L_1(X), \|f\| = 1, f \geq 0\})$

In the sequel we shall assume that $\mathcal{A}$ satisfies the following properties:

(i) $0 < m(A) < \infty$ for $A \in \mathcal{A}$

(ii) $A_1, A_2 \in \mathcal{A}$ implies $A_1 \cup A_2 \in \mathcal{A}$

(iii) There exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\cup A_n = X$.

A family satisfying $(i) - (iii)$ will be called **admissible.**

Definition 3.2. Let (X, Σ, m) and an admissible family $\mathcal{A} \subseteq \Sigma$ be given. A measurable function $f : X \to \mathbb{R}$ is called locally integrable, if

$$\int_A |f| dm < \infty \text{ for } A \in \mathcal{A} \ .$$

Theorem 3.1. *Let a measure space (X, Σ, m), an admissible family $\mathcal{A}$ and an integral Markov operator K be given. If K has no invariant density but there exists a positive locally integrable function f_* subinvariant with respect to K, then K is sweeping.*

Remark 3.1. Theorem 3.1. was proved in [4] for stochastic kernel operators $(\int_X K(x, y) dx = 1)$. But the proof is completely same for integral Markov operators.

We say that an integral Markov operator $K : L_1(X) \to L_1(X)$ satisfies a property (P) with respect to topology $\mathcal{T}$ on X, if

$$(\forall y \in X)(\exists B \in \Sigma \text{ with } m(B) > 0 \text{ such that } ((\forall x \in B)(\exists U_y^x \in \mathcal{T},$$

$\varepsilon_x > 0$ such that $y \in U_y^x$, and $\forall z \in U_y^x :\ K(x,z) > \varepsilon_x)))$

Theorem 3.2. *Let K be an integral Markov operator satisfying property (P) with respect to a topology $\mathcal{T}$. Let the measure m be locally finite (with respect to $\mathcal{T}$). Let the sets of $\mathcal{A}$ be compact. If K has no stationary density, then K is sweeping with respect to $\mathcal{A}$.*

Proof: Denote

$$\tilde{K}_C f = (Kf).1_C \ , \ \tilde{K}_D f = (Kf).1_D$$

$$f_C = f.1_C \ , \ f_D = f.1_D \ .$$

Now

$$\|\tilde{K}_D^l f_D\| = \|K\tilde{K}_D^l f_D\| = \|\tilde{K}_C\tilde{K}_D^l f_D\| + \|\tilde{K}_D^{l+1} f_D\| \ ,$$

hence

$$\|\tilde{K}_C\tilde{K}_D^l f_D\| = \|\tilde{K}_D^l f_D\| - \|\tilde{K}_D^{l+1} f_D\| \ ,$$

$$\sum_{l=k}^{n} \|\tilde{K}_C\tilde{K}_D^l f_D\| = \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^{n+1} f_D\| \qquad (3.1.)$$

Lemma 1. *Let $y \in X$. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap D} K_D^n f \, dm = 0$$

for every $f \in L_1(D)$.

Proof (of Lemma 1.): By Corollary 2.4.

$$0 < \sum_{n=0}^{\infty} K_D^n u(x) < \infty$$

for $u > 0$, hence the process K_D is dissipative. By Theorem 2.3. there exists a σ -finite subinvariant measure λ equivalent to $m \restriction D$.

Let $\mathcal{A}_\lambda$ be the family of all sets of finite measure (with respect to m) such that

$$\int_A \frac{d\lambda}{dm} dm < \infty \ \ \forall A \in \mathcal{A}_\lambda \ .$$

Since $\frac{d\lambda}{dm} < \infty$, the family $\mathcal{A}_\lambda$ is admissible. K_D is dissipative, hence by Theorem 3.1. K_D is sweeping with respect to $\mathcal{A}_\lambda$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $D \cap U$ has positive measure. By the assumption there is a set B $(m(B) > 0)$ such that for every $x \in B$

there is $U_y^x \in \mathcal{T}$ such that $K(x,z) > \varepsilon_x$ on U_y^x. No loss of generality we may assume that the sets U_y^x have finite measure.

Let $m(B \cap D) > 0$. Then $\forall x \in B \cap D$

$$g(x) \geq \int_{U_y^x \cap D} K(x,z) g(z) dz \geq \int_{U_y^x \cap D} \varepsilon_x g(z) dz \ ,$$

hence

$$\int_{U_y^x \cap D} g(z) dz \leq \frac{1}{\varepsilon_x} g(x) < \infty$$

for every $x \in B \cap D$ and $U_y^x \cap D \in \mathcal{A}_\lambda$, $\lim_{n\to\infty} \int_{U_y^x \cap D} K_D^n f dm = 0$ for every $x \in B \cap D$.

Let $m(B \cap D) = 0$. Let $\lim_{n\to\infty} \int_{U_y^x \cap D} K_D^n (f \restriction D) \neq 0$ for some $f \in L_1(X)$ and every $x \in B$. By Corollary 2.3.

$$K_D^n (f \restriction D) = (\tilde{K}_D^n f_D) \restriction D \ .$$

Then

$$\int_{U_y^x \cap D} \tilde{K}_D^n f_D(x) > \delta_x > 0$$

for some δ_x and infinitely many n,

$$\forall x \in B \quad \tilde{K}_C \tilde{K}_D^n f_D(x) > \delta_x . \varepsilon_x$$

for infinitely many n. By the Lebesgue Monotone Convergence Theorem

$$\infty = \| \sum_{n=0}^{\infty} \tilde{K}_C \tilde{K}_D^n f_D \| = \sum_{n=0}^{\infty} \| \tilde{K}_C \tilde{K}_D^n f_D \| \ ,$$

which contradicts (3.1.). $\square$

Lemma 2. *Let $y \in X$, let K_C has no stationary density. Then there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and*

$$\lim_{n\to\infty} \int_{U_y \cap C} K_C^n f dm = 0$$

for every $f \in L_1(C)$.

Proof (of Lemma 2.): By Corollary 2.1. and Theorem 2.2. K_C is Harris and there exists a function g, $0 < g < \infty$ such that $K_C g = g$.

Let y be such that for every neighbourhood $U \in \mathcal{T}$ of y the set $C \cap U$ has a positive measure. By the assumption there is a set B such that for every $x \in B$ there is $U_y^x \in \mathcal{T}$ such that $K(x,z) > \varepsilon_x$ on U_y^x. By Corollary 2.2. $K(x,z) = 0$ for $x \in D,\ z \in C$, hence $B \subseteq C$. Now

$$g(x) \geq \int_{U_y^x \cap C} K(x,z)g(z)dz \geq \int_{U_y^x \cap C} \varepsilon_x g(z)dz ,$$

hence

$$\int_{U_y^x \cap C} g(z)dz < \frac{1}{\varepsilon_x} g(x) < \infty$$

for some $x \in B$. Let $\mathcal{A}_g$ be the family of all sets of finite measure such that

$$\int_A gdm < \infty \quad \forall A \in \mathcal{A}_g \ .$$

Since $g < \infty$, the family $\mathcal{A}_g$ is admissible. Then $U_y^x \cap C \in \mathcal{A}_g$ and by Theorem 3.1.

$$\int_{U_y^x \cap C} K_C^n f dm \to 0 \quad \forall f \in L_1(C) \ . \quad \square$$

Lemma 3. *Let K_C has no stationary density, let $A \in \mathcal{A}$. Then*

$$\lim_{n\to\infty} \int_{A\cap C} K_C^n f_1 dm = 0 \ , \quad \lim_{n\to\infty} \int_{A\cap D} K_D^n f_2 dm = 0 \tag{3.2.}$$

for every $f_1 \in L_1(C),\ f_2 \in L_1(D)$.

Proof (of Lemma 3.): Let $y \in X$. By Lemma 1. there exists $U_1 \in \mathcal{T}$ such that $y \in U_1$ and

$$\lim_{n\to\infty} \int_{U_1\cap D} K_D^n f_2 dm = 0 \quad \forall f_2 \in L_1(D) \ .$$

By Lemma 2. there exists $U_2 \in \mathcal{T}$ such that $y \in U_2$ and

$$\lim_{n\to\infty} \int_{U_2\cap C} K_C^n f_1 dm = 0 \quad \forall f_1 \in L_1(C) \ .$$

Set $U_y = U_1 \cap U_2$. Then

$$\lim_{n\to\infty} \int_{U_y\cap C} K_C^n f_1 dm = 0 \ , \ \lim_{n\to\infty} \int_{U_y\cap D} K_D^n f_2 dm = 0 \tag{3.3.}$$

Thus we have proved that for every $y \in X$ there exists $U_y \in \mathcal{T}$ such that $y \in U_y$ and (3.3.) holds. Finally (3.2.) follows from compactness of A . $\square$

Proof (of Theorem 3.2.): By Lemma 3. K_D is sweeping, K_C is sweeping or has a stationary density.

Let K_C have a stationary density $\tilde{f}$. Let f_* be a function on X such that $f_* \restriction C = \tilde{f}, \quad f_* \restriction D = 0$. Then

$$(Kf_*) \restriction C = (K(f_*.1_C)) \restriction C + (K(f_*.1_D)) \restriction C = K_C\tilde{f} = \tilde{f} \ .$$

By Corollary 2.2. $(Kf_*) \restriction D = K_D(f_* \restriction D) = 0$, hence $Kf_* = f_*$. Let K_C be sweeping. We shall prove that K is sweeping.

Let $f \in L_1(X)$, then $f = f_C + f_D$, where $f_C = f.1_C, \ f_D = f.1_D$. By Corollary 2.3.

$$(K^n f_C) \restriction D = 0 \ , \ (K^n f) \restriction D = K_D^n(f \restriction D) \ .$$

By Lemma 3.

$$\int_{A \cap D} K^n f dm \to 0 \text{ for every } A \in \mathcal{A} \ .$$

Now it is enough to prove that

$$\int_{A \cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A} \ .$$

Clearly

$$\begin{aligned}
\tilde{K}_C f &= \tilde{K}_C(f_C + f_D), \ Kf = \tilde{K}_C f + \tilde{K}_D f \ , \\
\tilde{K}_C(Kf) &= \tilde{K}_C^2 f_C + \tilde{K}_C^2 f_D + \tilde{K}_C \tilde{K}_D f_D \\
\tilde{K}_C(K^2 f) &= \tilde{K}_C^3 f_C + \tilde{K}_C^3 f_D + \tilde{K}_C^2 \tilde{K}_D f_D + \tilde{K}_C \tilde{K}_D^2 f_D \\
&\dots \\
&\dots \\
K^n f.1_C &= \tilde{K}_C(K^{n-1} f) = \\
&= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots \\
&+ \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \ .
\end{aligned}$$

Take $1 < k < n$ and define:

$$\begin{aligned}
M_{k,n} f &= \tilde{K}_C^n f_C + \tilde{K}_C^n f_D + \tilde{K}_C^{n-1} \tilde{K}_D f_D + \dots + \tilde{K}_C^{n-k+1} \tilde{K}_D^{k-1} f_D \\
R_{k,n} f_D &= \tilde{K}_C^{n-k} \tilde{K}_D^k f_D + \dots + \tilde{K}_C \tilde{K}_D^{n-1} f_D \ .
\end{aligned}$$

$\tilde{K}_C$ is contraction, hence

$$\|R_{k,n} f_D\| \le \|\tilde{K}_C^{n-k} \tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \le$$
$$\le \|\tilde{K}_C \tilde{K}_D^k f_D\| + \cdots + \|\tilde{K}_C \tilde{K}_D^{n-1} f_D\| \ .$$

By (3.1.)

$$\|R_{k,n} f_D\| \le \|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| \ .$$

The sequence $\{\|\tilde{K}_D^n f\|\}$ is nonincreasing for $\tilde{K}_D$ being contraction. Thus

$$\|\tilde{K}_D^k f_D\| - \|\tilde{K}_D^n f_D\| < \frac{\varepsilon}{2} \text{ for } n, k \ge n_0(\varepsilon), \ n \ge k \ .$$

Now fix $k \ge n_0(\varepsilon), \ A \in \mathcal{A}$. $\tilde{K}_C$ be sweeping implies

$$\int_{A \cap C} M_{k,n} f dm < \frac{\varepsilon}{2}$$

for n sufficiently large, hence

$$\int_{A \cap C} K^n f dm \to 0 \text{ for } A \in \mathcal{A} \ . \quad \Box$$

4. Application

In the mathematical theory of the cell cycle an important role is played by the class of integral Markov operators of the form:

$$Kf(x) = \int_0^{\lambda(x)} K(x,y) f(y) dy \ ,$$

where

$$K(x,y) = -\frac{\partial}{\partial x} \exp\{-\int_y^{\lambda(x)} q(z) dz\} \ .$$

Assume the following conditions:
(i) $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is continuosly differentiable. Moreover,
$\lambda'(x) > 0$ for $x \ge 0, \ \lambda(0) = 0$, and $\lim_{x \to \infty} \lambda(x) = \infty$.
(ii) The function $q : \mathbb{R}^+ \to \mathbb{R}^+$ is locally integrable and $\int_0^\infty q(x) dx = \infty$.
Let $\mathcal{T}$ be the Euclidian metric topology, $\mathcal{A}$ the family of compact subsets of $\mathbb{R}^+$. Then it is quite easy to prove that K satisfies the property (P) with respect to $\mathcal{T}$:

$$K(x,y) = \lambda'(x) q(\lambda(x)) \exp\{-\int_y^{\lambda(x)} q(z) dz\} \ ,$$

$\lambda'(x) > 0$ for every x. Let $y_0 \in \mathbb{R}^+$, let

$$B = \{x : \ q(\lambda(x)) > 0 \ , \lambda(x) > y_0\} \ .$$

$m(B) > 0$ follows from $\int_0^\infty q(z)dz = \infty$. Further

$$\int_y^{\lambda(x)} q(z)dz \leq \int_0^{\lambda(x)} q(z)dz < \infty \ ,$$

hence

$$\exp\{-\int_y^{\lambda(x)} q(z)dz\} \geq \delta(x) > 0 \ ,$$

and

$$K(x,y) \geq \lambda'(x)q(\lambda(x))\delta(x) = \varepsilon(x) > 0 \qquad (4.1.)$$

on the set $\{x : q(\lambda(x)) > 0\}$. Now set $U_{y_0}^x = [0, \lambda(x))$ and the property (P) is fullfilled. By Theorem 3.2. K is sweeping with respect to $\mathcal{A}$ or has a stationary density.

Let K has stationary density f_*. We show that K is asymptotically stable. Denote $C = \text{supp } f_*$. Lemma 4.1. was proved in [1].

Definition 4.1. We say that a Markov process P overlaps supports if for every two densities $f, \ g$ there is a positive integer $n_0 = n_0(f,g)$ such that

$$\mu(\text{supp } P^{n_0} f \cap \text{supp } P^{n_0} g) > 0 \ .$$

Lemma 4.1. *Let $K : L_1(X, \Sigma, m) \to L_1(X, \Sigma, m)$ be a stochastic integral Markov operator which overlaps supports and has the invariant density f_*. Denote $C =$ supp f_*. If there is a $\delta > 0$ such that*

$$\sup_n \int_C K^n f dm \geq \delta$$

for every density f, then K is asymptotically stable.

Since

$$K(x,y) = \lambda'(x).q(\lambda(x)).\exp\{-\int_y^{\lambda(x)} q(z)dz\} \ ,$$

$q(\lambda(x)) > 0$ on C follows from

$$f_*(x) = \int_0^{\lambda(x)} K(x,y) f_*(y) dy \ .$$

Let

$$m_0 = \inf\{z : m((0,z) \cap C) > 0\} \ ,$$
$$m_1 = \lambda^{-1}(m_0) \ ,$$
$$m_2 = \lambda^{-1}(m_1) \ .$$

Then the set

$$((m_1, \infty) \cap \{z : q(\lambda(z)) > 0\}) \setminus C$$

has measure zero, since by (4.1.)

$$Kf_*(x) = \int_0^{\lambda(x)} K(x,y) f_*(y) dy \geq \int_{m_0}^{\lambda(x)} \varepsilon(x) f_*(y) dy > 0$$

on the set $(m_1, \infty) \cap \{z : q(\lambda(z)) > 0\}$.

Now $(0, m_1) \cup C \supset \operatorname{supp} Kf$ for every density f, and

$$\begin{aligned}
\int_C K^2 f(x) dx &= \int_C \left(\int_{\mathbb{R}^+} K(x,y) Kf(y) dy \right) dx = \\
&= \int_C \left(\int_{(0,m_1)\cup C} K(x,y) Kf(y) dy \right) dx = \\
&= \int_0^{m_1} \left(\int_C K(x,y) dx \right) Kf(y) dy \\
&+ \int_C \left(\int_{C \setminus (0,m_1)} K(x,y) Kf(y) dy \right) dx \ .
\end{aligned}$$

By (4.1.)

$$\begin{aligned}
\int_0^{m_1} \left(\int_C K(x,y) dx \right) Kf(y) dy &\geq \int_0^{m_1} \left(\int_{C \cap (m_2, \infty)} \varepsilon(x) dx \right) Kf(y) dy \geq \\
&\geq \delta \int_0^{m_1} Kf(y) dy \ . \qquad (4.2.)
\end{aligned}$$

For Markov operators $\operatorname{supp} f \subset C$ implies $\operatorname{supp} Kf \subset C$, if C is support of invariant density, hence

$$\begin{aligned}
\int_C \left(\int_{C \setminus (0,m_1)} K(x,y) Kf(y) dy \right) dx &= \int_X \left(\int_{C \setminus (0,m_1)} K(x,y) Kf(y) dy \right) dx = \\
&= \int_{C \setminus (0,m_1)} \int_X K(x,y) dx Kf(y) dy = \\
&= \int_{C \setminus (0,m_1)} Kf(y) dy \ . \qquad (4.3.)
\end{aligned}$$

Finally (4.2.) and (4.3.) imply that

$$\int_C K^2 f(x)dx \geq \delta \int_{(0,m_1)\cup C} Kf(x)dx = \delta \int_X Kf(x)dx = \delta$$

for every density f. By Lemma 4.1. it is enough to prove that K overlaps supports.

By (4.1.)

$$K(x,y) \geq \varepsilon(x) > 0$$

on the set

$$S = \{x : q(\lambda(x)) > 0\} \ .$$

Since $\int_0^\infty q(z)dz = \infty$ and q is locally integrable, the set $(k,\infty) \cap S$ has positive measure for every $k > 0$. If $f,\ g$ are arbitrary densities such that the sets

$$(0,k) \cap \text{ supp } f \quad \text{and} \quad (0,k) \cap \text{ supp } g$$

have positive measures, than on the set

$$(\lambda^{-1}(k), \infty) \cap S$$

$Kf > 0$ and $Kg > 0$, hence K overlaps supports. $\square$

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