

Dynamic Accumulation Model for the Second Pillar of the Slovak Pension System

S. Kilianová I. Melicherčík D. Ševčovič *

October 4, 2006

Corresponding author: Igor Melicherčík

email: igor.melichercik@fmph.uniba.sk

tel: +421 2 60295477, +421 903 653153

fax: +421 2 60295560

Address:

Department of Applied Mathematics and Statistics
Faculty of Mathematics, Physics and Informatics
Comenius University
Mlynská dolina
842 48 Bratislava, Slovakia

Home:

Hlaváčiková 2
841 05 Bratislava, Slovakia

*Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics
Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia.

Emails: {kilianova,igor.melichercik,sevcovic}@fmph.uniba.sk

DYNAMIC ACCUMULATION MODEL FOR THE SECOND PILLAR OF THE SLOVAK PENSION SYSTEM

Abstract

Since January 2005, pensions in Slovakia are operated by a three-pillar system proposed by the World Bank. The paper concentrates on the mandatory, fully funded second pillar. We present a dynamic accumulation model for determining the optimal switching strategy between pension funds with different risk profiles. The resulting strategies depend on individual risk preferences of future pensioners. Our results illustrated the fact that gradual decreasing of the risk during the saving for future pension is rational. Furthermore, we present several simulations of optimal fund switching strategies for various model parameter settings.

Od januára 2005 je na Slovensku 3-pilierový dôchodkový systém navrhnutý Svetovou Bankou. Článok sa zameriava na druhý (sporivý) pilier. Predkladáme dynamický model sporenia, ktorého výsledkom sú stratégie optimálneho prepínania medzi dôchodkovými fondami s rôznymi mierami rizika. Výsledné stratégie závisia na vzťahu jednotlivých sporiteľov k riziku. Naše výsledky naznačujú, že zákonné obmedzenia smerujúce k postupnému znižovaniu rizika pred dôchodkom sú rozumné. Ponúkame tiež simulácie prepínacích stratégií pre rôzne nastavenia modelu.

1. Introduction

Before January 2005, pensions in Slovakia were operated by the unfunded pay-as-you-go system. Mainly because of high unemployment and low contributions paid on behalf of unemployed by the government as well as high contribution evasions, the system generated deficits. The demography crisis was supposed to generate further pressure on the balance of the pay-as-you-go system. In April 2003 the government passed the Principles of the Pension Reform in the Slovak Republic. The goals of the pension reform were to secure a stable flow of high pensions to the beneficiaries, and sustainability and overall stability of the system. Corresponding legislation, as passed in December 2003, established a system based on three pillars:

1. the mandatory non-funded first pillar (pay-as-you-go pillar)
2. the mandatory fully funded second pillar
3. the voluntary fully funded third pillar

The contribution rates were set for the first pillar at 19.75% (old age 9%, disability and survival 6% and reserve fund 4.75%) and for the second pillar 9%. The total rate is about 0.75% higher than the old one. A thorough description of the Slovak pension reform with calculations of the balance of the pension system and expected level of pensions in the new system could be found in (Golias, 2003), (Melicherčík & Ungvarský 2004), (Thomay 2002).

Fund type	Stocks	Bonds and money market instruments
Growth Fund	up to 80%	at least 20%
Balanced Fund	up to 50%	at least 50%
Conservative Fund	no stocks	100%

Table 1: Limits for investment for the pension funds

Compared to Poland and Hungary, the Slovak second pillar is more substantial. Contribution rates are higher in Slovakia - compared to 7.3% in Poland and 6% (with possible future increase to 8%) in Hungary. A thorough description of the pension reforms in Hungary and Poland could be found in (Benczúr 1999), (Chlon et al. 1999), (Fultz 2002), (Palacios & Rocha 1998) and (Simonovits 2000).

The savings in the second pillar are managed by pension asset administrators. Each pension administrator manages three funds: Growth Fund, Balanced Fund and Conservative fund, each of them with different limits for investment (see Tab. 1). At the same time instant savers may hold assets in one fund only. In the last 15 years preceding retirement, the saver may not hold assets in the Growth Fund and in the last 7 years all assets must be in the Conservative Fund. Even with these restrictions the contributors have some space for individual decisions which fund is optimal in a specific situation (the age of the contributor, the saved amount, the past performance of the pension funds). The aim of this paper is to study whether the above restrictions for the funds can be illustrated by a mathematical model and to calculate optimal strategies of switching between the pension funds (Growth, Balanced and Conservative) having in mind the risk preferences of the contributors. Our model indicates that adopted pension fund regulations can be supported by means of a dynamic accumulation model.

The paper is organized as follows: In Section 2 we present a simple example supporting the idea of gradual decreasing of the risk during the saving for future pension. We also give a motivation for studying the dynamic accumulation models. Section 3 contains the formulation of the dynamic stochastic programming accumulation model and the numerical scheme for finding a solution of this model. In Section 4 we present the calculated results and we discuss sensitivity of fund switching strategies with respect to various scenarios of development of financial markets, wage growth development as well as individual risk preferences. At the end of the section we compare dynamic and static strategies using the mean-variance framework. The last section contains final remarks and conclusions.

2. First run a risk then secure savings

Pension funds usually hold portfolio consisting of bonds and equities. Limits for their weights in the portfolio may differ across the countries. In Slovakia, each pension company manages three funds: Growth Fund, Balanced Fund and Conservative fund, each of them with different limits for investment (see Tab. 1). As it was already mentioned in Introduction, instant

savers may hold assets in one fund only and they may not hold assets in the Growth Fund in the last 15 years preceding retirement. Moreover, all assets should be held in the Conservative Fund in the last 7 years preceding retirement. The intention of these restrictions and government regulations was to lower the risk of the value of savings to fall substantially shortly before retirement.

For the sake of simplicity let us consider a plain 2-period model of saving (the length of each period 1 year). At the beginning of each year the saver deposits an amount A . Suppose that the returns of the deposits are r_1 and r_2 in the first and second year, respectively. The saved amount M after 2 years is

$$M = A(1 + r_1)(1 + r_2) + A(1 + r_2) = A(2 + r_1)(1 + r_2).$$

Then for the sensitivity of M with respect to r_1 and r_2 we have

$$\frac{\partial M}{\partial r_1} = A(1 + r_2), \quad \frac{\partial M}{\partial r_2} = A(2 + r_1)$$

and therefore

$$\frac{\partial M}{\partial r_2} > \frac{\partial M}{\partial r_1} \tag{1}$$

for realistic asset returns r_1 and r_2 . This is in accord with intuition that the saved amount is more sensitive to later returns than to earlier ones. If the individual made just a single contribution at the start of his/her working career, the impact on his/her final pension wealth would be the same regardless of whether the asset price fall occurred early in life or just before retirement. But if a series of contributions throughout one's life is made, a fall in assets value early in life does not affect the future contributions, i.e. only part of one's future pension wealth is affected, while if it occurs close to retirement it affects all past accumulated contributions and returns on them, i.e. most of one's pension wealth.

Let us consider two funds:

1. a risky fund with normally distributed return with average 10% and standard deviation 10%
2. a secure fund with a certain return 5% .

Suppose that the saver deposits 1 unit in the first and 1 unit in the second period. Tab. 2 and Fig. 1 demonstrate a risk-return analysis of 5 different strategies. Strategy 1 assumes that in both periods the savings are invested to the secure fund. Strategy 2 is the most risky one - in both periods the savings are invested to the risky fund. This strategy has the highest expected value of the savings at the end of the second period $E(M)$ but also the highest standard deviation σ_M .

To decrease the risk, Strategies 3 and 4 invest into the secure fund in one of the periods. According to (1) the level of final savings is more sensitive to the second period asset returns. Therefore the risk (see the last 3 columns of the Tab. 2) connected with Strategy 4 (first year secure fund, second year risky fund) is higher than the risk connected with Strategy 3 (first year risky fund, second year secure fund).

Strategy 5 is a dynamic strategy where in the first year the savings are deposited to the risky fund and the decision in the second year is conditional: if the return in the first year is

No.	r_1	r_2	σ_1	σ_2	$E(M)$	σ_M	$E(M) - \sigma_M$	$E(M) - 2\sigma_M$
1	5%	5%	0%	0%	2.1525	0.00%	2.1525	2.1525
2	10%	10%	10%	10%	2.3100	23.73%	2.0727	1.8354
3	10%	5%	10%	0%	2.2050	10.50%	2.1000	1.9950
4	5%	10%	0%	10%	2.2550	20.50%	2.0500	1.8450
5	10%	cond.	10%	cond.	2.2753	18.73%	2.0880	1.9007

Table 2: Risk - return analysis of different strategies

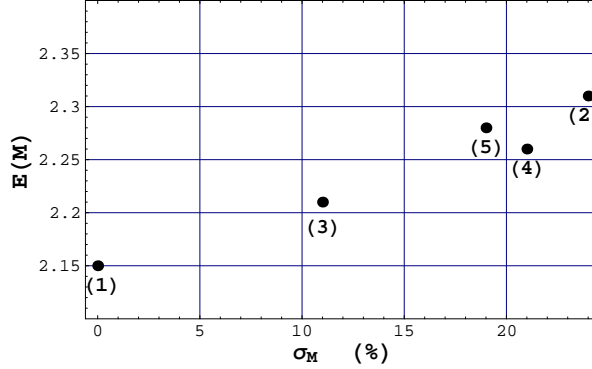


Figure 1: Risk - return analysis of different strategies

more than 15% then the secure fund is chosen in the second period, otherwise the risky fund is chosen again. Comparing to Strategy 4, this strategy is more efficient (see Tab. 2 and Fig. 1). Hence, by using closed-loop strategies, both the risk and return parameters could be improved.

3. The dynamic stochastic programming accumulation model

Suppose that the future pensioner deposits once a year a τ -part of his/her yearly salary w_t to a pension fund $j \in \{1, 2, \dots, m\}$. Denote by s_t , $t = 1, 2, \dots, T$ the accumulated sum at time t where T is the expected retirement time. Then the budget-constraint equations read as follows:

$$\begin{aligned}
 s_{t+1} &= s_t(1 + r_t^j) + w_{t+1}\tau, \quad t = 1, 2, \dots, T - 1, \\
 s_1 &= w_1\tau
 \end{aligned} \tag{2}$$

where r_t^j is the return of the fund j in the time period $[t, t + 1)$. When retiring the pensioner will strive to maintain his/her living standard in the level of the last salary. From this point of view, the saved sum s_T at the time of retirement T is not precisely what the future pensioner cares about. For a given life expectancy, the ratio of the cumulative sum s_T and the yearly salary w_T , i.e. $d_T = s_T/w_T$ is more important. Using the quantity $d_t = s_t/w_t$ one

can reformulate the budget-constraint equation (2):

$$\begin{aligned} d_{t+1} &= F_t(d_t, j), \quad t = 1, 2, \dots, T-1, \\ d_1 &= \tau \end{aligned} \quad (3)$$

where

$$F_t(d, j) = d \frac{1 + r_t^j}{1 + \varrho_t} + \tau, \quad t = 1, 2, \dots, T-1 \quad (4)$$

and ϱ_t denotes the wage growth defined by the equation

$$w_{t+1} = w_t(1 + \varrho_t).$$

Suppose that each year the saver has the possibility to choose a fund $j(t, I_t) \in \{1, 2, \dots, m\}$, where I_t denotes the information set consisted of the history of returns $r_{t'}^j$, $t' = 1, 2, \dots, t-1$, $j \in \{1, 2, \dots, m\}$ and the wage growth $\varrho_{t'}$, $t' = 1, 2, \dots, t-1$. Now suppose that the history of the wage growth ϱ_t , $t = 1, 2, \dots, T-1$ is deterministic and the returns r_t^j are assumed to be random and they are independent for different times $t = 1, 2, \dots, T-1$. Then the only relevant information is the quantity d_t . Hence $j(t, I_t) \equiv j(t, d_t)$. One can formulate a problem of dynamic stochastic programming:

$$\max_j E(U(d_T)) \quad (5)$$

with the following recurrent budget constraint:

$$\begin{aligned} d_{t+1} &= F_t(d_t, j(t, d_t)), \quad t = 1, 2, \dots, T-1, \\ d_1 &= \tau \end{aligned} \quad (6)$$

where the maximum is taken over all non-anticipative strategies $j = j(t, d_t)$. Here U stands for a given preferred utility function of wealth of the saver. Using the law of iterated expectations

$$E(U(d_T)) = E(E(U(d_T)|I_t)) = E(E(U(d_T)|d_t))$$

we conclude that $E(U(d_T)|d_t)$ should be maximal. Let us denote

$$V_t(d) = \max_j E(U(d_T)|d_t = d). \quad (7)$$

Then by using the law of iterated expectations

$$E(U(d_T)|d_t) = E(E(U(d_T)|d_{t+1})|d_t)$$

we obtain the Bellman equation

$$V_t(d) = \max_{j \in \{1, 2, \dots, m\}} E[V_{t+1}(F_t(d, j))] = E[V_{t+1}(F_t(d, j(t, d)))], \quad (8)$$

for $t = 1, 2, \dots, T-1$, where $V_T(d) = U(d)$. Using (8) the optimal feedback strategy $j(t, d_t)$ can be found backwards. This strategy gives the saver the decision for the optimal fund for

each time t and level of savings d_t . Suppose that the stochastic returns r_t^j are represented by their densities f_t^j . Then equation (8) can be rewritten in the form

$$\begin{aligned}
V_t(d) &= \max_{j \in \{1, 2, \dots, m\}} E[V_{t+1}(F_t(d, j))] \\
&= \max_{j \in \{1, 2, \dots, m\}} \int_{\mathbf{R}} V_{t+1} \left(d \frac{1+r}{1+\varrho_t} + \tau \right) f_t^j(r) dr \\
&= \max_{j \in \{1, 2, \dots, m\}} \int_{\mathbf{R}} V_{t+1}(y) f_t^j \left((y - \tau) \frac{1+\varrho_t}{d} - 1 \right) \frac{1+\varrho_t}{d} dy \\
&= \int_{\mathbf{R}} V_{t+1}(y) f_t^{j(t,d)} \left((y - \tau) \frac{1+\varrho_t}{d} - 1 \right) \frac{1+\varrho_t}{d} dy
\end{aligned} \tag{9}$$

where the substitution $y = d(1+r)(1+\varrho_t)^{-1} + \tau$ has been used and \mathbf{R} denotes the set of real numbers.

3.1. The constant relative risk aversion (CRRA) utility function

An important part of the problem (5)-(6) is the choice of the utility function U . The utility function varies across the investors and represents their attitude to the risk. A key role in defining the utility function plays the coefficient of relative risk aversion $C(x) = -xU''(x)/U'(x)$. Constant relative risk aversion implies that people hold a constant proportion of their wealth in any class of risky assets as their wealth varies (see e.g. (Friend & Blume 1975), (Pratt 1964) and (Young 1990)). In this case the utility function is of the form

$$\begin{aligned}
U(x) &= -Ax^{1-C} + B \quad \text{if } C > 1, \\
U(x) &= A \ln(x) + B \quad \text{if } C = 1, \\
U(x) &= Ax^{1-C} + B \quad \text{if } C < 1
\end{aligned} \tag{10}$$

where A, B are constants and $A > 0$. One can easily prove that, concerning the problem (5)-(6), the utility function is invariant to positive affine transformations, i.e. U and $K.U + L$ are equivalent.

In our case constant relative risk aversion implies that the utility functions $U(d)$ and $U(\kappa d)$ where κ is a constant lead to the same strategies. We use the constant relative risk aversion (CRRA) utility function

$$U(d) = \frac{1}{1-a} ((\kappa d)^{1-a} - 1) \tag{11}$$

where $a > 0$ is the constant coefficient of relative risk aversion. Using $\kappa = 1/12$ the utility function is "steeper" for reasonable values and the numerical procedure is more stable. Problem (5)-(6) then maximizes the expected utility of savings (compared to the last yearly salary) corresponding to 1/12 of the yearly benefits (i.e. the benefits for 1 month). It is clear that maximizing monthly benefits or yearly benefits should lead to the same strategy and therefore we can utilize the CRRA utility function.

The coefficient of relative risk aversion a plays an important role in many fields of economics. There is a consensus today, that the value should be less than 10 (see e.g (Mehra

and Prescott 1985)). In our typical results we considered values close to 9. It should be probably lower for lower equity premium. However, our goal was to formulate the mathematical model and to manage the numerical procedure. The reader can change the parameters and calibrate the model in his/her own way.

4. Pension portfolio simulations. Numerical experiments

The purpose of this section is to present results of pension portfolio simulations. The numerical approximation scheme is discussed in Appendix. The output of the numerical code is a matrix of size $(T = 40) \times (k = 200)$ allowing us to "browse" between different years (rows) t and different levels of d (columns). At a given cell of the table we can read the name of fund ($j = 1, \dots, m$) which has to be chosen. Plots of computed output matrices adjusted to the domain $\{(d, t), t \in (0, T), d \in (d_{min}, t/2)\}$ are depicted in this section.

Our results will be summarized in graphical plots of the so-called optimal choice function $j = j(t, d)$ as well as several tables discussing computed results of optimization. The role of the optimal choice function $j = j(t, d)$ is to provide an information when to switch between different funds for a given level of the ratio d of saved money and wage. We focus on two basic questions and problems: 1) what are the regions of constant values of $j(t, d)$; 2) what is the path of expected values of d_t .

Before presenting results of simulation we have to discuss input data such as e.g. fund structures and characteristics, the wage growth ϱ . Concerning the structure of funds we consider the present situation in Slovak Republic According to the adopted government regulation there are three funds (i.e. $m = 3$). Namely, the Growth, Balanced and Conservative fund (see Tab. 1). Hereafter, we shall suppose that these three funds are constructed from stocks (S) and secure bonds (B) where stocks are represented by S&P Poor's Index (Jan 1996 - June 2002) with average return $r_s = 0.1028$ and standard deviation $\sigma_s = 0.1690$ whereas the secure bonds are represented by 10 years US government bonds (Jan 1996 - June 2002) with the average return $r_b = 0.0516$ and standard deviation $\sigma_b = 0.0082$.¹ Using the historical data, the estimate of correlation between stocks and bonds is -0.1151. Stochastic asset returns are assumed to have normal distributions.²

We shall suppose that the structure of funds ($F_1 =$ Growth fund, $F_2 =$ Balanced fund, $F_3 =$ Conservative fund) of the second pension pillar in Slovak Republic is as follows:

$$\begin{aligned} F_1 &= 0.8 \times S + 0.2 \times B \\ F_2 &= 0.5 \times S + 0.5 \times B \\ F_3 &= B \end{aligned} \tag{12}$$

Both returns r_i and standard deviations $\sigma_i, i = 1, 2, 3$, of the above funds can be easily calculated from returns r_s, r_b , standard deviations σ_s, σ_b and the estimated correlation (see

¹We considered the estimated asset returns only for illustration of the model capability. We do not have an ambition to estimate future asset returns. However, in Section 4.3 we present a sensitivity analysis for different asset returns.

²The normal distribution is a simplification. There is a well known empirical evidence that stock returns exhibit asymmetry and heavy tails. However, the model presented in Section 3 allows different distributions.

Fund	Return	StdDev
F ₁	$r_1 = 0.0926$	$\sigma_1 = 0.1350$
F ₂	$r_2 = 0.0772$	$\sigma_2 = 0.0841$
F ₃	$r_3 = 0.0516$	$\sigma_3 = 0.0082$

Period	wage growth ($1 + \varrho_t$)
2006-2008	1.075
2009-2014	1.070
2015-2021	1.065
2022-2024	1.060
2025-2050	1.050

Table 3: Data used for computation. Fund returns and their standard deviations (left), expected wage growth for the period 2006–2050 (right).

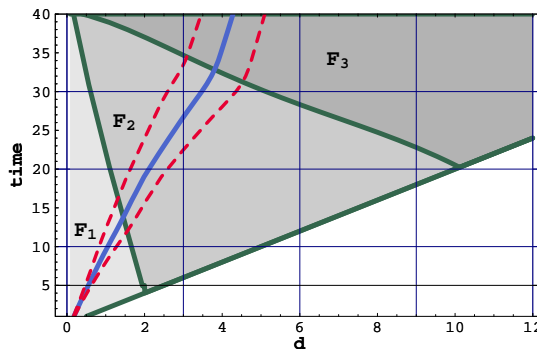


Figure 2: Regions of optimal choice and the path of average saved money to wage ratio ($a = 9$).

Tab. 3).

According to Slovak legislature the percentage of salary transferred each year to a pension fund is 9%. The law sets administrative costs of the second pillar at 1% of monthly contribution and 0.07% of the monthly asset value (i.e. 0.84% p.a.). Therefore, we considered effective contributions $\tau = 8.91\%$ ($= 9\% * 0.99$). The value 0.84% was subtracted from the asset returns in Tab. 3. We assumed the period of saving to be $T = 40$ years. The data for the expected wage growth ϱ are taken from Slovak Savings Bank (SLSP).³ The values are shown in Tab. 3.

4.1. Description of computed results and simulations

In Fig. 2 we present a typical result of our analysis with the coefficient of proportional risk aversion $a = 9$. It contains three distinct regions in the (d, t) plane determining the optimal choice $j = j(d, t)$ of a fund depending on time $t \in [1, T - 1]$ and the average saved money to wage ratio $d \in [d_{min}, d_{max}^t]$. For practical purposes we chose⁴ $d_{min} = 0.0891$ and $d_{max}^t = t/2$ for $t \geq 1$. In each year $t = 1, \dots, T - 1$ we invest the saved amount of money s_t uniquely corresponding with d_t to one of the funds $j = 1, 2, 3$ depending on the computed optimal

³The data were provided by the analysts of SLSP: Martin Barto and Juraj Kotian

⁴ $d_{min} = 0.0891$ because 8.91% is the effective 2nd pillar contribution rate.

mean	switch	switch
$E(d_T)$	$F_1 - F_2$	$F_2 - F_3$
4.28	14 (12-16)	33 (32-35)

Table 4: Summary of computation of the mean saved money to wage ratio d_T and switching times ($a = 9$).

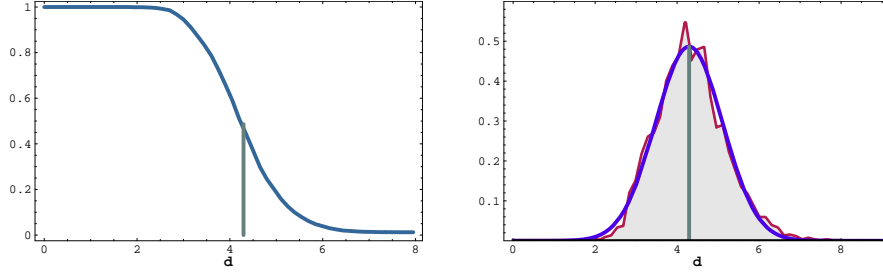


Figure 3: Cumulative distribution function $1 - F$ (left). Histogram of simulations and density function (right). Sample mean $E(d_T) = 4.28$ and standard deviation of $d_T = 0.82$

value $j = j(d, t)$. In the first year of saving we take $d_1 = d_{min}$.

The curvilinear solid line in Fig. 2 represents the path of the mean wealth $E(d_t)$, obtained by 10,000 simulations and here we use $a = 9$. Notice that, for $t > 1$, the ratio d_t is a random variable depending on (in our case normally distributed) random returns of the funds and on the computed optimal fund choice matrix $j(d, t'), t' < t$. The dashed curvilinear lines correspond to $E(d_t) \pm \sigma_t$ intervals where σ_t is the standard deviation of the random variable d_t .

In Tab. 4 we present the mean final wealth $E(d_T)$ as well as the so-called *switching-times* for mean path $E(d_t), t \in [1, T - 1]$, and the intervals (in brackets) of switching times for one standard deviation of the mean path. The normalized histogram resp. distribution function of the simulated final wealth is very similar to a normal distribution, as can be seen in Fig. 3.

In the next sections we pay our attention to the sensitivity of results when some parameters are changing.

4.2. Sensitivity analysis for varying risk aversions

Let us consider different risk aversion parameters a in the utility function: $a = 3, 5, 8, 9, 10$. It should be obvious that increasing risk aversion leads to a choice of less risky fund. Indeed, based on our computations, one can observe that increasing a (increasing risk aversion) causes that the switches between funds are shifted to an earlier time, i.e. we switch from F_1 to F_2 sooner, as well as from F_2 to F_3 . Obviously, for higher values of the risk aversion parameter a we obtain lower levels of the final wealth. Results for the experiments are displayed in Fig. 2, Fig. 4, Tab. 4 and Tab. 5.

The relation between different values of risk aversion parameter $a \in (0, 25)$ and the final

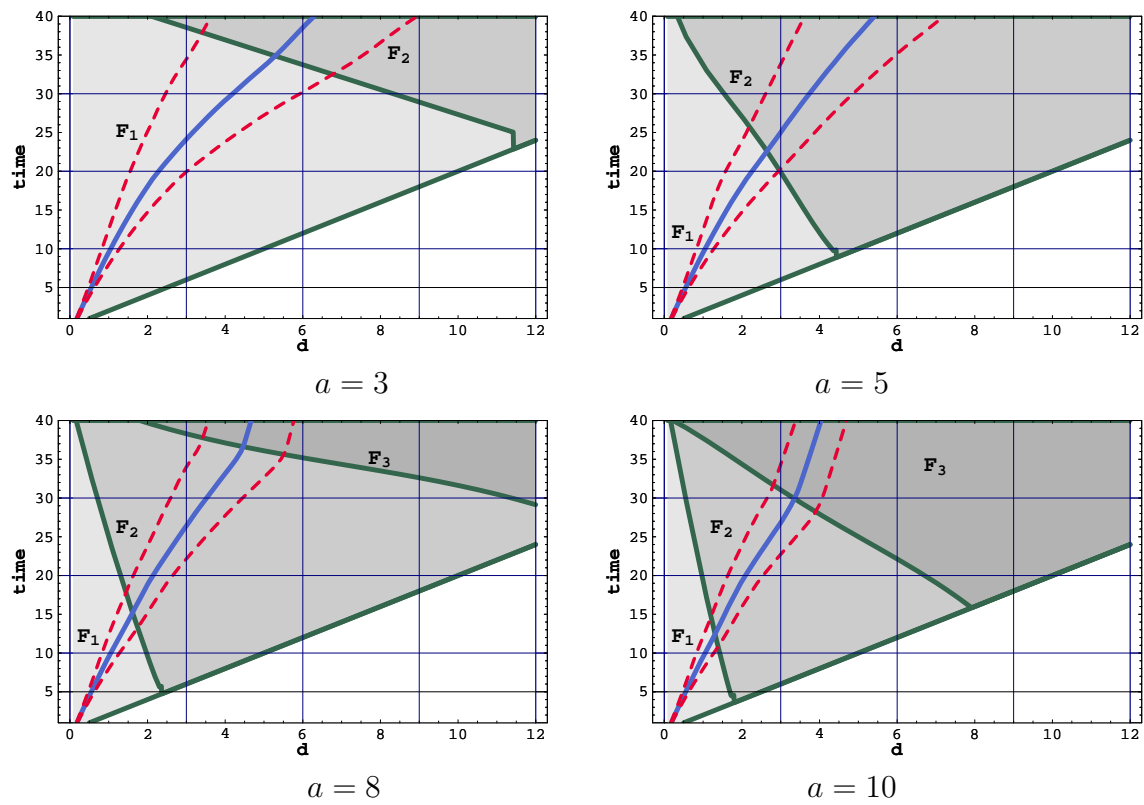


Figure 4: Sensitivity of regions of optimal choice with respect to different risk aversion values of the parameter $a = 3, 5, 8, 10$.

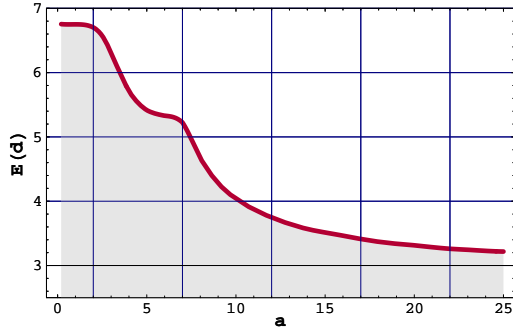


Figure 5: Relation between the risk aversion parameter a and the level of mean saved wealth $E(d_T)$.

a	mean $E(d_T)$	switch F_1-F_2	switch F_2-F_3
3	6.29	35 (32-38)	never
5	5.42	23 (20-26)	never
8	4.67	15 (13-18)	37 (36-38)
10	4.04	13 (11-15)	30 (29-32)

Table 5: Results for fixed wage growths, fixed returns and standard deviations (see values in Tab. 3), different risk aversion parameter a .

mean wealth to last wage ratio is shown in Fig. 5. We can see that the curve can be divided into three segments where the kinks separate ranges of the parameter a for which there are *no* switches, *one* switch, and *two* switches between funds in the optimal strategy.

One can see that results partially in accord with law regulations are reached for $a = 9$. This value is relatively high (see e.g. Mehra and Prescott, 1985). In the next section we show that the results are highly sensitive to asset returns and for lower stock returns the "typical" value of a should be lower.

4.3. Sensitivity analysis for various stock and bond returns

Now, let us examine the impact of the change in returns of funds on the optimal strategy and results. One can expect that if for example the return of stocks becomes higher, it will be more favorable to "stay" in F_1 or F_2 for a longer period. In our computations, we first fix the bond return and increase/decrease the stock return ($a = 9$ and other parameters are fixed). This change mirrors in the returns of the funds F_1 and F_2 . The results obtained show that a higher return of stocks implies a later switch from more risky to less risky funds. The wealth in the final period of savings is higher too. Secondly, we fix the stock return and increase/decrease the bond return. A higher return of bonds implies an earlier switch from

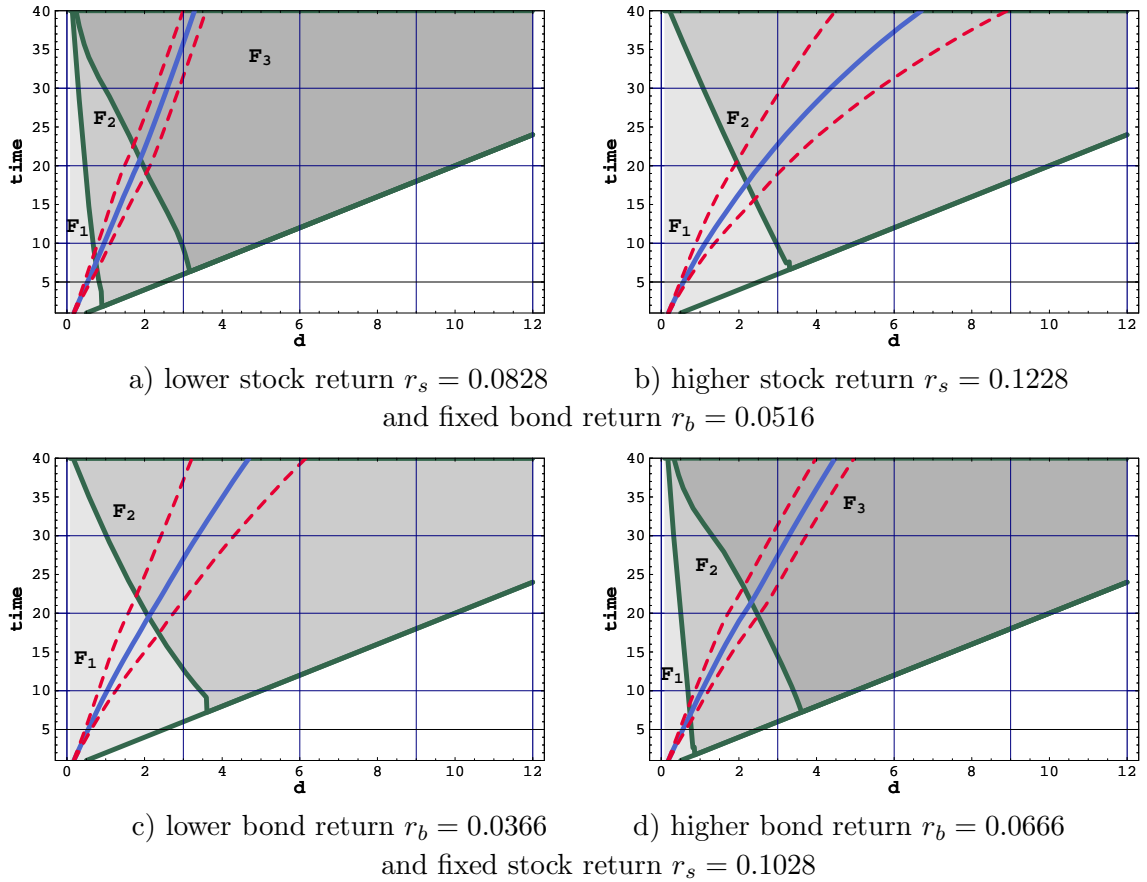


Figure 6: Sensitivity of regions of optimal choice for various expected values of stock and bond returns.

more risky to less risky funds.⁵ For overview of all results see Fig. 6 and Tab. 6.

4.4. Sensitivity analysis with respect to varying wages

Finally, we consider different wage growth rates. The intuition says that one can expect lower saved money to wage ratio d_t for higher wage growth ϱ .⁶ To examine the influence of this parameter on results, we considered the wage growth being raised (uniformly for all time periods) and lowered by 1 percentage point. We denote by $\varrho^{(+1pp)}$ ($\varrho^{(-1pp)}$) the wage growth development derived from Tab. 3 where ϱ_t has been increased by 1 pp (decreased by 1 pp) for each of five periods in Tab. 3. As we can see in Fig. 7 and Tab. 7, a higher wage growth leads to a lower wealth to last wage ratio, guided by shifting the switch-times

⁵We have merely varied means of returns distributions up or down, but kept standard deviations unchanged. As a result, the coefficient of variation (= standard deviation/mean) changes. If riskiness changed proportionately with returns, the results would differ much less.

⁶Although this increases the contributions (contribution rate τ unchanged), there is a steeper wage profile and hence lower savings to last wage ratio.

Stock & Bond returns	Fund returns	mean $E(d_T)$	switch F_1-F_2	switch F_2-F_3
$r_s = 0.1028$ $r_b = 0.0516$	$r_1 = 0.0926$ $r_2 = 0.0772$ $r_3 = 0.0516$	4.28	14 (12-16)	33 (32-35)
$r_s = 0.0828$ $r_b = 0.0516$	$r_1 = 0.0766$ $r_2 = 0.0672$ $r_3 = 0.0516$	3.29	8 (7-9)	21 (19-23)
$r_s = 0.1228$ $r_b = 0.0516$	$r_1 = 0.1086$ $r_2 = 0.0872$ $r_3 = 0.0516$	6.70	18 (16-20)	never
$r_s = 0.1028$ $r_b = 0.0366$	$r_1 = 0.0896$ $r_2 = 0.0697$ $r_3 = 0.0366$	4.69	19 (17-22)	never
$r_s = 0.1028$ $r_b = 0.0666$	$r_1 = 0.0956$ $r_2 = 0.0847$ $r_3 = 0.0666$	4.48	7 (6-8)	22 (20-24)

Table 6: Results for fixed wage growths, fixed $a = 9$, fixed standard deviations $\sigma_1 = 0.1350, \sigma_2 = 0.0841, \sigma_3 = 0.0082$, and various bond and stocks returns r_b and r_s , resp.

wage growth	mean $E(d_T)$	switch F_1-F_2	switch F_2-F_3
$\varrho^{(-1pp)}$	5.10	12 (10-14)	32 (30-34)
ϱ	4.28	14 (12-16)	33 (32-34)
$\varrho^{(+1pp)}$	3.63	16 (14-18)	34 (33-36)

Table 7: Results for fixed returns and standard deviations (see values in Tab. 3), fixed $a = 9$, and different wage grow rates.

to later moments.

4.5. The comparison of dynamic and static strategies

One can think about static strategies where the time instants where a contributor switches between the funds are determined at the beginning of the saving. The most risk averse contributor deposits the savings all the time to the Conservative Fund. The least risk averse investor contributes to the risky funds as long as the law permits it: in the first 25 years of saving to the Growth Fund (the total period of saving 40 years supposed), next 8 years to the Balanced Fund and the last 7 years to the Conservative Fund.

To compare the performance of dynamic and static strategies we have chosen two representatives of the static ones:

1. The most risky (accepting the law regulations) strategy with switching times 25 ($F_1 -$

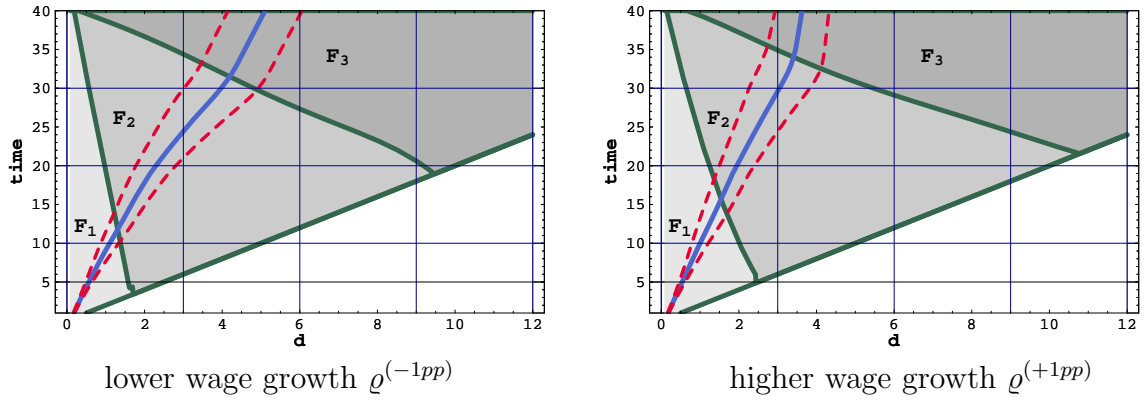


Figure 7: Sensitivity of regions of optimal choice for various wage growth ρ scenarios.

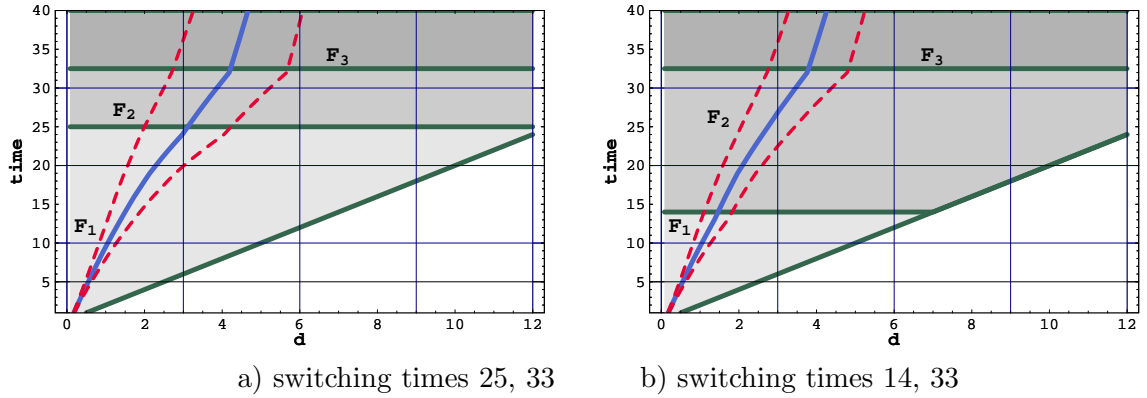


Figure 8: Static strategies - the development of savings.

F_2) and 33 ($F_2 - F_3$).

2. The strategy with switching times 14 and 33 similar to a typical representative of dynamic strategies with the risk aversion parameter $a = 9$.

In Fig. 8 one can see the average d_t development and $E(d_t) \pm \sigma_t$ intervals for chosen static strategies. The strategy with switching times 14 and 33 has the same $E(d_T) = 4.67$ comparing to a dynamic one with $a = 8$ but significantly higher the standard deviation of d_T , $\sigma_T = 1.41$ (comparing to 1.10 for the dynamic strategy with $a = 8$). A mean-variance analysis of dynamic strategies with different risk aversion (represented by the curve - efficient frontier) and the two static ones is depicted in Fig. 9. The static strategies are clearly inefficient.

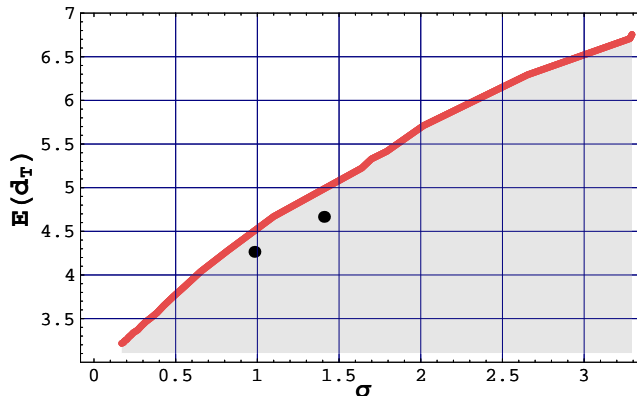


Figure 9: Mean-variance analysis of dynamic and strategic strategies .

5. Conclusions

We have presented a dynamic accumulation model for determining optimal switching strategies for choosing pension funds with different risk profiles. It turned out that dynamic strategies could be more efficient comparing to static ones. The results of simulations of a mathematical model have illustrated that gradual decreasing of the risk (incorporated in the corresponding legislation) is reasonable and can be supported by means of a dynamic accumulation model. The resulting strategies depend on individual risk preferences of the future pensioners represented by their individual utility functions. In accord with common intuition, a higher wage growth implies lower performance of the funded pillar relative to the pay-as-you-go pillar. Since it is very difficult to predict the future asset returns, the results were calculated for various means of asset returns distributions.

Appendix: Numerical approximation scheme

In this section we discuss a numerical approximation scheme we used in our pension portfolio simulations. The principal difficulty in computing the Bellman integral (9) is due to significant oscillations in the integrand function. More precisely, it may attain both large values as well as low values of the order one. Therefore a scaling technique is needed when computing the integral (9). The idea of scaling is rather standard and is widely used in similar circumstances.

Let $H_t(d)$ be any bounded positive function for $t = 1, 2, \dots, T$. We scale the function V_t by H_t , i.e. we define a new auxiliary function

$$W_t(d) = H_t(d)V_t(d).$$

Clearly, the original function $V_t(d)$ can be easily calculated from $W_t(d)$ as follows: $V_t(d) =$

$W_t(d)/H_t(d)$. Then, for each time step t from $t = T$ down to $t = 2$ we have

$$\begin{aligned}
W_T(d) &= H_T(d)V_T(d) \quad \text{and} \\
W_{t-1}(d) &= H_{t-1}(d)V_{t-1}(d) \\
&= \max_{j \in \{1,2,\dots,m\}} \int_{\mathbf{R}} H_{t-1}(d)V_t \left(\frac{d}{1+\varrho_t}(1+r) + \tau \right) f_t^j(r) dr \\
&= \max_{j \in \{1,2,\dots,m\}} \int_{\mathbf{R}} \frac{H_{t-1}(d)W_t \left(\frac{d}{1+\varrho_t}(1+r) + \tau \right)}{H_t \left(\frac{d}{1+\varrho_t}(1+r) + \tau \right)} f_t^j(r) dr \\
&= \max_{j \in \{1,2,\dots,m\}} \int_{\mathbf{R}} \frac{H_{t-1}(d)W_t(y)}{H_t(y)} f_t^j \left((y-\tau) \frac{1+\varrho_t}{d} - 1 \right) \frac{1+\varrho_t}{d} dy.
\end{aligned} \tag{13}$$

It is worthwhile noting that any choice of the family $H_t, t = 1, \dots, T$, of positive bounded scaling function does not change the result. It may however significantly improve the stability of numerical computation.

In order to capture both large and small values of V_t we recursively define the scaling functions $H_t, t = T, T-1, \dots, 2, 1$, depending on the previously computed solution V_{t+1} as follows:

$$H_T = \frac{1}{\sqrt{1+V_T^2}}, \quad \text{and} \quad H_t = \frac{1}{\sqrt{1+V_{t+1}^2}} \quad \text{for } t = T-1, \dots, 1. \tag{14}$$

In our algorithm we compute values of the function $W_t = W_t(d)$ for discrete values of d from the time dependent interval $d \in (d_{min}, t/2)$, where we use $d_{min} = 0.0891$. The upper bound $t/2$ has been chosen with respect to maximal expected values of the savings to yearly salary ratio d . In each time level $t = T$ down to $t = 1$ we choose a uniform spatial discretization of the interval $(d_{min}, t/2)$ consisting of $k = 200$ mesh points. Stochastic fund returns r_t^j were assumed to have normal distributions with densities f_t^j having constant in-time means \bar{r}^j and standard deviations $\sigma^j, j = 1, \dots, m$. In order to compute numerically the Bellman type integral with normal distribution densities f_t^j we used the Simpson rule with 11 equidistant grid points covering the essential interval $(\bar{r}^j - \sigma^j, \bar{r}^j + \sigma^j)$.

Acknowledgment: We are indebted to Prof. P. Brunovský for his valuable comments and ideas that significantly improved the quality of this paper.

References

- [1] BENCZÚR, P. (1999): *Changes in the Implicit Debt Burden of the Hungarian Social Security*. National Bank of Hungary, Working Paper (August 1999).
- [2] CHLON, A. – GÓRA, M. – RUTKOWSKI, M. (1999): *Shaping Pension Reform in Poland: Security Through Diversity*. Social Protection Discussion Paper Series, The World Bank, (August 1999).

- [3] FRIEND, I. – BLUME, M.E. (1975): *The Demand for Risky Assets*. The American Economic Review, vol. 65, No. 5, pp. 900–922.
- [4] FULTZ, E. (2002): *Pension Reform in Central and Eastern Europe*. Vol. 1, Budapest, ILO.
- [5] GOLIAŠ, P. (2003): *Pension Calculations for the PAYG and the Funded Pension System in Slovakia*. Academia Istropolitana Nova, Professional Programme in Applied Economics and Finance, (August 2003).
- [6] MEHRA, R. – PRESCOTT, E. (1985): *The Equity Premium: a Puzzle*. Journal of Monetary Economics, vol. 15, pp. 145–161.
- [7] MELICHERČÍK, I. – UNGVARSKÝ, C. (2004): *Pension Reform in Slovakia: Perspectives of the Fiscal Debt and Pension Level*. Finance a úvér, vol. 9-10, pp. 391–404.
- [8] PALACIOS, R. – ROCHA, R. (1998): *The Hungarian Pension System in Transition*. In: L. Bokros and J. Dethier (eds.): *Public Finance Reform during the Transition: The Experience of Hungary*, The World Bank Washington, D.C., 1998, pp. 177–216.
- [9] PRATT, J.W. (1964): *Risk Aversion in the Small and in the Large*. Econometrica, vol. 32, No. 1-2, pp. 122–136.
- [10] SIMONOVITS, A. (2000): *Partial Privatization of a Pension System: Lessons from Hungary*. International Journal of Development, vol. 12, pp. 519–529.
- [11] THOMAY, M. – ŠVEJNA, I. – ORAVEC, J. *Koncepcia reformy dôchodkoého systému*. The F. A. Hayek Foundation, Bratislava, (June 2002).
- [12] YOUNG, H.P. (1990): *Progressive Taxation and Equal Sacrifice*. The American Economic Review, vol. 80, No. 1, pp. 253–266.