Some useful optimization problems in portfolio theory

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Abstract. Some mathematical methods contemporary used in portfolio management are presented. Among the oldest and still used counts the one-period problem formulated by Markowitz ([3]). The problem is presented with its extension concerning transaction costs. It could be considered in a broader framework of one-period models maximizing expected utility of the wealth at a defined time horizon. Recently, an interest in the development of multi-period models of portfolio management has been observed. These models suppose portfolio rearrangement before the time horizon according to the development of asset prices. They lead to problems of dynamic stochastic programming. In addition to a general principle of dynamic portfolio management, the specific problem concerning an optimal portfolio composition of a saver in the second pillar of the Slovak pension system is presented.

Keywords: Portfolio management, mean-variance approach, utility function, multi-period models, pension system in Slovakia

1 Introduction

Financial institutions face the problem of optimal portfolio decisions under uncertainty. The mathematical framework for optimizing the portfolio decisions could be found in several models.

One important class of models represent single period models based on the idea of mean-variance optimization of Markowitz ([3]). These models quantify the risk associated with uncertain portfolio returns by the variance of the final wealth. Mathematically they lead to problems of quadratic programming. Mean-variance models belong to a broader class of models maximizing the expected utility. The utility function is different for each investor and explains better the balance of returns and risks due to personal preferences than the expected return of the portfolio which is the input of the mean-variance approach.

In reality asset managers think dynamically. They often rearrange managed portfolios using a new information coming from finacial markets. The idea of future rearrangement leads to dynamic asset allocation. In dynamic models a decision rule indicates precisely how a portfolio has to be altered as a function of a new information (e.g. realized asset returns). The decision takes into account possible future asset returns and future rebalancing according to a new information. Mathematically this approach leads to problems of dynamic stochastic programming.

Paper is organized as follows. Section 2 explains mean-variance and utility approach used in one-period models. In Section 3 we present a general principle of dynamic stochastic programming used in portfolio management. Section 4 contains a dynamic model for pension savings management in the funded pillar of the Slovak pension system. Paper is concluded with recommendation for further reading.

2 One-period optimization

2.1 Mean-variance approach

Among the oldest and still used models of portfolio optimization counts the one-period problem formulated by Markowitz. The optimal portfolio is characterized by first two moments of the distribution of the end period wealth. The output of this approach is the set of *efficient portfolios*. The efficient portfolio is defined as follows: given a defined level of the expected wealth $\bar{W}$ at the end of the optimization period we choose the portfolio with minimal risk quantified by the variance of the end-period wealth. The corresponding problem of quadratic programming is

$$\begin{aligned} &\min_x x^\top V x \\ \text{s.t.}\ \ & x^\top (\mathbf{1} + \bar{r}) = \bar{W}\,, \\ & x^\top \mathbf{1} = W_{ini} \end{aligned} \tag{1}$$

where $\bar{r}$ is the vector of expected returns, V is the covariance matrix of the returns, x is composition of the initial portfolio, $\mathbf{1} = (1, 1, \ldots, 1)^\top$ and W_{ini} is the initial wealth. Denote by $u = \frac{x}{W_{ini}}$ the weights of assets in the initial portolio and $\bar{r}_p = \frac{\bar{W}}{W_{ini}} - 1$ the expected return of the porfolio. One can prove easily that (1) can be formulated equivalently as

$$\begin{aligned} &\min_u u^\top V u \\ \text{s.t.}\ \ & u^\top \bar{r} = \bar{r}_p\,, \\ & u^\top \mathbf{1} = 1\,. \end{aligned} \tag{2}$$

The advantage of formulation (2) is that it does not depend on the level W_{ini} of the initial wealth. Suppose that the covariance matrix V is regular (i.e. the asset returns are linearly independent) and there exist two assets i, j with different expected returns $\bar{r}_i \neq \bar{r}_j$. In this case one can calculate a unique solution of (2). Denote

$A = \mathbf{1}^\top V^{-1} \bar{r} = \bar{r}^\top V^{-1} \mathbf{1}\,,$

$B = \bar{r}^\top V^{-1}\bar{r} > 0\,,$
$C = \mathbf{1}^\top V^{-1}\mathbf{1} > 0\,,$
$D = BC - A^2.$

One can easily calculate (see [4] for details) optimal weights u_p:

$$u_p = g + h\bar{r}_p \tag{3}$$

where

$$g = \frac{1}{D}[B(V^{-1}\mathbf{1}) - A(V^{-1}\bar{r})]\,,$$
$$h = \frac{1}{D}[C(V^{-1}\bar{r}) - A(V^{-1}\mathbf{1})]\,.$$

Variance of the optimal portfolio could be calculated as

$$\begin{aligned}\mathrm{Var}(r_p) &= w_p^\top V w_p = (g + h\bar{r}_p)^\top V (g + h\bar{r}_p)\\ &= g^\top V g + (h^\top V g)\bar{r}_p + (g^\top V h)\bar{r}_p + (h^\top V h)\bar{r}_p^2\,.\end{aligned} \tag{4}$$

One can see that $\mathrm{Var}(r_p)$ is a quadratic function of $\bar{r}_p$. The set of optimal portfolios is illustrated in Fig. 1. It is obvious, that efficient are only portfolios from the "upper" part of the curve which is called *Efficient frontier*. It is worth to note, that in practical applications short positions ($u_i < 0$ for some i) are forbidden. Therefore, the constraints $u_i \geq 0$, $i = 1, 2, \ldots n$ are added. In this case we still have a problem of quadratic programming, but we loose the explicit solution.

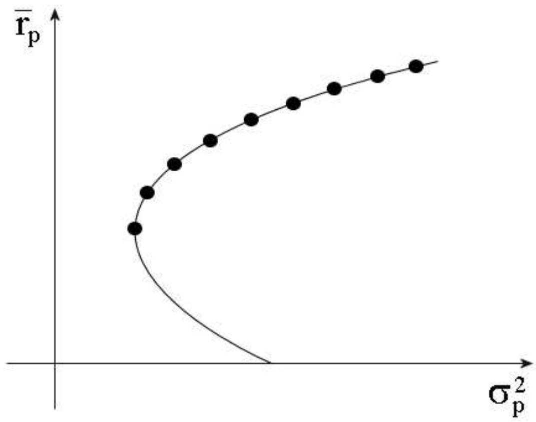

Fig. 1. Efficient frontier.

Problem (1) (and equivalent formulation (2)) ignores transaction costs associated with buying and selling assets. Considering the transaction costs we cannot ignore the composition of initial portfolio. Denote by x_{ini} the composition of initial portfolio. The portfolio consists of cash $x_{ini,0}$ and risky assets $x_{ini,1}, \ldots, x_{ini,n}$. The cash is supposed to have risk-free return r_0. The assets $i = 1, 2, \ldots, n$ have risky returns with means $\bar{r} = (\bar{r}_1, \ldots, \bar{r}_n)^\top$ and covariance matrix V. Formulation (1) could be extended to the case with transaction costs as follows:

$$
\begin{aligned}
& \min_{x, v^+, v^-} \sum_{i,j=1}^{n} x_i x_j V_{ij} \\
& \text{s.t. } x_0(1 + r_0) + \sum_{i=1}^{n} x_i(1 + \bar{r}_i) = \bar{W} \,, \\
& x_{ini,i} + v_i^+ - v_i^- = x_i \,,\ i = 1, 2, \ldots, n \,, \\
& x_{ini,0} - \sum_{i=1}^{n} (1 + d_i) v_i^+ + \sum_{i=1}^{n} (1 - c_i) v_i^- = x_0 \,, \\
& v^+, v^- \geq 0
\end{aligned} \tag{5}
$$

where v^+, v^- represent the value of bought and sold risky assets respectively, d and c the proportional transaction costs associated with buying and selling. In the case of forbidden short positions, the constraints $x_i \geq 0$, $i = 1, 2, \ldots, n$ have to be added. One can observe that (5) is (alike (1)) a problem of quadratic programming.

2.2 Utility based approach

In the mean-variance approach risk preferences are given through the expected return $\bar{r}_p$ of the portfolio. The higher $\bar{r}_p$ the lower the aversion to risk. An alternative approach widely used in finance is the one based on utility function. In this approach investor's preferences are given through the utility function U. Optimal portfolio is the one which maximizes the expected utility of the final wealth W through all considered strategies:

$$
\begin{aligned}
& \max_u E(U(W)) \\
& W = F(W_{ini}, u) \,.
\end{aligned} \tag{6}
$$

Here W is a random variable representing final wealth at the end of the period depending on the initial welath W_{ini} at the beginning of the period and a trading strategy u.

The utility function U is usually different for different investors. It represents the investor's aversion to risk. One can prove that for risk averse investor the utility function U has to be increasing and concave (see e.g. [4]). Widely used is a standard class of utility functions with constant coefficient of relative risk aversion (CRRA functions) $C = -xU''(x)/U'(x)$. In this case the utility function is of the form

$$U(x) = -Ax^{1-C} + B \quad \text{if } C > 1,$$
$$U(x) = A\ln(x) + B \quad \text{if } C = 1\,,$$
$$U(x) = Ax^{1-C} + B \quad \text{if } C < 1 \tag{7}$$

where A, B are constants and $A > 0$. One can easily prove that, concerning the problem (6), the utility function is invariant to positive affine transformations, i.e. U and $K.U + L$ are equivalent. It is worth to note that in the case of CRRA functions the solution of (6) does not depend on the level of initial wealth W_{ini}.

Example 1. Consider a situation where investor has a choice of m funds with random returns $r_j, j = 1, 2, \ldots, m$. He /she wants to invest the initial wealth W_{ini} to one of the funds. Using the utility based approach the investor solves the problem

$$\max_{j\in\{1,2,\ldots,m\}} E(U(W_{ini}(1 + r_j)))\,.$$

The values $E(U(W_{ini}(1+r_j)))$ have to be calculated for all funds $j \in \{1, 2, \ldots, m\}$. The solution is the fund for which this value is maximal.

Example 2. As an another example suppose that investor has a choice between a risk-free asset with value e^{rt} for $t \geq 0$, where r is the risk-free rate and a risky asset with value satisfying a stochastic differential equation

$$dS_t/S_t = \mu\, dt + \sigma\, dB_t$$

where μ and $\sigma > 0$ are constants representing the drift and the volatility of the asset and B_t is a standard Brownian motion (Wiener process). Consider a class of strategies with constant proportion $0 \leq u$ of the risky asset (i.e. proportion $(1 - u)$ is hold in the risk-free asset) in the whole period $[0, T]$. Denote by W_t value of the porfolio at time $0 \leq t \leq T$. One can calculate

$$dW_t/W_t = (r + u(\mu - r))\, dt + u\sigma\, dB_t\,.$$

Using Itô's lema we have

$$W_t = W_0 \exp((r + u(\mu - r) - \frac{1}{2}u^2\sigma^2)t + u\sigma B_t)\,.$$

Suppose that investor's preferences are represented by a CRRA utility function with coefficient of relative risk aversion $C > 1$

$$U(W) = \frac{W^{1-C}}{(1 - C)}\,.$$

Since B_T has a normal distribution with zero mean and variance T one has

$$E(U(W_T)) = \frac{P_0^{1-C}}{1 - C} \exp((1-C)(r+u(\mu-r)-\frac{1}{2}u^2\sigma^2)t+\frac{1}{2}(1-C)^2u^2\sigma^2 t)\,. \tag{8}$$

Folowing standard calculations we have that (8) is maximal for

$$u = \frac{\mu - r}{C\sigma^2}\,.$$

2.3 Relation between mean-variance and utility based approaches

Recall the mean-variance problem (1). Using the utility approach one can formulate a similar problem:

$$\max_x E(U(\sum_{i=1}^{n} x_i(1+r_i)))$$
$$\text{s.t.} \;\; x^\top \mathbf{1} = W_{ini}\,. \tag{9}$$

Note that (9) does not contain expected return $\bar{r}_p$ of the portfolio. The risk preferences are included in the utility function U which is specific for a conrete investor. The question is: What is the relation between the mean-variance and utility approaches? The answer is that they are not in contradiction in two cases:

- the returns are normally distruted or
- the utility function U is quadratic.

We shall present only basic idea of the proof. Take a Taylor series of the utility function in $\bar{W}$ (expected value of the wealth W):

$$U(W) = U(\bar{W}) + U'(\bar{W})(W - \bar{W}) + \frac{1}{2}U''(\bar{W})(W - \bar{W})^2 + R^3$$

where R^3 are terms of degree more than 2. Using this we have

$$E(U(W)) = U(\bar{W}) + \frac{1}{2}U''(\bar{W})Var(W) + E(R^3)\,.$$

If we neglect $E(R^3)$, for fixed $\bar{W}$ variance of W should be minimal. This implies that mean-variance and utility approaches are not in contradiction. For quadratic utility function $R^3 = 0$. The idea of proof for normal distribution of returns is that all higher moments of the normal distribution could be calculated from first and second ones. The quadratic utility function is not increasing, which is in contradiction with the basic property of utility functions. In reality the quadratic utility function could be used if realistic values are from the region, where U is increasing.

Fig. 2 illustrates the relation between mean-variance and utility based approaches. Indifferent curves represent portfolios with the same expected utility $E(U(W))$. Curves disjoint with efficient frontier represent unreachable portfolios (with given W_{ini} and random returns r_i). The common solution of mean-variance and utility based approach is the intersection of efficient frontier and tangent curve.

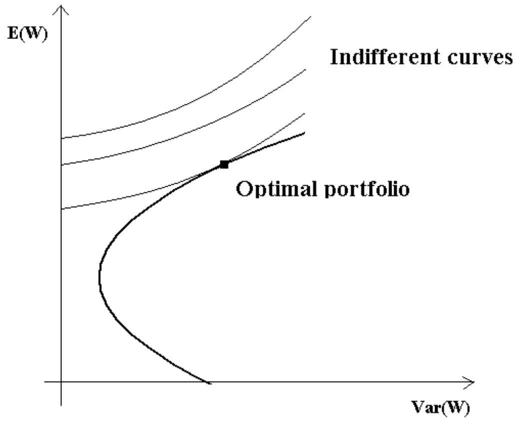

Fig. 2. Efficient frontier with indifferent curves.

One can also formulate an equivalent of mean-variance approach with transaction costs (5) in the utility framework:

$$
\begin{aligned}
\max_{x,v^+,v^-} \; & E(U(\sum_{i=0}^{n} x_i(1+r_i))) \\
& x_{ini,i} + v_i^+ - v_i^- = x_i \,,\; i = 1,2,\ldots,n \,, \\
x_{ini,0} - \sum_{i=1}^{n}(1+d_i)v_i^+ + & \sum_{i=1}^{n}(1-c_i)v_i^- = x_0 \,, \\
& v^+, v^- \geq 0 \,.
\end{aligned} \tag{10}
$$

3 Multi-period asset allocation

In previous sections one-period models of portfolio management have been considered. In these models the decision (asset allocation) is done at the beginning of the period and no future corrections are possible. However, in reality asset managers typically suppose that the decision could be corrected in the future using a new information from financial markets.

A model with future portfolio rebalancing could look as follows. Consider a portfolio management problem with n assets. A manager is rebalancing (without transaction costs) a portfolio with initial value W_{ini} at times $t = 0, 1, \ldots, T-1$. At each time $t = 0, 1, \ldots, T-1$ the manager applies a decision $u = u_t(I_t)$ depending on information I_t at time t. Assets returns are supposed to be random. Therefore, the wealth W_{t+1} is a random variable depending on the wealth W_t and the decision u_t: $W_{t+1} = F_t(W_t, u_t)$. The goal is to maximize the expected utility of the wealth at time horizon T. Mathematically the problem reads as follows:

$$\max_u E(U(W_T))$$
$$W_{t+1} = F_t(W_t, u_t)\,,\; t = 0, 1, \ldots, T-1\,. \tag{11}$$

One can prove that if the returns r_t are independent for different times $t = 0, 1, \ldots, T-1$, then the only information relevant at time t is the wealth W_t. Therefore, in this case $I_t \equiv W_t$.

The optimal strategy u_t is a solution of the Bellman equation. Using the law of iterated expectations

$$E(U(W_T)) = E(E(U(W_T)|I_t)) = E(E(U(W_T)|W_t))$$

we conclude that $E(U(W_T)|W_t)$ should be maximal. Let us denote

$$V_t(W) = \max_{u_t, u_{t+1}, \ldots, u_{T-1}} E(U(W_T)|W_t = W)\,. \tag{12}$$

Then by using the law of iterated expectations

$$E(U(W_T)|W_t) = E(E(U(W_T)|W_{t+1})|W_t)$$

we obtain the Bellman equation

$$V_t(W) = \max_{u_t} E[V_{t+1}(W_{t+1})|W_t = W] = \max_{u_t} E[V_{t+1}(F_t(W, u_t))]\,, \tag{13}$$

for $t = 0, 1, \ldots, T-1$, where $V_T(W) = U(W)$. Using (13) the optimal feedback strategy u can be found backwards. For complete calculations the distribution of returns r_t should be given. Next section contains a concrete application.

4 Dynamic accumulation model for the second pillar of the Slovak pension system

Suppose that a future pensioner deposits once a year a τ-part of his/her yearly salary G_t to a pension fund $j \in \{1, 2, \ldots, m\}$. Denote by W_t, $t = 1, 2, \ldots T$ the accumulated sum at time t where T is the expected retirement time. Then the budget-constraint equations read as follows:

$$W_{t+1} = W_t(1 + r_t^j) + G_{t+1}\tau\,,\; t = 1, 2, \ldots, T-1\,,$$
$$W_1 = G_1\tau \tag{14}$$

where r_t^j is the return of the fund j in the time period $[t, t+1)$. When retiring the pensioner will strive to maintain his/her living standard in the level of the last salary. From this point of view, the saved sum W_T at the time of retirement T is not precisely what the future pensioner cares about. For a given life expectancy, the ratio of the cumulative sum W_T and the yearly salary G_T, i.e. $d_T = W_T/G_T$

is more important. Using the quantity $d_t = W_t/G_t$ one can reformulate the budget-constraint equation (14):

$$\begin{aligned} d_{t+1} &= F_t(d_t, j)\,,\ t = 1, 2, \ldots, T-1\,, \\ d_1 &= \tau \end{aligned} \tag{15}$$

where

$$F_t(d, j) = d\,\frac{1 + r_t^j}{1 + \varrho_t} + \tau\,,\ t = 1, 2, \ldots, T-1 \tag{16}$$

and ϱ_t denotes the wage growth defined by the equation

$$G_{t+1} = G_t(1 + \varrho_t)\,.$$

Suppose that each year the saver has the possibility to choose a fund $j(t, I_t) \in \{1, 2, \ldots, m\}$, where I_t denotes the information set consisted of the history of returns $r_{t'}^j$, $t' = 1, 2, \ldots, t-1$, $j \in \{1, 2, \ldots, m\}$ and the wage growth $\varrho_{t'}$, $t' = 1, 2, \ldots, t-1$. Now suppose that the history of the wage growth ϱ_t, $t = 1, 2, \ldots, T-1$ is deterministic and the returns r_t^j are assumed to be random and they are independent for different times $t = 1, 2, \ldots, T-1$. Then the only relevant information is the quantity d_t. Hence $j(t, I_t) \equiv j(t, d_t)$. One can formulate a problem of dynamic stochastic programming:

$$\max_j E(U(d_T)) \tag{17}$$

with the following recurrent budget constraint:

$$\begin{aligned} d_{t+1} &= F_t(d_t, j(t, d_t))\,,\ t = 1, 2, \ldots, T-1\,, \\ d_1 &= \tau \end{aligned} \tag{18}$$

where the maximum is taken over all non-anticipative strategies $j = j(t, d_t)$. Here U stands for a given preferred utility function of wealth of the saver. Problem (17 - 18) could be solved by same method as (11). Let us define equivalent quantity to (12):

$$V_t(d) = \max_j E(U(d_T) | d_t = d)\,. \tag{19}$$

Using the law of iterated expectations we obtain the Bellman equation

$$V_t(d) = \max_{j \in \{1,2,\ldots,m\}} E[V_{t+1}(F_t(d, j))] = E[V_{t+1}(F_t(d, j(t, d)))]\,, \tag{20}$$

for $t = 1, 2, \ldots, T-1$, where $V_T(d) = U(d)$. Using (20) the optimal feedback strategy $j(t, d_t)$ can be found backwards. This strategy gives the saver the decision for the optimal fund for each time t and level of savings d_t. Suppose that the stochastic returns r_t^j are represented by their densities f_t^j. Then equation

(20) can be rewritten in the form

$$
\begin{aligned}
V_t(d) &= \max_{j\in\{1,2,\dots,m\}} E[V_{t+1}(F_t(d,j))] \\
&= \max_{j\in\{1,2,\dots,m\}} \int_{\mathbf{R}} V_{t+1}\left(d\,\frac{1+r}{1+\varrho_t}+\tau\right) f_t^j(r)\,dr \\
&= \max_{j\in\{1,2,\dots,m\}} \int_{\mathbf{R}} V_{t+1}(y) f_t^j\left((y-\tau)\frac{1+\varrho_t}{d}-1\right)\frac{1+\varrho_t}{d}\,dy \\
&= \int_{\mathbf{R}} V_{t+1}(y) f_t^{j(t,d)}\left((y-\tau)\frac{1+\varrho_t}{d}-1\right)\frac{1+\varrho_t}{d}\,dy \qquad (21)
\end{aligned}
$$

where the substitution $y = d\,(1+r)(1+\varrho_t)^{-1}+\tau$ has been used and $\mathbf{R}$ denotes the set of real numbers. In our calculations we consider standard class of CRRA utility functions with constant coefficient of relative risk aversion $C = -xU''(x)/U'(x)$.

Concerning the structure of funds we consider the situation in Slovak Republic after stablishing system based on three pillars (2005). According to the adopted government regulation there were three funds (i.e. $m = 3$). Namely, the Growth, Balanced and Conservative fund. The funds are assumed to have normal distributions. Returns r_i and standard deviations $\sigma_i, i = 1,2,3$, used for the calculations could be found in Tab. 1. The data have been taken from [1].

Fund	Return	StdDev
F_1	$r_1 = 0.0926$	$\sigma_1 = 0.1350$
F_2	$r_2 = 0.0772$	$\sigma_2 = 0.0841$
F_3	$r_3 = 0.0516$	$\sigma_3 = 0.0082$

Period	wage growth $(1+\varrho_t)$
2006-2008	1.075
2009-2014	1.070
2015-2021	1.065
2022-2024	1.060
2025-2050	1.050

Table 1. Data used for computation. Fund returns and their standard deviations (left), expected wage growth for the period 2006–2050 (right).

According to Slovak legislature the percentage of salary transferred each year to a pension fund is 9%. The law sets administrative costs of the second pillar at 1% of monthly contribution and 0.07% of the monthly asset value (i.e. 0.84% p.a.). Therefore, we considered effective contributions $\tau = 8.91\%$ $(= 9\% * 0.99)$. The value 0.84% was subtracted from the asset returns in Tab. 1. We assumed the period of saving to be $T = 40$ years. The data for the expected wage growth ϱ are taken from [1]. The values are shown in Tab. 1.

The details of numerical approximation scheme could be found in [1]. The output of the numerical code is a matrix allowing us to "browse" between different years (rows) t and different levels of d (columns). At a given cell of the table we can read the name of fund ($j = 1, ..., m$) which has to be chosen. In Fig. 3 we present a typical result of our analysis with the coefficient of proportional risk aversion $C = 9$. It contains three distinct regions in the (d, t) plane determining the optimal choice $j = j(d, t)$ of a fund depending on time $t \in [1, T-1]$ and the average saved money to wage ratio $d \in [d_{min}, d^t_{max}]$. For practical purposes we chose $d_{min} = 0.0891$ (the effective 2nd pillar contribution rate) and $d^t_{max} = t/2$ for $t \geq 1$. In each year $t = 1, ..., T-1$ we invest the saved amount of money W_t (uniquely corresponding with d_t) to one of the funds $j = 1, 2, 3$ depending on the computed optimal value $j = j(d, t)$. In the first year of saving we take $d_1 = d_{min}$.

The curvilinear solid line in Fig. 3 represents the path of the mean wealth $E(d_t)$, obtained by 10,000 simulations and here we use $C = 9$. Notice that, for $t > 1$, the ratio d_t is a random variable depending on (in our case normally distributed) random returns of the funds and on the computed optimal fund choice matrix $j(d, t'), t' < t$. The dashed curvilinear lines correspond to $E(d_t) \pm \sigma_t$ intervals where σ_t is the standard deviation of the random variable d_t. In Tab. 2 we present the mean final wealth $E(d_T)$ as well as the so-called *switching-times* for mean path $E(d_t), t \in [1, T-1]$, and the intervals (in brackets) of switching times for one standard deviation of the mean path.

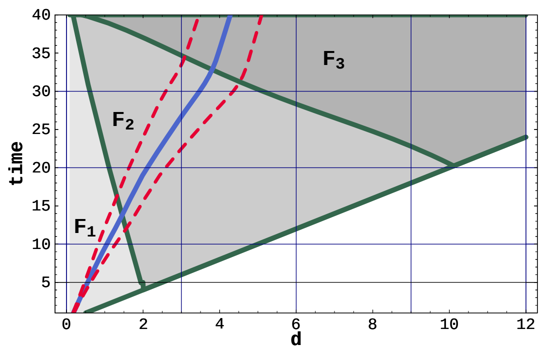

Fig. 3. Regions of optimal choice and the path of average saved money to wage ratio ($C = 9$).

Further reading

A concise overview of sigle-period and multi-period models could be found in [6] and [7]. Concerning single-period mean-variance and utility approach we recommend [2] or [4] (in Slovak). For details of model presented in Section 4 we refer to [1]. More general dynamic stochastic accumulation model for saving in the

mean $E(d_T)$	switch $\mathrm{F}_1 - \mathrm{F}_2$	switch $\mathrm{F}_2 - \mathrm{F}_3$
4.28	14 (12-16)	33 (32-35)

Table 2. Summary of computation of the mean saved money to wage ratio d_T and switching times ($C = 9$).

funded pillar of pension system in Slovakia with stochastic interest rates could be found in [5].

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