Pension saving strategies based on Samuelson's lifecycle theory

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We consider a model for an individual pension saving with gradual contributions, using expected utility as the optimality criterion. Simulations capture a phenomenon known in pension finance as stochastic lifestyling, a term coined by Cairns et al. (2006), whereby it is optimal early on to invest the accumulated savings in stocks and then gradually switch the investment into bonds and safe deposits as the retirement approaches and the total amount of savings increases. Thus the optimal strategy behaves as if the risk-aversion coefficient were lower for low levels of accumulated funds.
Suppose, that a saver has an opportunity to invest to *d* risky assets, whose dynamics are given by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

where *B* denotes a *d*-dimensional Brownian motion, $\mu \in R^d$ and $\sigma \in R^{d\times d}$ is a regular matrix. Moreover, he/she can invest to a riskless asset with the price process $S_t^0 = e^{rt}$. The goal is to invest the savings so as to maximize the expected utility of the terminal value of the saved wealth. We suppose, that investor uses the utility function with a constant relative risk aversion $\gamma > 0$ which has the form:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1$$

For $\gamma = 1$, $U(x) = \ln(x)$. Samuelson (1969), Merton (1969, 1971) and Hakanson (1970) showed, that in the case of a one-time investment, weights of the risky assets are independent of time and saved wealth and can be calculated as:

$$w = \frac{1}{\gamma}(\mu - r)^{\mathrm{T}}(\sigma\sigma^{\mathrm{T}})^{-1} \tag{1}$$

Equation (1) allows short positions in the risky assets. In the case of pension saving, short positions are typically forbidden. Therefore, we suppose that the weights of risky assets should be in the convex set C. Nutz (2010) proved that in this case, the weights can be calculated as:

$$w = \arg\max_{z\in \mathsf{C}}\{z(\mu - r) - \frac{\gamma}{2} z(\sigma\sigma^{\mathrm{T}})z^{\mathrm{T}}\} \tag{2}$$

In the case when short positions are forbidden, one has $\mathsf{C} = \{z \in R^d: z \geq 0 \ \& \ \sum_{i=1}^{d} z_i \leq 1\}$.
However, retirement savings are usually not a one-time investment. In the case of the old-age pension saving scheme in Slovakia, the future pensioner contributes a defined part of the gross monthly salary through the Social Insurance Agency to a maximum of two personal pension accounts maintained by the pension management company. Let's assume that the future contributions are deterministic (i.e, not random) and at the same time we have the opportunity to borrow them. Let C_t^{NPV} be the present value of future contributions. Then one can add C_t^{NPV} to the saved amount and convert the problem to the one-time investment. We can then use one of equations (1)-(2) to calculate optimal weights. Ayres and Nalebuff (2013) also suggest other leveraging options (e.g. using derivatives). However, leveraging cannot be considered as a realistic investment method when considering the investment restrictions in the old-age savings scheme in Slovakia.
Kilianová and Ševčovič (2013) derived a Hamilton-Jacobi-Bellman (HJB) partial differential equation to solve the problem of maximizing expected utility with gradual contributions and forbidden short positions of assets. They also proposed a numerical method for its solution. Černý and Melicherčík (2013) proposed an approximation of the exact strategy resulting from the solution of the HJB equation. Moreover, they pointed out that it is optimal to invest more

into stocks in earlier periods and later gradually switch the investment into bonds and safe deposits. The simplified strategy is calculated using formula (2), with

$$C= \{z \in R^d : z \geq 0 \ \& \ \sum_{i=1}^{d} z_i \leq \alpha\}, \quad \alpha = \frac{U_t}{U_t + C_t^{NPV}} \tag{3}$$

where U_t is the sum saved at time *t* and C_t^{NPV} the present value of future contributions. The weights *w* calculated according to (2) then refer to the whole wealth (including future contributions) $U_t + C_t^{NPV}$. The weights related to the real saved wealth can be then calculated as w/α.

When applying the model to the system of retirement savings in Slovakia, we will only consider the possibility of allocating between two types of risky funds, one of which will be equity (unsecured) and one bond (guaranteed) pension fund. In essence, they represent a financial instrument for the contributor, in which he/she allocates the regular contributions. It will not be possible to hold part of the cash savings or invest in risk-free assets. In this case, the constraint (3) comes to the form:

$$C= \{z \in R^2 : z \geq 0 \ \& \ z_1 + z_2 = \alpha\}, \quad \alpha = \frac{U_t}{U_t + C_t^{NPV}} \tag{4}$$

One can simplify the model of personal wage development and use the approximation $\alpha = j/k$, where *j* is the number of years during which the future pensioner contributes to the pension system and *k* represents the total number of years of saving.

Adequacy of pension savings can be assessed in several ways. A natural criterion (we denote it as *Perf*) is the ratio of a saved sum *M* to payed contributions *C*:

$$Perf = \frac{M}{C}$$

The criterion *Perf* does not take into account wage growth and inflation. Kilianová et al. (2006), introduced a retirement-months indicator (*PM*) in their work. The *PM* indicator shows the number of monthly earnings on the pension at the time of retirement. It is calculated as the ratio of the sum *M* saved at the time of retirement and the average monthly wage $\bar{W}$ for the last 250 business days before retirement:

$$PM = M/\bar{W}$$

Melicherčík, Szücs and Vilček (2015) pointed out that this indicator can be easily recalculated to the replacement rate (the ratio of the first pension to $\bar{W}$).

The last presented criterion compares the efficiency of the second and first pillars. One can calculate the sum *Target* to be saved in order to compensate the reduction of the first pillar pension as a consequence of entering the second pillar:

$$Target = \frac{P_I - P}{AR}$$

where P_I denotes the pension without entering the second pillar, *P* the adjusted pension when entering the second pillar and *AR* is the annuity rate calculated as the ratio of the annual amount of the pension paid to the pensioner at retirement to the amount saved at the end of the saving horizon. The criterion to assess the level of savings is then

$$TR = \frac{M}{Target} - 1$$

Strategy based on formula (4) will be compared (using criteria *Perf*, *PM* and *TR*) to benchmark strategies represented by conservative strategy investing all the time to bonds with short maturity and risky strategy represented by pure stock investments. The simulations will be done using historical data of asset returns, inflation and wage growths.

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