Optimal Asset Allocation Towards the End of the Life Cycle: To Annuitize or Not To Annuitize?

by

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Brief Title To Annuitize or Not?

Abstract

This paper develops a *normative* model that will provide a relevant framework for the choice between asset allocation and discretionary annuitization at retirement. The paradigm will be rich enough to accommodate the altruistic desire for bequest as well as the fundamental pre-occupation with consumption security. Our methodology deviates from the traditional financial economic approach to asset allocation by focusing on the *probability of consumption shortfall as* the operational objective function. Our model allows individuals to input their own estimates for stochastic market performance and obtain the "optimal" age at which to annuitize, based on a probabilistic tolerance level.

As a byproduct, using our own estimates, we are able to confirm the intuition shared by many in the financial planning community. Given the empirical evidence on the cost structure of annuities, the adverse selection implicit in annuity mortality tables together with the long-run propensity for equities to outperform fixed income investments, it makes very little sense for consumers under the age of 75-80 to voluntarily annuitize. The exception is in the event interest rates are extraordinarily high or when consumers have private health information.

Key Words and Phrases: Investments, Insurance, Personal Finance

1 Introduction and Motivation:

> "... *It is a well known fact that annuity contracts, other than in the form of group insurance* through pension systems, *are extremely rare. Why this should be so is a subject of considerable current interest. It is still ill-understood. Adverse selection, causing an unfavorable payout, and the fact that some utility may be derived from bequest are, presumable, an important part of the answer . ...*" Frenco Modigliani, December 9, 1985 - Nobel Prize acceptance speech in Stockholm, Sweden. [2]

Most individuals must decide how much, if any, of their wealth should be annuitized at about the time they retire. For many individuals a large portion of wealth is forcefully annuitized, for example pensions and government social security. In other cases consumers have discretion in the matter. In its most general form, purchasing a life annuity involves paying a non-refundable lump sum to an insurance company in exchange for a guaranteed constant life-long consumption stream that <u>can not be outlived</u>. The natural alternative to annuitization is individual strategic asset allocation amongst the various investment classes, such as equity, fixed income and real estate, together with (a fixed) periodic consumption from capital, dividends and interest. This “do it yourself” strategy runs the financial risk of under-funding retirement in the event of long-run inferior investment returns in conjunction with

[2] Reprinted in the *American Economic Review*, Modigliani (1986).

unexpected human longevity.

As Modigliani (1986), Friedman and Warshawsky (1990), Mirrer (1994) and many others have pointed out, very few people consciously choose to annuitize their remaining marketable (i.e. liquid or discretionary) wealth. This phenomena is especially puzzling within the paradigm of the Ando and Modigliani (1963) Life Cycle Hypothesis (LCH), under which individuals would seek to smooth their life-time consumption by annuitizing wealth. What better way is there to "smooth" and "guarantee" consumption for the rest of ones natural life? The most common answer is to simply abandon the strict form of the life-cycle hypothesis and declare that individuals have strong bequest motives, as Bernheim (1991), Hurd (1989) and many others have pointed out.

Another approach, which we prefer, is to argue that even when individuals have negligible bequest motives, annuities are simply *too expensive,* as was demonstrated by Warshawsky (1988) and Friedman and Warshawsky (1990). This means that the implied rates of return from life annuities are much lower, as a result of transaction costs or "loads"), than those available from other investment assets, even taking into account the life-long consumption guarantee which they provide. We take this approach one step further by giving practical advice in the face of these costs. We answer the question: At what point do "mortality credits"[3] outweigh the inferior

[3]Defined and explained in detail in the body of the paper.

returns?

Albeit the *positivist* economic question of *why* people refrain from purchasing annuities is a very interesting and important research topic, this paper will start by focusing on the *normative* aspects of *who* should annuitize and *when* they should do so, despite the implied costs. In the process we will attempt to shed light on the positivist issues.

Our methodology will deviate from the traditional financial economic utility maximizing approach to asset allocation by focusing on the *probability of* consumption *shortfall as* the operational objective function. Our goal is to help retiring individuals decide if and when to purchase (additional) annuities, without requiring much in the way of risk aversion parameters, inter-temporal rates of substitution, personal discount rates and elasticity of marginal utility. Needless to say, these macroeconomic parameters are difficult, if not impossible to measure accurately, and are predicated on the existence of well defined utility functions.

Shortfall relative to a target as a measure of risk was introduced into finance by Roy (1952) and Kataoka (1963), expanded by Fishburn (1977, 1984) and widely applied to investment asset allocation by Leibowitz, Bader and Kogelman (1996). The Shortfall-Target approach assumes that the investor has an exogenous threshold rate of return that must be achieved. Shortfall deviations from the target return are avoided at all costs.

Our (relatively easy to implement) mathematical model requires financial planners and/or individuals to input their own (subjective or historical) estimates for future (stochastic) market performance, and obtain the “optimal” age at which to annuitize. The main concept underlying our approach is to compute the probability of “beating” the consumption stream generated by an annuity. The higher this probability, the more it makes sense to wait before annuitizing, especially if there is a bequest motive. Consequently, we let the individual decide where he or she feels most comfortable in the shortfall/bequest trade-off. A typical presentation of this decision would involve computing the probability of beating the annuity stream and contrasting it with the expected bequest.

Furthermore, using our own estimates we are able to rigorously confirm the intuition shared by many in the financial planning community. Namely, given the empirical evidence on the cost structure of (immediate) life annuities, the adverse selection implicit in annuity mortality tables together with the long-run propensity for equities to outperform fixed income investments (i.e. time diversification) it makes very little sense for consumers under the age of 75-80 to annuitize any additional marketable wealth. The exception to this rule would be in the event that interest rates are extraordinarily high (cheap annuities) **or when the consumer has private** information that would lead him or her to believe that they are much healthier than the general population.

The above convictions are further re-enforced by the inability of the private consumer to acquire (at a reasonable cost) real indexed annuities that protect consumption against inflation, something that equity markets are able to do quite effectively over long horizons.[4]

The remainder of this paper is organized as follows. In Section 2 we will provide an historical perspective on optimal asset allocation and insurance over the life-cycle together with a brief literature review of the academic research that has been done on the subject of annuities *vis a vis* the traditional financial economic utility maximizing paradigm. In addition we review the research that has been done on the subject of shortfall risk which forms the theoreticad backbone of our approach. Section 3 will provide a simple pedagogical (four period) model of annuity pricing in which we review the main insurance concepts and introduce the foundation for our shortfall met methodology. In Section 4 we present the full-fledged model in which the dynamics of equity markets **and interest rates are incorporated to arrive at a** *probability of consumption shortfall.* In the same section we illustrate how the individual or financial planner can use this approach to optimize the trade-off between consumption and

[4]Although the interaction between equity markets and inflation in the short run is still debated in the academic literature, we refer the interested reader to the comprehensive work by Jeremy Seigel (1995) *Stocks for the Long Run* in which he documents the 200 year history of North American financial markets and concludes that equities consistently outperform inflation and fixed income securities over the long run.

bequest. The empirical Section 5 uses Canadian annuity payout rates (provided by *Cannex)* to estimate the non-actuarial transaction cost "loads" of annuities in contrast to the rates of return available from other fixed income investment products. Using reasonable capital market parameters we conclude that for relatively young individuals, annuities rates can easily be "beaten". Finally, Section 6 reviews the main conclusions of our approach and illustrates directions for further research in which a richer universe of annuity products is investigated together with the relatively complex tax implications.

2 Literature Review and Classical Approach:

It is common practice in the financial economic literature to analyze all normative problems that involve decision under uncertainty by postulating the existence of a von-Neuman Morgenstern (1947) utility function. Microeconomic decisions can then be "solved" by locating the course of action that will maximize the expected level of utility experienced by the rational agent. Optimal asset allocations, insurance decisions, consumption patterns and many other choices are thus established by deriving first order conditions subject to exogenous resource and budget constraints.

Within the context of our paper, the classic utility approach would involve solving the following dynamic stochastic optimization problem,

$$\max_{\{C_t,\alpha_t,A_t\}} E\left[\int_0^T e^{-\rho t}U(C_t)dt + e^{-\gamma T}B(W_T)\right] \tag{1}$$

where $U(C_t)$ **denotes the instantaneous utility of consumption,** is the personal **discount rate for consumption,** $B\ (W_T)$ denotes **the** utility **of bequest,** γ **is the discount** rate for bequest, T is the stochastic time of death, A_t is the amount of annuities, C_t is the rate of consumption, W_t is the level of marketable wealth and α_t is the asset allocation vector. Implicit in $U(C_t)$ and $B\ (W_t)$ is a functional form that will involve at least two other parameters, namely the marginal utility of consumption and bequest. See the work originated by Samuelson (1969) and **the book by Merton (1993) in which the basics of the above methodology are laid out in great detail. Richard (1975) was the first to amalgamate the** portfolio/utility optimization techniques of Merton and Samuelson to the arena of insurance products. Finally, see the paper by Sethi (1995) for a recent survey of continuous time consumption/investment problems allowing for the possibility of bankruptcy prior to death. (Additional **parameters required.)**

More specifically, within the context of annuities (and life insurance), Yaari (1965) and Ficher (1973) solved a somewhat simplified version of equation (1) and demonstrated that in a perfect capital market individuals with no **utility of bequest will** annuitize all of their marketable wealth. Likewise, with a utility of bequest, the optimal amount annuitized depends on the relative magnitude of the many parameters mentioned above. It is quite important to note than in the setting of imperfect capital markets, Yagi and Nishigaki (1993) have demonstrated that where annuities are constrained to be constant, investors *do* maintain some marketable wealth even without a bequest motive.

Consequently, the financial theorist can suggest optimal policies for all decisions involving uncertainty, provided, and this is the crucial part, that the economic agent in question, can precisely describe his or her utility function. Unfortunately, as many scholars (most of them psychologists) have pointed out, individuals are not able to describe their risk preferences very clearly and hence can not provide the financial theorist with a well defined von-Neuman Morgenstern utility function.[5] "Not to worry" has been the response of some economists, "individuals behave according to rational utility functions, even though they can't describe them". The solution is to create elaborate questioners that can properly gauge the exact level **or risk aversion or** *time preference.*

Even this proposition has come under fire recently with studies indicating that people do not behave according to the expected utility paradigm. Of course, the psychology literature has long known that most risk and uncertainty questions are answered based on the way in which the proposition was phrased. See Kahneman and Tversky (1979) and other articles in the book by Eatwell, Milgate and Newman

[5]Some recent work by Hanna, Fan and Chang (1995) has attempted to bridge this "gap" by illustrating what the various utility function forms would imply for investor behavior. This way consumers can "see" if their consumption/savings path is rational in the classical sense.

(1990). In fact, it is very easy to create hypothetical scenarios in which individuals consistently violate the von-Neuman Morgenstern axioms of rational choice, even when there is real money at stake!

2.1 The Shortfall Approach:

Our approach is quite simple. An individual who is about to retire will obtain an annuity quote (payout ratio) from an insurance company which will determine a "baseline" fixed (nominal) annual consumption stream that can not be outlived. The individual, perhaps with the help of a financial planner, will compute (a) the probability of being able to purchase the exact same annuity consumption stream sometime in the future and (b) the average bequest[6] that would result from not annuitizing. The investor/consumer in question would then compare (a) to (b), the risk return trade-off, and decide whether or not it is worthwhile to annuitize.

Example 1 *Consider a single female aged 65 who has $500,000 in liquid (tax sheltered) marketable wealth which she is considering annuitizing. A competitive insurante broker quotes her a payout ratio of 132-to-1. This translates into a guaranteed monthly payment of* $\frac{\$500000}{132} = \$3,787$ *for the rest of her life. If she buys the annuity at this time, her average bequest will be zero because all payments cease upon death of the annuitant. On the other hand, if she decides to wait-and-see, by investing her*

[6]We could just as easily have focused on a utility or median bequest.

funds in a well diversified portfolio of equities, there is a 93% percent chance that she can continue to consume $3,787 per month from her liquid wealth and purchase the exact same annuity some time in the future.[7] *The benefit of this strategy is that the average bequest will be* $112,000 *upon death. This is no sleight of hand, rather, by deferring the decision to annuitize the consumer maintains control over her wealth and leaves open the possibility of a bequest. Of course there is 7% (risk) chance that she will never be able to purchase the exact same consumption stream. This is the trade-off that the consumer can ponder and act upon.*

As **mentioned in the introduction, the concept of a subjective threshold rate of** return that investors would like to maintain or achieve is not new. The theoretical foundation was laid down by Roy (1952) and Kataoka (1963). The only novelty in our approach is that the retiree fixes a consumption stream that he or she would like to maintain, based on the current annuity payout ratio. The risk is in the probability of shortfall, namely that this consumption level can not be sustained without annuitizing. From a theoretical point of view, we base ourselves on the work by Fishburn (1977) in which he demonstrated that it is economically *rationa/* for individuals to have a subjective target rate of return which they want to achieve. This same idea was applied to investments by Van Harlow (1991), corporate finance

[7]The computation to derive the probability-bequest numbers and the exact “time to annuitize” is derived in the next section.

by Tse, Uppal and White (1993) and used in personal finance by Milevsky, Ho and Robinson (1996).

Crucial to our argument is the notion of Time Diversification, discussed in Reichenstein (1995) , Thorley (1995) and Marshall (1994). The longer the equity holding period the greater is the probability of dominating the risk free asset. In our context, the younger the investor, the greater is the probability of "beating" the annuity rate of return, which is priced off the risk free term structure.

3 Deterministic Analysis of Annuity Pricing:

This section will focus on deterministic annuity pricing[8] with the help of a simplified four period model. (We will take this opportunity to review the basics of annuity pricing as well.) Finally, we will generalize the discussion to a continuous consumption stream.

3.1 Discrete Model

The first step in the annuity comparison is to construct the mortality assumptions. We begin with N_0. individuals who are alive at time period zero, N_{11} are alive at time

[8] Strictly speaking the word 'deterministic' is a misnomer since, by definition, annuity prices depend on stochastic mortality rates. However, we use the word in reference to the underlying rates of return.

period one, N_2 are alive at time period two, N_3 are alive at time period three and they all die by time period four. ($N_4 = 0$). The probability that an individual alive at time zero will survive to time one is: N_1/N_0, the probability y that an individual alive at time zero will survive to time two is: N_2/N_0, etc.

At this stage, we start with a constant deterministic market interest rate denoted by R which is used for all time-value-of-money calculations. Annuity payouts take place one instant before the end of the period. The vital parameter L encompasses all non-actuarial transaction costs, commissions and fees charged by the life insurance Company.[9] Therefore, in our simple model, the price at time zero for an immediate one dollar life annuity is:

$$A_0 = \frac{(N_1/N_0)}{(1+R-L)} + \frac{(N_2/N_0)}{(1+R-L)^2} + \frac{(N_3/N_0)}{(1+R-L)^3}. \tag{2}$$

An individual with initial wealth W_0 would be able to "purchase" a lifetime guaranteed periodic consumption stream of $W_0/A_0 = C$. The annuity is *actuarially fair,* if the load L *is* equal to zero.

Likewise, for illustrative purposes, the price at time one for an immediate one dollar life annuity is:

$$A_1 = \frac{(N_2/N_1)}{(1+R-L)} + \frac{(N_3/N_1)}{(1+R-L)^2}. \tag{3}$$

[8] We will derive some estimates of this number in Section 5. In particular, we are modelling the load L to be a fixed percentage off the interest rate term structure as opposed to a fixed percentage off the mortality rates.

Question: What if the individual decides not to purchase the annuity at time zero, but rather invest at the market rate R, consume C, and then buy the annuity at time one? Will he or she be able to afford the same consumption stream C?

Mathematically we ask; is $W_0(1 + R) - C$ greater than (enough to buy the same consumption stream) CA_1 at time period one?

The answer depends on the "load" L charged by the insurance company. We must solve, as a function of L, the inequality:

$$W_0(1+R) - C \geq CA_1. \tag{4}$$

Using the definition that $W_0/C = A_0$ and dividing thru by C, we reduce equation (4) to solve the inequality,

$$A_0(1+R) - 1 \geq A_1 \tag{5}$$

as an explicit function of L. Substituting from equation (3) and (2) we obtain that the above inequalities are satisfied if and only if

$$\frac{(N_1/N_0)(1+R)}{(1+R-L)} + \frac{(N_2/N_0)(1+R)}{(1+R-L)^2} + \frac{(N_3/N_0)(1+R)}{(1+R-L)^3}$$
$$\geq \frac{(N_2/N_1)}{(1+R-L)} + \frac{(N_3/N_1)}{(1+R-L)^2} + 1.$$

Simplifying the above expression we obtain that it is worth while to *wait* at least one period (if not more), when:

$$L \geq \frac{(N_0 - N_1)(1+R)}{N_0} = q_0(1+R), \tag{6}$$

where q_0 denotes the probability of "dying" before time period one. When the load is higher than the right hand side of equation (6), one can safely "bake" the annuity at home and purchase the exact same or better consumption stream at time period one. Relatively speaking, when q_0 is low (younger person) compared to the load factor, it makes sense to avoid the annuity. On the other hand, when q_0 is high (older person) relative to the load factor, the insurance becomes quite valuable. Naturally, the exact same issue can be analyzed at time period one as well. Is it worth while to annuitize or do it yourself? The answer depends, once again, on the relationship between the load factor and the probability of death. It will make sense to avoid annuitizing if and only if

$$L \geq \frac{(N_1 - N_2)(1+R)}{N_1} = q_1(1+R),$$

where q_1 denotes the probability y that an individual who is alive at time period one, dies before time period two. In most mortality tables, the one-period probabilities of death (q_i) increase in time.

We can generalize the three period model to many periods during which the decision to annuitize can be evaluated based on probabilities. The following is the *Johansen* (1996) Individual Annuity Basic Table for Males/Females. We also assume a risk free rate of 8% per annum.

With an average industry load factor of approximately 125-175 basis points off the zero coupon term structure of interest rates, *it appears* that males and females will not be able to "beat" the annuity after age 70 and 75 respectively, the point at which $q_i(1.08)$ becomes larger than the load factor. The "mortality credits" are large enough to swamp out the unfavorable discount rate cum load factor.

M/Age	$1000q_i$	R	$q_i(1+R)$	F/Age	$1000q_i$	R	$q_i(1+R)$
55	5.411	0.08	0.0058	55	2.526	0.08	0.0027
60	7.616	0.08	0.0082	60	3.949	0.08	0.0043
65	11.691	0.08	0.0126	65	6.475	0.08	0.0070
70	19.958	0.08	0.0216	70	10.291	0.08	0.0111
75	33.093	0.08	0.0357	75	18.194	0.08	0.0196
80	53.775	0.08	0.0581	80	33.224	0.08	0.0359
85	85.600	0.08	0.0924	85	59.601	0.08	0.0644

Table Implied Annuity Load Threshold

Fortunately, the preceding discussion is only half of the story. Indeed, if the individual earns the same risk-free rate used by the insurance company to discount cash flows, the load factor threshold is $q_i(1+R)$. But what if the individual, who does not purchase the annuity this period, decides to take some investment "risk"? In this case we can reformulate equation (4) and (5) to take into account a more favorable rate of return from the individuals non-annuitized investment portfolio. Formally, we ask, what are the conditions on the excess return (risk premium), denoted by $\widetilde{K}$, *vis* *a vis* the load L, so that the following inequality holds?[10]

$$E\left[A_0(1+R+\widetilde{K})-1\right] \geq E\left[A_1\right] \tag{7}$$

Substituting, once again, from equation (2) and (3) we obtain that the rate of return in excess of the risk free rate (i.e. the risk premium) must satisfy the relation,

$$E\left[\widetilde{K}\right] \geq K := \frac{q(1+R)}{1-q} - \frac{L}{1-q}. \tag{8}$$

Thus, if the investor can earn (on average) at least K in excess of the risk free rate, it is worth while to wait and not annuitize. The following table illustrates some numbers for the minimum excess return K needed in order to "beat" the annuity for some fixed exogenous load factor L. If the investor can earn (on average) a rate of return greater than $R + K$, it makes perfect sense to delay the decision to annuitize.

For example, a 70 year old female would simply have to earn (on average) 62 basis point more than the risk free rate in order to beat an annuity from an insurance company that charges a (relatively small) load of 50 basis points. On the other hand, if the insurance company charges a (more common) load on the order of 150 basis points, the 70 year old female can earn a negative 39 basis point spread (i.e. less than the risk free rate) and still beat the annuity.

[10]Strictly speaking one needs risk-neutrality for the above mentioned argument to hold, which contradicts the essence of annuities to begin with. This issue will be dealt with in the next section. For now we simply focus on a return threshold that must be achieved in order to "beat" the annuity rate.

	Male			Female		
Age	**$1000q_i$**	**L=50 bp**	**L=150 bp**	**$1000q_i$**	**L=50 bp**	**L=150 bp**
55	5.411	*0.08%*	*-0.92%*	2.526	*-0.23%*	*-1.23%*
60	7.616	*0.33%*	*-0.68%*	3.949	*-0.07%*	*-1.08%*
65	11.691	*0.77%*	*-0.24%*	6.475	*0.20%*	*-0.81%*
70	19.958	*1.69%*	*0.67%*	10.291	*0.62%*	*-0.39%*
75	33.093	*3.18%*	*2.15%*	18.194	*1.49%*	*0.47%*
80	53.775	*5.61%*	*4.55%*	33.224	*3.19%*	*2.16%*
85	85.600	*9.56%*	*8.47%*	59.601	*6.31%*	*5.25%*
90	130.14	*15.58%*	*14.43%*	106.16	*12.27%*	*11.15%*

Table B: Break-Even Risk Premium K

As we will shall see in the empirical section, a 125-175 basis point spread seems to be the norm in the aunuity industry. This result combined with a (conservative) historical risk premium of about 4% would support the notion that a female can "beat" the annuity up to and including age 85, while the male can do so until age 80.

3.2 Continuous Model:

Following the actuarial literature and the recent work on annuity pricing by Frees, Carriere and Valdez (1996), we model mortality as a two-parameter Gompertz distribution in which the conditional survival probability is:

$$p(m, b, x, t) = \frac{s(m, b, x+t)}{s(m, b, x)}, \tag{9}$$

where the unconditional survival function is defined to be:

$$s(m, b, y) = \exp\left(\exp(-m/b)(1 - \exp(y/b))\right), \tag{lo}$$

which simplifies to:

$$p(m, b, x, t) = \exp\left(\exp(-m/b)(\exp(x/b) - \exp((x+t)/b))\right) \tag{11}$$

The parameter *"m"* is the mode of the (conditional) probability distribution and the parameter *"b"* is the scale measure of the (conditional probability) distribution. See the textbook by Bowers, Gerber, Hickman, Jones and Nesbitt (1986) for an elaborate discussion of alternative analytic mortality laws.

Likewise, using continuous compounding, we denote the present value of a dollar to be received t time units from now (discount bond), by the function

$$PV(r, t) := \exp(-rt). \tag{12}$$

The market price of a *continuous* one dollar life annuity for au individuad at age x, is:

$$A_x = \int_0 \quad PV(r - l, t) p(m, b, x, t) dt \tag{13}$$

where r denotes the continuously compounded risk free rate and 1 denotes the continuously compounded version of the load factor. The price of the annuity at age x, is the present value, using the rate $r - l$, of a 1$ consumption stream conditional on survival. Likewise, the price of the annuity at age $x + s$, is

$$A_{x+s} = \int_0^\infty PV(r-l,t)p(m,b,x+s,t)dt. \quad (14)$$

Solving the integral in equation (13) or equation (14) is impossible in closed form. Were forced to use numerical methods. The price of the annuity is decreasing, ceteris paribus, in the variable x, because a life-time guaranteed 1$ consumption stream becomes cheaper as one ages.

Example 2 *Consider a single male aged 65 who has $500,000 in liquid (tax sheltered) marketable wealth which he is considering annuitizing. Using equation (13) with a market interest rate of* $r = 4\%$ *and a load factor of* $l = 1\%$, *we obtain that the price of a* $1 *lifetime yearly annuity is* $A_{65} = 13.72$ *which translates roughly into* $\frac{500000}{13.72} = \$36,443$ *per year. Likewise, using equation (13) when the market interest rate is* $r = 8\%$ *and the load factor is* $l = 1\%$, *we obtain that the price of a* $1 *lifetime yearly annuity is* $A_{65} = 9.67$ *which translates roughly into* $\frac{500000}{9.67} = \$51,706$ *per year. (In both of these cases the actuary is assumed to use the mortality parameters* $m = 86.4$ *and* $b = 9.8$ *in the Gompertz function.)*

Figure 1 displays the cost of a one-dollar life annuity for a male when the continuously compounded load factor is equal to 100 basis points, (1 = 0.01) and the continuously compounded interest rate is equal to 7%, 6%, *5%, 4%* respectively. As

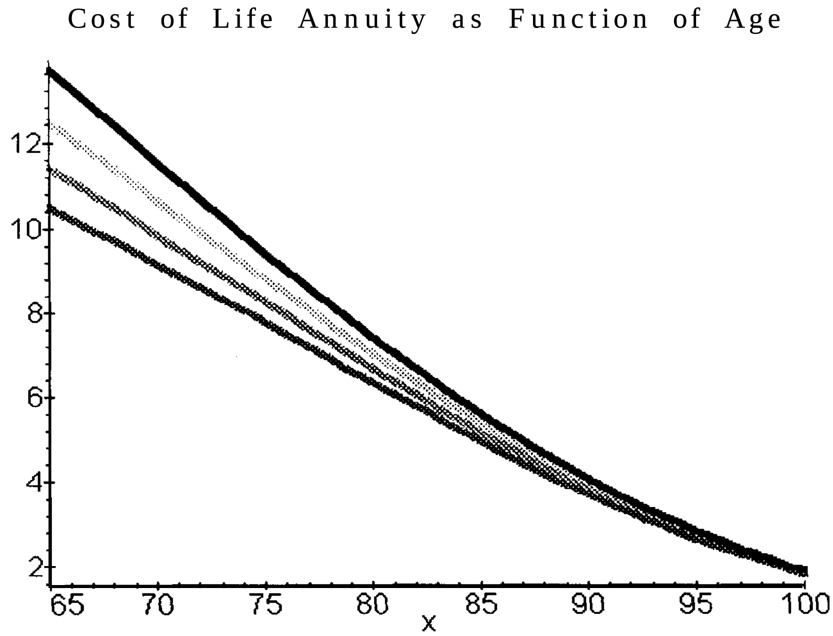

Figure 1: Price of Annuity at 7%, 6%, 5%, 4%

our intuition dictates, the cost of the $1 consumption stream decreases as one ages and increases with lower interest rates. In theory, it would always make sense to "wait" because you can purchase the annuit y for less next year. However, this ignores the fact that you must consume in the meantime.

To summarize, an individual with marketable wealth $W_0 = w$ at age x, can choose to annuitize all of this wealth by purchasing $c_x^w := \frac{w}{A_{-}}$ life-time consumption units. The notation c_x^w denotes the consumption units that can be obtained at age x with a wealth of $W_0 = w$.

Alternatively, the individual can invest the $W_0 = w$ *in* a “balanced” portfolio, earning a rate of return $r+k$, and consuming the exact same amount c_x^w from capital, interest and dividends. Of course, marketable assets will be depleted and by construct ion the investors wealth will obey the ordinary differential equation,

$$dW_t = ((r+k)W_t - c_x^w)\,dt \qquad\qquad W_0 = w, \tag{15}$$

which can be solved using elementary techniques to yield:

$$W_t = \left(w - \frac{c_x^w}{r+k}\right)\exp((r+k)t) + \frac{c_x^w}{r+k} \qquad \forall\, W_t \geq 0. \tag{16}$$

From a qualitative point of view, the constant multiplying the exponential function in equation (16) will be negative whenever the annuity payment c_x^w is greater than the perpetuity consumption defined by w *(r + k)*. This negativity implies that at some point in the future the (negative) exponential term will overpower $\frac{c_x^w}{r+k}$ and W_t will hit zero. Understandably, if the risk-premium is high enough, $w(r+k) > c_x^w$, one can consume for-ever because the constant multiplying the exponential function in equation (16) would keep W_t greater than zero.

Figure (2) shows how the risk premium afffects the future path of wealth when the individual (starting with exactly $9.67) decides *not to* annuitize, but consumes the one dollar per year all the same. Using our notation, $A_{65} = 9.67$, $r = 0.08,1 = 0.01$ and $k = 0.01,0.02,0.03$ respectively. In the event of a three percent risk premium, the investor can continue to consumer for ever. On the other hand, when $k = 0.01$, the investor will run out of funds in about twenty years.

Evolution of Wealth as Function of Risk Premium

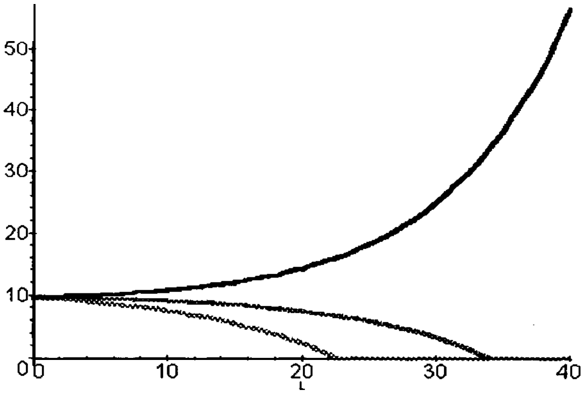

Figure 2: Evolution of Non-Annuitized Wealth.

Of course, we are entirely ignoring the volatility risk which will be the subject of the next section. However, we do observe the comforting phenomena that the (long run) risk-premium does not have to be that high for the annuity to be "beaten".

Under the do-it-yourself strategy, marketable wealth will be depleted at the point in time at which inf {t, W_t = O}. Solving for t^* in equation (16) we obtain:

$$t^* = \frac{\ln\left[\frac{c_x^w}{c_x^w - w(r+k)}\right]}{r+k} = \frac{-\ln\left[1 - A_x(r+k)\right]}{r+k}. \quad \mathbf{(17)}$$

Of particular interest is the fact that t* will become infinite, which means that wealth is never depleted, when the argument in the logarithm is zero. In other words, when the risk premium satisfies,

$$k \geq k^* = \frac{1}{A_x} + r, \quad \mathbf{(18)}$$

the investor can safely beat the annuity for ever. On the other hand, when $k < k^*$, "Shortfall" is certain conditional on being alive. The unconditional probability of' consumption shortfall is the probability y of surviving to time t*, as per equation (9):

$$p(b, m, x, t^*) \tag{19}$$

Finally, the expected bequest from the d-it-yourself strategy is simply the instantaneous probability y of death multiplied by the wealth level integrated from time zero until the money runs out. The expected bequest depends on the mortality parameters *m, b,* the current age *x,* the initial level of wealth w, the desired (annuity) consump tion stream, the total rate of return *r + k* and the time at **which the money runs out. Mathematically:**

$$E[B(x, w, c_x^w, r+k, t^*)] = \int_0^{t^*} W_t p(m, b, x, t)\xi(m, b, x+t)dt, \tag{20}$$

where $\xi(m, b, y)$ denotes the force of mortality, using equation (9):

$$\xi(m, b, y) = \frac{\frac{-\partial}{\partial y} s(m, b, y)}{s(m, b, y)}.$$

The integral in equation (20) can not be solved analytically and we must resort to numerical methods.[11] Once again we refer the interested reader to the textbook by Bowers, Gerber, Hickman, Jones and Nesbitt (1986) for further details of the actuarial calculations. Note: We use the same mortality rates for pricing (net of the loads) as

[11] In particular, the authors used the symbolic computational language MAPLE V.4 to solve the above and other integrals

we use for expected bequests. This conservatism may underestimate the true level of wealth bequeathed..

Example 3 *Consider, once again, the single male aged 65 who has \$500,000 in liquid (tax sheltered) marketable wealth which he is considering annuitizing. Using equation (13) when the market interest rate is* $r = 8\%$ *and the load factor is* $l = 1\%$, *we obtain that the price of a* \$1 *lifetime yearly annuity is* $A_{65} = 9.67$ *which translates roughly into* $\frac{500000}{9.67} = \$51,706$ *per year. Question: When would this individual run-out of money if he decided (instead of annuitizing) to invest the \$500,000 in a portfolio with a return of* $r + k = 8\% + 2\% = 10\%$ *and an annual consumption of exactly \$51,706 ? Answer: Using equation (17) and (19) we obtain* $t^* = 34.1$ *years and the probability of shortfall is 3%. The expected bequest from this strategy, using equation (20), is \$361,100. Question: What if* $r + k = 8\% + 1\% = 9\%$ *and an annual consumption of exactly \$51,706 ? Answer: Using equation (17) and (19) we obtain* $t^* = 22.6$ *years and the probability of shortfall is 36%. The expected bequest from this strategy, using equation (20), is \$181,200.*

We have described the *buy-annuity* strategy and the *do-it-yourself* strategy, we know move our attention to the *do-it-yourself and then switch* strategy. The main question of interest now becomes, at what time *s* will the marketable wealth from equation (16) be equal to $c_x^w \times A_{x+s}$, the price of a continued life-time consumption stream c_x^w. This will be the point at which it is worth while to annuitize wealth. In other words, for the first few years the consumer can earn a rate of return that exceeds that of' the insurance company. Eventually, the "mortality credits" become so large that it becomes worth while to annuitize. The point in time will be such that the individual can purchase the exact same annuity stream he or she was contemplating a few years ago.

Mathematically, we are searching for the value of s that satisfies the following equation:

$$W_s = \mathbf{c}_x^w A_{x+s} \qquad\qquad 0 < s < \infty,$$

Or by using equation (16) and (13) we solve:

$$\left(w - \frac{w/A_x}{r+k}\right)\exp((r+k)s) + \frac{w/A_x}{r+k} = \frac{w}{A_x}A_{x+s}$$

which can be simplified to:

$$((r+k)A_x - 1)\exp((r+k)s) + 1 = (r+k)A_{x+s}$$

or

$$s = \frac{1}{r+k}\ln\left[\frac{(r+k)A_{x+s} - 1}{(r+k)A_x - 1}\right] \qquad \textbf{(21)}$$

Once again, due to the complexity of the expression this must be done numerically because the *s* is on both sides of the equation. However, solving for s is quite easy with the use of a spreadsheet (or MAPLE V.4) when the future annuity prices can be stated with certainty. In fact, most insurance companies provide an annuity schedule for various ages. One simply has to compute the right hand side of equation (21) and

Evolution of Wealth v.s. cost of Annuity

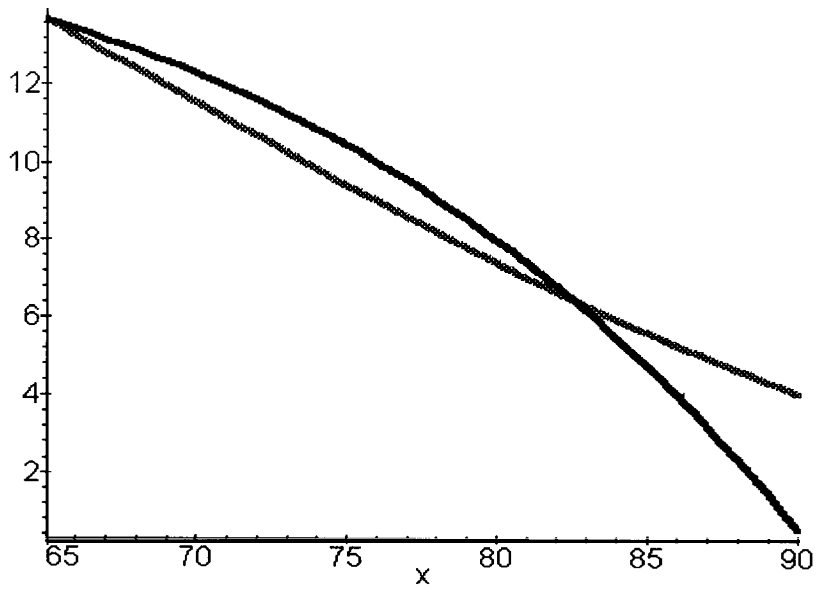

Figure 3: Optimal Time to Annuitize

find the "fixed point". Figure (3) illustrates this idea graphically for the parameters $r = 4\%, l = 1\%, k =$ 1.5% and age 65 thru 90.

Finally, the expected bequest from the *do-it-yourself and then switch* strategy can be obtained using equation (20) where the integration takes place until time *s*. **Mathematically:**

$$E[B(x, w, c_x^w, r + k, s)] = \int_0^s W_t p(m, b, x, t)\xi(m, b, x + t)dt.$$

Naturally, the expected bequest will be lower (than the do-it-yourself strategy) as a result of the eventual purchase of the life annuity.

Example 4 *Consider, once again, the single male aged 65 who has $500,000 in liquid (tax sheltered) marketable wealth which he is considering annuitizing. Using equation (13) when the market interest rate is r = 4% and the load factor is l =* 1%, *we obtain that the price of a* $1 *lifetime yearly annuity is* A_{65} = 13.72 *which translates roughly into* $\frac{500000}{13.72} = \$36,443$ *per year. The individual decides to do-it-himself by investing the $500,000 in a portfolio earning r + k =* 4% + 1.5% = 5.5% *per annum and consuming* $36,443 *from wealth. If indeed the individual would like to annuitize, the best time to do so would be, according to equation (21), at age* 82½, *the point at which the same* $\$36,443$ *annuity would cost* $c_{65}^{500000} \times A_{82} = \$36,443 \times 6.5 = 236,879$. *The benefit in waiting is an expected bequest of $155,600 that would be lost by naively annuitizing at age 65.*

We reiterate the relative crudeness of the above approximation in that it does not take into full account the stochastic nature of both interest rates and equity returns. Fortunately, the basic intuition remains the same even in our simplistic world. Namely, insurance companies charge loads that are "tagged on" to the risk free rate while the consumer has the ability to (take some risk and) earn a risk-premium. Therefore, in some cases it is worthwhile to wait.

4 Continuous Time Stochastic Model:

In this section we relax two key assumption from the previous discussion. (a) The interest rate used by the insurance company is constant over the entire life horizon and (b) the risk premium earned by the individual is deterministic over the entire life horizon. In particular, we now assume that the annuity is priced by the insurance company using the entire term structure of interest rates driven by the short rate which follow a mean reverting Ornstein-Uhlenbeck continuous time diffusion process, originally introduced by Vasicek (1977).12 In addition, we assume that the investment portfolio mutual fund (earning the risk premium) follows a Geometric Brownian Motion process. See the textbook by Hull (1996) for details on the application of these processes to financial economics.

From a technical point of view, we denote the price of one unit of a risky investment mutual fund by the symbol S_t, normalized to one at time t = 0, for convenience. The dynamics of the fund obey the Stochastic Differential Equation (SDE):

$$dS_t = \mu_s S_t dt + \sigma_s S_t dZ_t^s \qquad t \geq 0, S_0 = 1, \tag{22}$$

[12] Strictly speaking the Vasicek (1997) model does not represent an ideal model of reality because it allows negative interest rates, (albeit with very low probability y). In addition, most "equilibrium" models of the term structure do not fit the yield curve very well. However, in its defense, we are not pricing derivative securities and thus, for the purpose of this article, we believe that these shortcomings are not detriment al.

where μ_s, represent the growth rat e of return, σ_s represents the diffusion volatility parameter and Z_t^s is a standard Brownian motion driving the mutual fund price process. It is taken for granted that the parameters (μ_s, σ_s) are on the "efficient frontier" and are thus consistent with capital market equilibrium.. Practically speaking the individual will have to decide on a suitable risk-return trade-off given the investment horizon in question. A more complete model would actually locate an optimal (μ_s, σ_s).

Likewise, the short-term interest rate obeys the Stochastic Differential Equation:

$$dr_t = \gamma(\bar{r} - r_t) + \sigma_r dZ_t^r \quad (23)$$

where $\bar{r}$ is the long-run average level of the short-term interest rate, γ is the speed of adjustment (the rate at which the interest rate is being pulled towards its long-mm average), σ_r is the diffusion volatility parameter and Z_t^r is a standard Brownian motion driving the interest rate process.[13] Finally, $\rho = d\langle Z^r, Z^s \rangle$ denotes the correlation coefficient between the two Brownian motions. More details on the properties of these diffusions can be found in the book by Oksendal (1991).

Based on the above model for interest rates and the risk-neutral valuation technique described in Hull (1996), the time t price of a discount bond paying one dollar at time T is:

$$PV(r_t, t, T) = G(t, T) \exp(-H(t, T) r_t), \quad (24)$$

[13] Common values for these parameters are: $\gamma = 0.18, \bar{r} = 0.08, \sigma_r = 0.02$ See the empirical work by Chan, Karolyi, Longstaff and Sanders (1992) for detailed parameter estimations.

where:

$$H(t,T) = \frac{1}{\gamma}(1 - \exp(-\gamma(T-t))) \quad (25)$$

and

$$G(t,T) = \exp\left[\frac{(H(t,T) - T + t)(\gamma^2\bar{r} - \sigma_r^2/2)}{\gamma^2} - \frac{\sigma_r^2 H^2(t,T)}{4\gamma}\right]. \quad (26)$$

For the purpose of annuity pricing, equation (24) takes the place of equation (12) in section 3.2. Therefore, as per equation (13), the price of the whole-life annuity becomes a stochastic process that can be expressed as:

$$\widetilde{A}_{(x+t)} = \int_t^\infty PV(r_t - l, t, T)p(m, b, x+t, T)dT \qquad \widetilde{A}_{(x+0)} = A_x \quad (27)$$

It is important to note that r_t in equation (27) is a stochastic variable that will vary over time. Therefore we do not know what the price of the annuity will be in the future. This is in sharp contrast to the discussion of section 3.2 in which the function A_x was shown to be a decreasing function of age for all interest rates. In fact, if **current interest rates are very high (e.g. $r_t >> \bar{r}$), it is quite possible that the price** of the annuity will be more expensive next year, even though the annuitant ages. As in the deterministic case, it is impossible to obtain a closed form expression for $A_{(x+t)}$ in equation (27). Once again we must resort to numerical methods for actual values and solutions of the stochastic differential equations. The mathematical details can be found in the books and Kushner and Dupuis (1992).

As before we assume that **the retiree, with wealth w, can purchase a one dollar** whole-life annuity at age x for a price of A_x. This translates into a nominal consumption stream of $c_x^w = \frac{w}{A_x}$ which can not be outlived. If on the other hand the retiree chooses the d-it-yourself strategy the wealth dynamics will be, in analogue to equation (15):

$$d\widetilde{W}_t = \left(\mu_s \widetilde{W}_t - c_x^w\right) dt + \sigma_s \widetilde{W}_t dZ_t^s \qquad W_0 = w \tag{28}$$

Consequently, we are interested in evolution of two continuous time stochastic processes, $\widetilde{W}_t$ and $c_x^w \times \widetilde{A}_{(x+t)}$. From a technical point of view we obtain the density functions using Kolmogorov's backward equation.

We now compute three important quantities. The expected bequest of an individual aged x with initial wealth w who decides to annuitize at age $x + \tau$, by investing for the next τ years, is denoted by:

$$\int_0 \tau \widetilde{W}_t p(m, b, x, t) \xi(m, b, x + t) dt, \tag{29}$$

The probability of consumption shortfall for an individual aged x with initial wealth w who decides to annuitize at age $x + \tau$, by investing for the next τ years, denoted by:

$$P_x \left| W\tau < c_x^w A_{(x+\tau)} \right| p(m, b, x, \tau). \tag{30}$$

Note that we multiply the probability of wealth shortfall by the independent probability of being alive at time τ, to obtain the unconditional probability of shortfall.

Finally: the expected consumption shortfall magnitude for an individual aged x with initial wealth w who decides to annuitize at age $x+\tau$, by investing for the next τ years, denoted by:

$$E_x\left[\left|c_x^w \widetilde{A}\tau - \widetilde{W}\tau\right|^{+}\right] p(m,b,x,\tau). \tag{31}$$

The symbol $|a|^{+}$ denotes the positive part of a. Once again, we have to condition on being alive.

4.1 Numerical Results:

The three relevant quantities were computed using numerical integration routines in MAPLE V.3 and are displayed in the following summary table.[14]

$\tau+65$	**P(S)**	**P(S)**	**E[B]**	**E[B]**	**E[S]**	**E[S]**
	M.	**F.**	**M.**	**F.**	**M.**	**F.**
65	0.00	0.00	$0.00 \times w$	$0.00 \times w$	$0.04 \times c_x^w$	$0.02 \times c_x^w$
70	0.08	0.06	$0.11 \times w$	$0.12 \times w$	$0.06 \times c_x^w$	$0.04 \times c_x^w$
75	0.09	0.07	$0.14 \times w$	$0.18 \times w$	$0.10 \times c_x^w$	$0.07 \times c_x^w$
80	0.12	0.11	$0.19 \times w$	$0.23 \times w$	$0.13 \times c_x^w$	$0.13 \times c_x^w$
85	0.26	0.22	$0.29 \times w$	$0.33 \times w$	$0.21 \times c_x^w$	$0.18 \times c_x^w$
90	0.42	0.37	$0.36 \times w$	$0.41 \times w$	$0.25 \times c_x^w$	$0.23 \times c_x^w$

Summary Results: $r=$ %, $k=1\%, \bar{r}=7.5\%, \mu=12\%, \sigma_s=18\%, \gamma=18\%, \sigma_r=2\%, \rho=0$.

[14] **Algorithm and other details available upon request from the author.**

For example: A Female age 65 who can purchase a c_x^w annuity consumption stream, has a 89% chance of acquiring the exact same annuity at age 80. The expected bequest from waiting to age 80 is 23% of her current wealth. In the unfavorable event (11% percent chance) that her net wealth is not enough to purchase the annuity at age 80, **the expected consumption shortfall would be 13% of c_x^w.**

Empirical Data and Parameter Estimates:

The load-factor (L in discrete time and l in continuous time) is one of the most crucial parameters in determining the optimality of annuitization. In this section we perform an empirical analysis on actual annuity quotes in order to extract the embedded load factor.

The data was supplied by Cannex Financial Exchanges Limited, an intermediary that compiles the payout rates for the ten most competitive annuity providers in Canada. The data is in the form of a time series of payout ratios per $50,000 (Canadian) of tax sheltered funds (within a Registered Retirement Savings Plan). It is interesting to note that one can obtain slightly more favorable payout rates for tax sheltered (or qualified) funds due to the possible mitigation of adverse selection. Cannex compiles and reports (to the financial press) the payout rates for a Male or Female 65 year old with a ten year guarantee from the "top ten" best insurance companies in Canada.

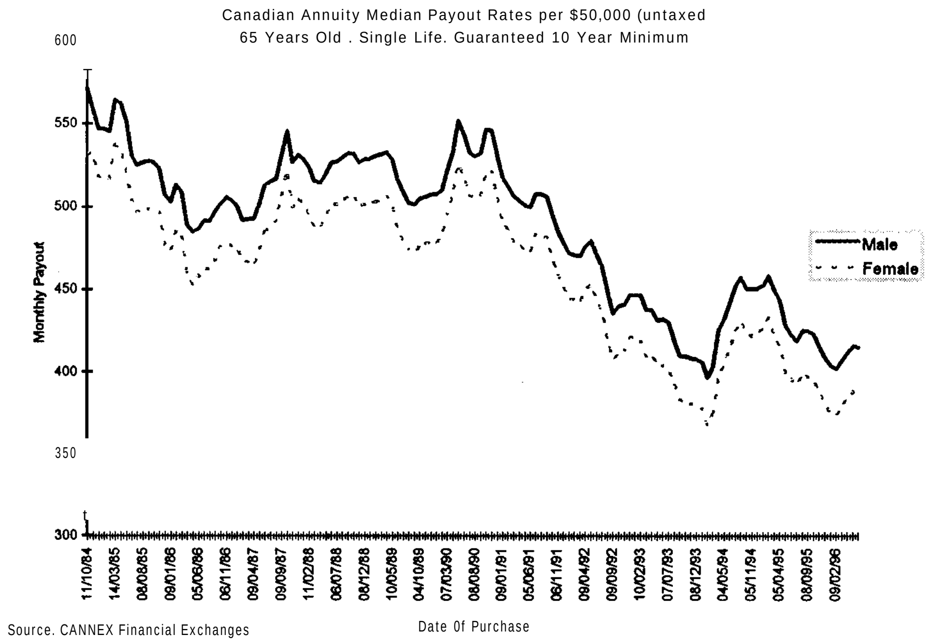

Figure 4: Annuity Payout Rates

We were able to obtain monthly rates for the for the last 12 years. The highest, lowest and median payout rate (per $50,000) and cd-per-monthly-dollar of lifetime consumption during the 19841996 period is reported in Table C and displayed in Figure 4 and[15] Figure 5. It is quite evident that annuity payout rates, for a 65 year old, fluctuate over time, being directly related to the prevailing interest rates in the market.

[15] Females age 65 pay approximately 7% percent more than males age 65. This “markup” is purely a function of female longevity but varies over time as Figure 5 demonstrates.

		Male	Female
Best	Payout:	$571	$537
	$1 Cost:	87.5	93.1
Worst	Payout:	$396	$367
	$1 Cost:	126.3	136.2
Median	Payout:	$501	$474
	$1 Cost:	99.8	105.5
Range	Payout:	$175	$170

Table C nnuity Summary Data: 1984-1996

From here on we will focus on the time-series of median *payout rates* from these ten firms and compare them to the government of Canada term structure of interest rates as reported in the Globe and Mail. We partitioned the term structure of interest rates into annual segments and "solved" for the implied *constant* load factor L using the formula,

$$A_{65}^{p} = \sum_{i=1}^{10} \frac{1}{(1+R_i-L)^i} + \sum_{i=11}^{45} \frac{{}_ip_{65}}{(1+R_i-L)^i} \tag{32}$$

where A_{65}^{p} is the market payout rate per dollar, L is the load factor, R_i is the appropriate interest rate applicable to each maturity and ${}_ip_{65}$ is the probability that a 65 year old (male and/or female) will survive i more years. The mortality numbers

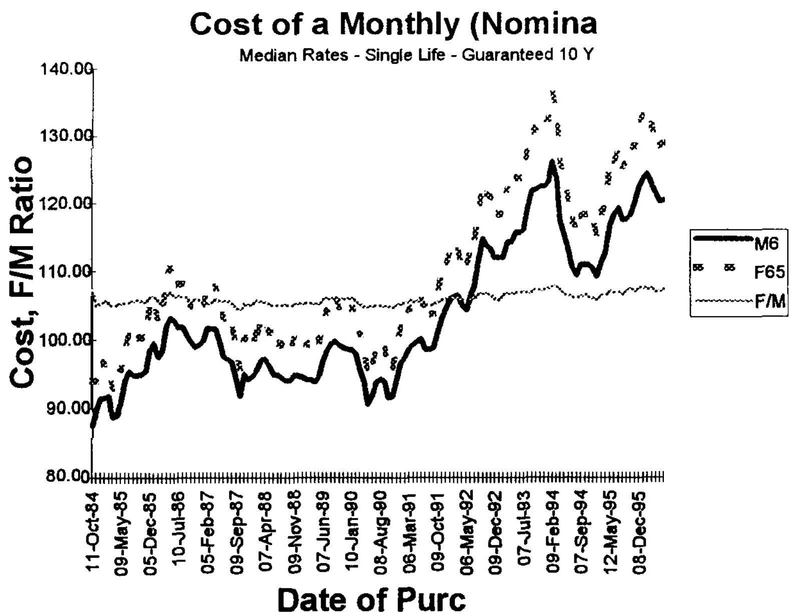

Figure 5: Time Series Cost of Monthly Dollar

were taken from the 1996 Johansen Individual Annuity Basic Table[16], truncated at age 110.

The rational behind the two segments in equation (32) is that the first ten years are guaranteed, thus there is no life insurance component in the numerator. If indeed

[16] The 1996 Individual Annuity Basic Table was created by applying Scale G (for thirteen years) to update the industry standard 1983 Individual Annuity Basic Table. We used the 1996 Individual Annuity Basic **Table (Johansen, Appendix D) and not the 1996 Individual Annuity Mortality Table (Johansen, Appendix E), which is the former table with a 10% loading subtracted and regraduated. The reason being that we wanted to estimate the embedded load factor which ends up being higher than 10% in all cases.**

we were dealing with a straight life annuity, the equation would simply be: $A^{p}_{65} = \sum_{i=1}^{45} \frac{_{i}p_{65}}{(1+R_i-L)^i}$.

The appropriate annual zero coupon rates (on a monthly basis) were extracted from the yield curve using the standard bootstrap method as described in Hull (1993). On a technical note, the Canadian yield curve extends only up to thirty years, thus the remaining fifteen rates (needed for the forty five years of possible annuity consumption) were taken as constant 30 year rates. Once the zero rates were obtained, a computer program (procedure) was written in Splus that performed a non-linear minimization on equation (32) to find a value for L that would make both sides equal.[17]

Although our data set is relatively small, we did notice a tendency for the load factor to increase when interest rates are lower and decrease when. interest rates are higher. Also, there is somewhat of a lag from the time interest rates increase to the time annuity prices decrease and vice versa which could explain the above phenomena.

Table D displays the statistical results for the implied load factors. The high values were observed around the time interest rates moved up sharply and presumably insurance companies had not yet increased their payout ratios (reduced their prices).

[17]This **could have also been done by solving for the root of a 40'th order polynomial in** L, **but we ran into uniqueness problems and decided to minimize the distance between the market price and the actuarial adjusted present value of cash flows.**

High (Max)	**217bp**
Low (Min)	**83bp**
Average (Mean)	**142bp**
Standard Deviation	**22bp**

Table D: Time Series Load Results: 19841996

Likewise, the low values were observed around the time interest rates fell sharply and insurance companies had not yet decreased their payout ratios (increased their prices).[18] In conclusion, the average 142 basis point load translates into a lucrative incentive to wait for males and females under the age of 75 and 80 respectively.

5 Conclusion and Directions for Research:

The objective of this paper was twofold, first we developed a model to help the decision maker choose the optimal time at which to annuitize discretionary marketable wealth. Second we have argued that it makes very little sense to annuitize up to and including age 75-80 because of the fairly low probability of consumption shortfall from a do-it-yourself and switch strategy.

[18] A practical by-product of this "lag" is that consumers can act quickly and get relatively high annuity payout rates when interest rates suddenly **fall.**

5.1 Further Questions:

There are some un-answered issues that we must ponder.

(a) We have focused our discussion in a tax-free vacuum. Indeed, in the presence of Income Taxes there are benefits in receiving life annuity payments because of the favorable (short life expectancy) mortality tables used by the tax authorities. The annuity payouts consist of interest and the return of capital. A shorter assumed lifespan implies an accelerated return of capital which implies a lower present value of taxes. We therefore ask: perhaps it becomes harder to "beat" the annuity on an after-tax basis?

(b) Nominal annuities are not protected against inflation. How does inflation affect the decision to annuitize? The are some insurance companies that offer pseudo-indexed (increasing) annuities. These annuities simply increase their payout by l%-3% every year. Other more sophisticated annuities (hedged using real real return bonds) actually increase their annual payment based on the realized value of the Consumer Price Index. How does their cost structure compare with those of regular annuities? Can one beat the *real* rate of return from an annuity?

(c) Variable Immediate Annuities (VIA) provide some form of market participation in conjunction with life-time protection at the expense of fluctuating payouts. See Daily (1994) for an excellent description. At fist glance, VIA'S seem to have the best of both worlds, aside from the bequest issue. Can one "beat" the return from a

Variable Immediate Annuity? Under what conditions does it make sense to purchase a VIA? What are the probabilities of shortfall?

(d) Unlike marketable fixed income instruments such as Government Bonds and Treasury Bills, whole life annuities can not be "sold" in a secondary market once they have been purchased by the annuitant. In essence an annuity is an irreversible investment. Consequently, purchasing an annuity can be analyzed as a real *option*. Perhaps it makes sense to delay the decision as a result of the *option-to--wait*. The benefits of waiting are that interest rates may go up, alternative investments may yield more and the individual may die, (or get very ill). Can the real option framework enable us to further quantify the benefits of waiting, even within the classical utility maximizing paradigm?

In the time honored tradition, we leave these issues open for further research.

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