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Optimal Asset Allocation Towards The End of the Life Cycle: To Annuitize or Not to Annuitize?

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ABSTRACT

Most individuals must decide how much of their marketable wealth should be annuitized at retirement. The natural alternative to annuitization is investing the wealth and withdrawing the exact same consumption stream as the annuity would have provided. Of course, this strategy risks under-funding retirement in the event of below average investment returns with above average longevity. This paper develops the framework for a third alternative. We propose a model in which retirees defer annuitization, via a "do-it-yourself" scheme, until it is no longer possible to beat the mortality-adjusted rate of return from a life annuity. We make use of a unique Canadian database to calibrate the insurance loads and interest rate parameters. We conclude that in the current environment, a sixty five year old female (male) has a ninety percent (eighty-five percent) chance of beating the rate of return from a life annuity, until age eighty.

INTRODUCTION

> *"...It is a well known fact that annuity contracts, other than in the form of group insurance through pension systems, are extremely rare. Why this should be so is a subject of considerable current interest. It is still ill-understood. Adverse selection, causing an unfavorable payout, and the fact that some utility may be derived from bequest are, presumably, an important part of the answer…."* Franco Modigliani, December 9, 1985 - Nobel Prize acceptance speech in Stockholm, Sweden.[1]

Most individuals must decide how much, if any, of their wealth should be annuitized near the time they retire. For many individuals a large portion of wealth is forcefully annuitized; for example, pensions and government social security. In other cases they have discretion in the matter. In its most general form, purchasing a life annuity involves paying a non-refundable lump sum to an insurance company in exchange for a guaranteed constant life-long consumption stream that cannot be

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[1]Reprinted in the *American Economic Review*, Modigliani (1986).

outlived. The natural alternative to annuitization is investment amongst the various asset classes, such as equity, fixed income and real estate, together with a fixed periodic withdrawal equivalent to the consumption stream generated by the annuity. This do-it-yourself strategy incurs the financial risk of under-funding retirement in the event of long-run inferior investment returns in conjunction with unexpected human longevity.

As Modigliani (1986), Friedman and Warshawsky (1990), Mirer (1994), Poterba and Wise (1996) and many others have pointed out, very few people consciously choose to annuitize their marketable (liquid or discretionary) wealth, as evidenced by the comprehensive Health and Retirement Survey (HRS), conducted in the United States.[2] Only 1.57 percent of the HRS respondents reported annuity income. Likewise, only 8.0 percent of HRS respondents with a defined contribution pension plan selected an annuity payout. This phenomenon is especially puzzling within the paradigm of the Ando and Modigliani (1963) Life Cycle Hypothesis (LCH), under which individuals seek to smooth their lifetime consumption by annuitizing wealth. What better way is there to "smooth" and "guarantee" consumption for the rest of one's natural life? The most common answer is to simply abandon the strict form of the life-cycle hypothesis and declare that individuals have strong bequest motives, as Bernheim (1991), Hurd (1989) and many others have suggested.

Another approach, which we prefer, is to argue that even when individuals have negligible bequest motives, annuities are simply too expensive, as was hypothesized by Warshawsky (1988) and Friedman and Warshawsky (1990). This means that the implied rates of return from life annuities are much lower as a result of transaction costs, or loads, than those available from other investment assets, considering the life-long consumption guarantee which they provide. The contribution of this paper is to give prescriptive advice in the face of these costs. We suggest that most individuals should defer annuitization, via the do-it-yourself scheme, until is no longer possible to beat the mortality adjusted rate of return from the life annuity. We call this approach the "do-it-yourself-and-then-switch" strategy.

Our methodology deviates from the traditional financial economic utility maximizing approach to asset allocation by focusing on the probability of consumption shortfall, as the measure of risk. The classic utility-based approach would involve solving the following dynamic stochastic optimization problem:

$$\max_{\{C_t,\alpha_t,a_t\}} E\left[\int_0^T e^{-\rho t}U(C_t)dt + e^{-\gamma T}B(W_T)\right], \tag{1}$$

where $U(C_t)$ denotes the instantaneous utility of consumption; ρ is the personal discount rate for consumption; $B(W_T)$ denotes the utility of bequest; γ is the discount rate for bequest; T is the stochastic time of death; a_t is the amount of

[2]The HRS is an ongoing survey of 12,600 individuals between the ages of fifty-one and sixty-one in 1992.

annuities; C_t is the rate of consumption; W_t is the level of marketable wealth and α_t is the asset allocation vector. Implicit in $U(C_t)$ and $B(W_t)$ is a functional form that will involve at least two other parameters, namely the marginal utility of consumption and bequest.[3] Unfortunately, from a practical point of view, most individuals are not able to describe their utility based risk preferences very clearly and hence cannot provide the financial practitioner with a well defined von-Neuman Morgenstern utility function to "plug into" equation (1). Furthermore, recent psychology studies, starting with Kahneman and Tversky (1979), have demonstrated that people do not behave according to the expected utility paradigm. These issues cast a shadow on the model's positive as well as normative abilities.

In contrast, our goal is to help retiring individuals decide if, and when, to purchase (additional) annuities without requiring much in the way of risk aversion parameters, inter-temporal rates of substitution, personal discount rates and elasticity of marginal utility. We do this through shortfall probability which, unlike the utility function, is a tangible, intuitive and practical measure of risk that most individuals can understand and employ in their decision making process.

Specifically, an individual who is contemplating annuitizing will obtain an annuity quote (payout ratio) from an insurance company that will determine a "baseline" fixed (nominal) annual consumption stream that cannot be outlived. The retiree, perhaps with the help of a financial planner, will then compute the probability of being able to purchase the same exact annuity in 5, 10, 15 and 20 years, while at the same time investing and withdrawing the "baseline" consumption stream. In the event that the probability of a successful deferral is high enough, i.e. the probability of shortfall is low enough, the retiree may decide to defer annuitization and thus reap the benefits of liquidity, flexibility and bequest. The decision to "defer" the purchase or simply "lock in" the current annuity rate would depend on the probability of shortfall, *vis a vis* the risk tolerance of the individual in question.

An additional benefit from deferring annuitization is a manifestation of the classic "adverse selection" problem in insurance.[4] Individuals who perceive themselves to be in poor health will be less inclined to annuitize. Although it is somewhat difficult to quantify this effect, if there is a positive probability that the individual will learn more about their subjective mortality status at the end of the waiting period, then waiting buys additional information. Brugiavini (1993) developed a model in which the demand for annuities is related to the uncertainty about one's own mortality type. His model predicts that people in the later stages of their life will refrain from buying annuities because of the adverse selection problem.

Shortfall relative to a target as a measure of risk was introduced into finance by Roy (1952) and Kataoka (1963), expanded by Fishburn (1977) and widely applied to investment asset allocation by Leibowitz, Bader and Kogelman (1996). The Shortfall-Target approach assumes that the investor wants to maximize the growth rate of their investment portfolio, but at the same time wants to control the

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[3]See the work originated by Samuelson (1969), Merton (1993) and Richards (1975) in which the above methodology is laid out in great detail.

[4]We thank an anonymous referee for stressing this point.

occurrence of shortfall from the target by placing an upper bound on its probability. This idea has been applied to investments by Van Harlow (1991), and to personal finance by Milevsky, Ho and Robinson (1997). The probability of financial regret has been applied to (self) insurance by Brockett, Cox and Witt (1984).

Agenda

The remainder of this paper is organized as follows: The next section will briefly review the academic literature on the demand for life annuities; we will then provide a simple pedagogical model of annuity pricing where we review the main insurance concepts and introduce the foundation for our shortfall methodology. The following section will present the full-fledged stochastic simulation model in which the dynamics of equity markets and interest rates are combined to arrive at a probability of a successful deferral. The final section analyzes a Canadian annuity time-series database to obtain estimates of (a) the insurance loads on life annuities and (b) the internal rates of return (IRR) from life annuities, net of insurance loads.

LITERATURE REVIEW

Within the context of annuities (and life insurance), Yaari (1965) and Fischer (1973) solve a somewhat simplified version of the utility maximizing equation (1) and demonstrate that, in a perfect capital market, individuals with no utility of bequest will annuitize all of their marketable wealth. Likewise, with a utility of bequest, the optimal amount annuitized depends on the relative magnitude of the many parameters mentioned above. It is quite important to note that in the setting of imperfect capital markets, Yagi and Nishigaki (1993) have demonstrated that, where annuities are constrained to be constant, investors do maintain some marketable wealth, even without a bequest motive. (See Sinha [1986] for additional research on the interaction between mortality rates, interest rates and the demand for annuities within the classical utility framework.) Williams (1986) presents some experimental evidence to argue that higher interest rates and longer life expectancies would decrease the demand for annuities. Broverman (1986) examines the statistical distribution of the internal rate of return from a life annuity.

On a related theme Cherin and Hutchins (1987) examine the rate of return from universal life insurance and conclude that loading and expense charges make it more profitable to buy term and invest the difference. Our results are similar in spirit in that we suggest people consume term and then invest the difference, where term (perhaps abused) is the consumption stream provided by the annuity.

Poterba and Wise (1996) and Mitchell, Poterba and Warshawsky (1997) examine the market for annuities, both empirically and theoretically, using the utility maximizing "wealth equivalence" approach. They conclude that even if insurance loads are as high as thirty percent, for reasonable utility function parameters, the individual is "better off" annuitizing.

ANNUITY PRICING

In this section we briefly review annuity pricing, *vis a vis* the concept of "beating" the mortality adjusted return from a life annuity. First we assume that individual portfolio investment returns and interest rates are non-stochastic. Later, we add randomness to our model, which can then be contrasted with the deterministic results from this section. The first sub-section will focus on the discrete time case; the second sub-section will look at continuous time. The authors believe that both approaches are necessary, as the former will assist in acquiring the basic pedagogical intuition, while the latter will introduce the platform and technology for the stochastic model presented in the next section.

Discrete Time

The basic market pricing definition of a one-dollar per year Fixed Immediate Annuity (FIA) in discrete time, is:

$$a_x = (1+L_x)\left(\sum_{i=1}^{\infty}\frac{{}_i p_x}{(1+R)^i}\right) \tag{2}$$

where R denotes the (risk-free)[5] rate of interest, or internal rate of return, used by the insurance company to discount cash flows; ${}_i p_x$ denotes the conditional probability that an individual aged x will attain age $(x+i)$, where it is understood that ${}_j p_n = 0$ for a large enough value of j; and L_x denotes the insurance load charged at issue age (x). The proportional insurance load, L_x, incorporates all expenses, taxes, commissions and distribution fees, and is multiplied by the pure actuarial premium to arrive at a market price, a_x. Mitchell, Poterba and Warshawsky (1997) have provided strong evidence to suggest that L_x increases with issue age.

Consistent with our main theme, the retiree may decide to defer purchasing the life annuity at age (x), and instead invest the funds (a_x), and purchase the exact same one-dollar life annuity at age $(x+1)$. In order to afford the exact same life annuity stream in one year, the annual investment return, K, earned by the retiree must be such that:

$$a_x(1+K)-1 \geq a_{x+1}. \tag{3}$$

In other words, the life annuity premium at age (x) invested at a rate K, minus the one-dollar consumption at the end of the year, must be greater than, or equal to, the market price of the annuity at age $(x+1)$. Re-arranging equation (3) in terms of

[5]Some insurance companies discount cash flows, for pricing purposes, at a slightly higher corporate bond rate; others use their cost of capital.

the portfolio investment return K, the condition for beating the rate of return from the annuity, over one year, is:

$$K \geq K^{*} = \frac{a_{x+1}}{a_x} + \frac{1}{a_x} - 1. \tag{4}$$

We refer to K* as the threshold annual investment return necessary for a successful deferral. In general, using the actuarial identity, ${}_i p_{x+n} = \frac{{}_{n+i} p_x}{{}_n p_x}$,[6] we can re-write a_{x+1} in terms of a_x using equation (2), and then re-write the condition for beating the rate of return on the annuity, using equation (4), as:

$$K \geq K^{*} = \frac{1+R}{{}_1 p_x}\left(\frac{1+L_{x+1}}{1+L_x}\right) - \frac{L_{x+1}}{a_x} - 1. \tag{5}$$

Equation (5) is crucial to our main thesis. When the insurance loads, L_x and L_{x+1}, in equation (5) are set equal to zero, the condition for beating the annuity is simply: $K \geq K^{*}(1+R)/({}_1 p_x) - 1$. Since the term $({}_1 p_x)$ is strictly less than one, the threshold return on investment K^{*} is greater than the rate R. The term $({}_1 p_x)^{-1}$ is referred to as "mortality credits", because they enhance the return R. The lower the probability of survival, the higher the mortality credits. In general, a higher insurance load tends to reduce the threshold rate K^{*}.[7]

Continuous Time

Using continuous compounding, the market price of a continuous one-dollar life annuity for an individual at age (x), is:

$$a_x = (1 + L_x) \int_0^{\infty} e^{-rt} \, {}_t p_x \, dt , \tag{6}$$

where r denotes the continuously compounded internal rate of return; ${}_t p_x$ is the conditional probability that an individual aged (x) survives to age $(x+t)$ and L_x denotes the insurance load charged at issue age (x).

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[6]See the textbook by Bowers, Gerber, Hickman, Jones and Nesbit (1986), Chapter 3, for details.

[7]Equation (5) also reveals that for a young enough individual (x) and a high enough insurance load L_x, the investment return threshold K* could, in theory, be lower than R. In this (peculiar) case, one can beat the mortality adjusted return from the life annuity by simply investing in the exact same assets used by the insurance company to discount cash flows.

Following the actuarial literature and recent work on annuity pricing by Frees, Carriere and Valdez (1996), we model mortality as a two-parameter Gompertz[8] distribution in which the conditional survival probability is:

$$ {}_t p_x = \frac{\exp\left(e^{-m/b}\left(1-e^{(x+t)/b}\right)\right)}{\exp\left(e^{-m/b}\left(1-e^{x/b}\right)\right)}. \tag{7} $$

The parameter "m" is the mode and the parameter "b" is the scale measure of the probability distribution.

Solving the integral in equation (6), with mortality defined by equation (7), we obtain a closed form (tractable) expression for the price of the life annuity:

$$ a_x = (1+L_x)\, b e^{(x-m)r} \exp\left(e^{(x-m)/b}\right) \Gamma\left(-br, e^{(x-m)/b}\right), \tag{8} $$

where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function.[9]

In our model, an individual with marketable wealth $W_0 = w$, at age x, can choose to annuitize all of this wealth by purchasing $c_x^w = \frac{w}{a_x}$ lifetime consumption units.

Alternatively, the individual can invest the $W_0 = w$ in a portfolio, earning a (continuously compounded) rate of return δ, and consuming the exact same (life annuity) amount c_x^w, until the individual runs out of money at some future time t^*, which may be infinite. By construction, the investors' wealth will obey the ordinary differential equation,

$$ W_t = \begin{cases} \left(w - \frac{c_x^w}{\delta}\right) e^{\delta t} + \frac{c_x^w}{\delta} & \text{for all } t < t^* \\ 0 & \text{for all } t \geq t^* \end{cases} \tag{9} $$

From a qualitative point of view, the constant multiplying the exponential function in equation (9) will be negative whenever the annuity payment c_x^w is greater than the perpetuity consumption defined by $w\delta$. The "negativity" implies that at some point in the future, the exponential term will overpower $+c_x^w/\delta$ and W_t will "hit" zero. Understandably, if the return δ is high enough, in other words $w > c_x^w/\delta$, then $t^* = \infty$, and one can consume forever. Solving for t^*, in terms of

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[8]One can "fit" the Gompertz distribution to the 1983 IAM table, the most common mortality table, to within 0.25 percent deviation in probability of death. Arguably, this approximation is good enough for our purposes, given the uncertainty we face in future investment returns and the analytic tractability of the Gompertz function.

[9]The Incomplete Gamma function is defined to be: $\Gamma(x,y) = \int_y^\infty e^{-t} t^{(x-1)} dt$, and is available in most mathematical software and spreadsheet packages.

the investment return δ in equation (9), and then substituting $a_x = w/c_x^w$, we obtain:

$$t^* = \begin{cases} -(\delta)^{-1}\ln[1-a_x\delta], & \text{for all } \delta < (a_x)^{-1} \\ \infty, & \text{for all } \delta \geq (a_x)^{-1} \end{cases}. \tag{10}$$

Of particular interest is when $\delta \geq (a_x)^{-1}$, the investor can safely beat the annuity indefinitely. In contrast, when $\delta < (a_x)^{-1}$, "shortfall" is certain conditional on being alive. The unconditional probability of consumption "shortfall" is the probability of surviving to time t^*, which, as per equation (7), is ${}_{t^*}p_x$.

Figure 1
Deterministic Evolution of Net Wealth
Consumption Equivalent to Annuity at 6%

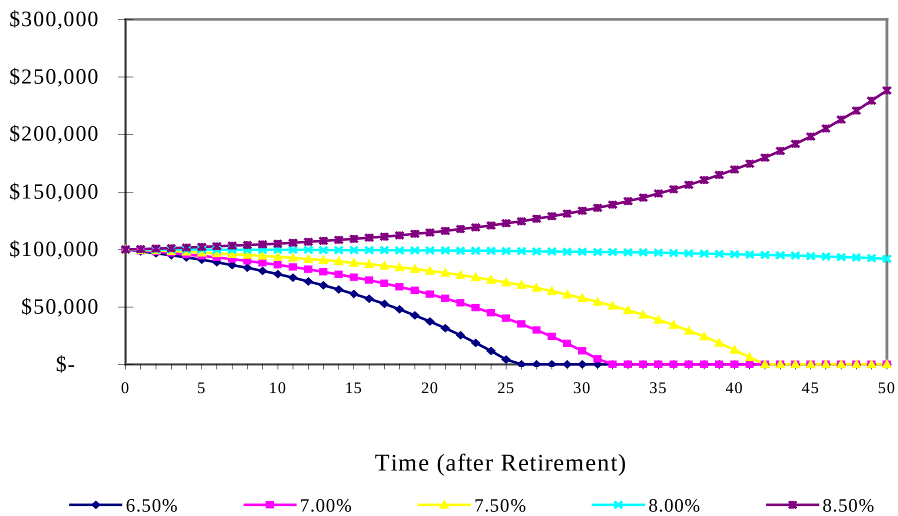

Figure 1 illustrates the dynamic evolution of net wealth, as per equation (9), using four different values for the parameter δ. For example, when $w = \$100{,}000$ and a_{65} = 12.459, the consumption rate $c_{65}^w = \frac{w}{a_{65}} = \$8{,}026$ per annum. Consequently, if the funds are invested at a rate δ = 7%, then t^* 30, as per equation (10), and the individual will run-out of funds in 30 years. Alternatively, when $\delta = 8.5\%$, the individual will be able to consume forever, and a bit more.

We have described the "buy-annuity" strategy and the "do-it-yourself" strategy. We now move our attention to the "do-it-yourself-and-then-switch" strategy. The main question of interest becomes, at what time s will the marketable wealth from equation (9) be equal to $c_x^w a_{x+s}$, the price of a continued lifetime consumption stream, c_x^w? This will be the point at which the individual should "switch" and annuitize wealth. In other words, for the first few years the consumer can earn more than the mortality adjusted return. Eventually, the "mortality credits" become so large that it becomes worthwhile to annuitize.

Mathematically, we are searching for the (waiting period) value of s, as an implicit function of the investment return, δ, that satisfies:

$$\begin{array}{ll} \underset{0\le s\le\infty}{\text{The Largest Value of :}} & \{s\} \\ s.t. & W_s / a_{x+s} \ge c_x^w \end{array} . \tag{11}$$

Equation (11) states that the individual "defers" annuitization until the last possible moment, the point in time, s^*, at which the original consumption stream is no longer affordable in the annuity market. Using equation (9) together with the technical condition that $s^* < t^*$, we can re-write equation (11) as:

$$\begin{array}{ll} \underset{0\le s\le t^*}{\text{The Largest Value of :}} & \{s\} \\ s.t. & e^{\delta s} \ge \frac{1/\delta - a_{x+s}}{1/\delta - a_x} \end{array} . \tag{12}$$

Now, due to the monotonicity (increasing in s) of $e^{\delta s}$ and (decreasing in s) of a_{x+s}, the optimization problem (12) can be solved to yield:

$$s^* = \begin{cases} \frac{1}{\delta}\ln\left[\frac{1/\delta - a_{x+s^*}}{1/\delta - a_x}\right], & \text{for all } \delta < (a_x)^{-1} \\ \infty, & \text{for all } \delta \ge (a_x)^{-1} \end{cases} . \tag{13}$$

Although the variable s^* appears on "both sides" of equation (13), solving for s^* is quite easy with the use of a spreadsheet or symbolic computational language, when the future annuity prices a_{x+s^*} can be stated with certainty.

STOCHASTIC MODEL

In practice, the decision to postpone the purchase of a life annuity, and the implicit formulation of the previous section, is hampered by three major sources of uncertainty: (1) stochastic *investment* returns, (2) stochastic *interest rates* and (3) stochastic *mortality* rates.[10] The stochastic investment return implies that the evolution of (non-annuitized) wealth does not obey the ordinary differential equation stipulated in equation (9). Thus, we do not know with certainty whether the investor will have enough money to purchase the exact same annuity in the future. The stochastic interest rate means that the discount factor applicable in the market and used by the insurance company to price annuities, fluctuates over time. The price of the same exact annuity in 5, 10, 15 or 20 years is uncertain. Finally, we do not know exactly what mortality table the insurance company will use when pricing the annuity in 5, 10, 15 or 20 years.

The above mentioned randomness can only be dealt with by imposing an exogenous structure on the *ex-ante* probability distribution governing financial uncertainty. With such a model in place, we can look into the future and compute the probability of being able to successfully defer annuitization. Of course, the parameters of such a model are country specific. Consequently, we decided to conduct our stochastic simulations using Canadian parameter estimates for investment returns, mortality and interest rates.[11]

Stochastic Investment Returns

Stochastic investment returns imply that we cannot assume a constant δ in the evolution of wealth dynamics. We therefore model continuously compounded investment returns, during any period in time, as normally distributed. This assumption is standard in financial economics and can be traced back to Boyle (1976) in the actuarial, risk and insurance literature. Consequently, in sharp contrast to the deterministic equation (9), the investors' portfolio will obey a Stochastic Differential Equation denoted by:

$$d\tilde{W}_t = \left(\mu\tilde{W}_t - c_x^w\right)dt + \sigma\tilde{W}_t dZ_t^k, \quad \tilde{W}_0 = w\,; \tag{14}$$

where μ is the growth rate of the portfolio (akin to δ in the deterministic case); σ is the "volatility" of investment returns; and dZ_t^k is the Brownian Motion (white noise) driving the uncertainty in investment returns, in other words: $Z_t^k - Z_0^k \sim N\left(0,\sqrt{t}\right)$. Equation (14) can be viewed as an Ordinary Differential Equation (ODE) perturbed by a (random) noise term proportional to $\tilde{W}_t$. Indeed, when the volatility term σ is zero, equation (14) collapses to its ODE counterpart, equation (9). The solution to equation (14) is:

$$\tilde{W}_t = e^{\left(\left(\mu-\sigma^2/2\right)t+\sigma Z_t^k\right)}\left(w - c_x^w \int e^{-\left(\left(\mu-\sigma^2/2\right)s+\sigma Z_s^k\right)}ds\right). \tag{15}$$

[10] We are grateful to an anonymous referee for pointing out the importance of the third source of uncertainty.

[11] This decision was made due to (a) the author's nationality and country of origin and (b) the availability of Canadian annuity data, described in detail in the section "Analysis of Canadian Life Annuity Quotes."

At any point in time t the stochastic wealth will exhibit a statistical distribution corresponding to the right hand side of equation (15). The actual parameters μ, σ depend on the composition of the investors' portfolio. However, it is taken for granted that they are efficient in the mean-variance sense. In the simulations, we used two categories of investment portfolio: conservative and aggressive. The aggressive investment assumes a well-diversified portfolio of equities, similar in composition to an index fund. The conservative investment assumes a 60:40 mix of equities and fixed income bonds.[12] The (historical) returns from both portfolios, net of management expense ratios and transaction costs, were computed[13] (and therefore assumed) to be $\mu_{aggr} = 13\%, \sigma_{aggr} = 17\%$, and $\mu_{cons} = 10\%, \sigma_{cons} = 12\%$.

Stochastic Interest Rates

In this paper we assume that the interest rate used by the insurance company to discount cash flows, the "pure" internal rate of return from the life annuity, obeys a mean reverting Stochastic Differential Equation:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma_r \sqrt{r_t}\, dZ_t^r , \tag{16}$$

where $\bar{r}$ is the long-run average level of the interest rate; γ is the speed of adjustment (the rate at which the interest rate is being pulled towards its long-run average); σ_r is the volatility of interest rates; Z_t^r is Brownian motion driving the interest rate process and $\rho = d\langle Z^r, Z^k \rangle$ denotes the correlation coefficient between investment returns and interest rates.[14]

This model of interest rate behavior was introduced by Cox, Ingersoll and Ross (1985) and has been applied widely in financial economics. According to this view of the world, interest rates follow a mean-reverting cyclical pattern over time in which the volatility of the interest rate is proportional to the square root of the level of interest rates. (See Chan, Karolyi, Longstaff and Sanders [1992] for detailed parameter estimates.)[15]

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[12]A more complete model would actually locate an optimal (μ, σ) so as to maximize the probability of being able to purchase the annuity in the future.

[13]Source: (a) Ibbotson Associates, (b) The Research Foundation of the Institute of Chartered Financial Analysts: *Canadian Stocks, Bonds, Bills and Inflation*: by J.E. Hatch and R.W. White, and (c) *The Financial Post* quarterly Mutual Fund Survey.

[14]The twelve years of monthly IRR's extracted from annuity prices were regressed against monthly stock index returns for the same period. The result was a statistically insignificant correlation coefficient. Therefore, in most of the simulations we conducted, we assumed that the correlation coefficient between equity returns and the annuity interest rate is zero. This is consistent with Ibbotson data, which show a very low correlation between Canadian equity returns and Government Bonds. In fact, we obtained virtually identical results when ρ was assumed to be anywhere between -20 percent and 20 percent.

[15]The empirical evidence is somewhat inconclusive with respect to the CIR diffusion as a model for interest rates, especially when applied to pricing derivative securities where a one percent error can translate into millions of dollars. However, we feel that for the purpose of our model, the CIR diffusion captures the essence of cyclical IRR dynamics.

Equation (16) can be solved in closed form to yield the following conditional distribution for interest rates at time s, given the interest rate at time $t < s$:

$$r_s \sim \chi^2\left[2b_1 r_t; 2b_3 + 2, 2b_2\right], \tag{17}$$

where

$$b_1 = \frac{2\gamma}{\sigma_r^2\left(1 - e^{-\gamma(s-t)}\right)}, \quad b_2 = \frac{2\gamma r_t e^{-\gamma(s-t)}}{\sigma_r^2\left(1 - e^{-\gamma(s-t)}\right)}, \quad b_3 = \frac{2\gamma\bar{r}}{\sigma_r^2} - 1.$$

The distribution function is a non-central chi-square with ($2b_3 + 2$) degrees of freedom and non-centrality parameter $2b_2$. For large values of s, the distribution will approach the gamma distribution.

Stochastic Mortality Rates

One does not know with certainty what mortality table the insurance company will use, in the event of the decision to postpone. Thus, we can only estimate the price of the exact same annuity in the future. In the simulation, we simplify the possibility of stochastic mortality rates by projecting the current annuity mortality table forward using the Scale G improvement factor. This conforms to the methodology used by Robert Johansen (1996). The stochastic model will thus "select" the appropriate mortality table for the deferral period in question. It is important to note that the individual does not have to estimate his, or her, own subjective mortality rate. The only mortality considerations lie in the pricing of the annuity, which is calculated with a population mortality table. Finally, consistent with the empirical evidence provided by Mitchell, Poterba and Warshawsky (1997), the insurance annuity loads, L_x, were assumed to increase with the age of the annuitant.[16]

Description of the Monte Carlo Simulation

With uncertainty in the model, one can never know if the decision to defer will be successful. The best we can do is to compute the probability of a successful deferral. In fact, if the retiree is willing to accept the ε chance (the risk) that the deferral will not be successful, we can compute the longest possible waiting period for which the probability of success is greater than $1 - \varepsilon$. The tolerance level ε will depend on the risk-aversion of the individual in question, but can easily be fixed at a conventional level of confidence equal to five percent. The crucial issue, therefore, is to compute the distribution of the random variable

[16]The tables provided by MPW (1997) indicate that the load can increase by amounts ranging from 0.25 percent to one percent per year, after age sixty-five. Most of our simulations were conducted using a 0.5 percent increase per annum. However, the results were not very sensitive to the exact "rate." The crucial variable was the initial load.

$$\tilde{c}(s) \equiv \frac{\tilde{W}_s}{\tilde{a}_{x+s}}, \tag{18}$$

which is the attainable consumption, at time s. Once we have the distribution of attainable consumption, we can compute the probability that it is greater than the original consumption level. In fact, we can go one step further and compute the probability that the attainable consumption is greater than some multiple, h, of the original consumption level. Mathematically we are interested in:

$$\Pr\left[\tilde{c}(s) \geq (h)\frac{W_0}{a_x} = (h)c_x^w\right]. \tag{19}$$

For example, the probability that the individual will be able to afford at least seventy-five percent of the original consumption level in ten years is denoted by $\Pr[\tilde{c}(10) \geq (0.75)c_x^w]$. Likewise, the probability that individual will be able to afford at least 125 percent of the original consumption level in fifteen years is denoted by $\Pr[\tilde{c}(15) \geq (1.25)c_x^w]$. By varying the parameter h around the level of $h = 1$, we gain information on the magnitude, in addition to the probability, of the shortfall risk. It should be intuitive that $\Pr[\tilde{c}(s) \geq (h)c_x^w]$ is a decreasing function of h because of the higher levels of wealth needed to fund the extra consumption multiple.

Due to the analytic complexity of the stochastic process $\tilde{c}(s)$ defined by equation (18), we performed a Monte Carlo simulation to obtain an empirical density function of $\tilde{c}(s)$ for values of s = 5, 10, 15, 20, 25 and 30. In particular, for each simulation run, our algorithm started with an initial wealth of $W_0 = 1$ and generated a vector of 25,000 random numbers for $\tilde{W}_s$, and a vector of 25,000 random numbers for $\tilde{a}_{x+s}$. The procedure then took the element-by-element ratio of the two vectors to obtain 25,000 random samples from the density function $\tilde{c}(s)$. The program then counted the number of elements in the random sample that were greater than 75 percent, 100 percent and 125 percent times the original consumption level c_x^w; thus providing an empirical estimate $\Pr[\tilde{c}(s) \geq (h)c_x^w]$.

Table 1

Probability of Success

65 Year Old Female: Probability of Successful Deferral with Aggressive Portfolio. Current Annuity Price: a_{65} = \$14.510, Internal Rate of Return: $r_o = 5\%.$, Insurance Load: $L_x = 0.10 + (0.005)s$; Portfolio Parameters: $\mu = 13\%$,   = 17%.

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
s	Prob.	Prob.	Prob.
5 yrs.	0.9972	0.8104	0.0004
10 yrs.	0.9552	0.8629	0.6208
15 yrs.	0.9307	0.8916	0.8313
20 yrs.	0.9223	0.9064	0.8873
25 yrs.	0.9177	0.9120	0.9058
30 yrs.	0.9124	0.9180	0.9023

65 Year Old Male: Probability of Successful Deferral with Aggressive Portfolio. Current Annuity Price: a_{65} = \$12.5553, Internal Rate of Return: $r_o = 5\%$, Insurance Load: $L_x = 0.10 + (0.005)s$; Portfolio Parameters: $\mu = 13\%$,   = 17%.

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
s	Prob.	Prob.	Prob.
5 yrs.	0.9928	0.7962	0.0139
10 yrs.	0.9264	0.8344	0.6512
15 yrs.	0.8896	0.8527	0.8042
20 yrs.	0.8710	0.8572	0.8418
25 yrs.	0.8564	0.8520	0.8475
30 yrs.	0.8423	0.8413	0.8403

Table 1 is a display of one of the simulation runs. In an $r_0 = 5\%$ interest rate environment, with an $L_{65} = 10\%$ insurance load, a sixty-five year old female can purchase a one-dollar per annum life annuity for a_{65} = \$14.5410. This translates into a consumption rate of $c_{65}^1 = \frac{1}{14.5410} = 0.0687$ dollars per annum, per dollar of initial wealth. If, on the other hand, the sixty-five year old female decides to (defer) and invest the initial wealth in an "aggressive" investment portfolio with expected return $\mu = 13\%$ and volatility $\sigma = 17\%$, the probability of a successful deferral is reported for various waiting periods and three different definitions of "success." For example, in fifteen years, the probability that she can afford an annuity that will provide at least 75 percent of the initial consumption rate, is 93.07 percent. The probability that she can afford an annuity that will provide at least 100 percent of the initial consumption rate, is 89.16 percent. This translates into an 11.84 percent probability of consumption shortfall. Finally, the probability that she can afford an annuity that will provide at least 125 percent of the initial consumption rate, is 83.13 percent. In other words, there is a six out of seven chance that she will able to buy a twenty-five percent increase in annuitized consumption, by waiting fifteen years.

An examination of Table 1 yields the following stylized observations. First, the probability of success is uniformly higher for females than for males. This is due to the cheaper annuity prices, (compare $a_{65} = 14.5410$ to $a_{65} = 12.5553$) which translate into a higher consumption rates for males. The higher consumption rate, in turn, means that there is a larger consumption flow from the do-it-yourself investment portfolio, which leaves a lower net wealth at the end of the deferral period.

A further point of interest is the asymptotic probabilities. Interestingly, as the waiting period s increases, the quantity $\Pr\left[\tilde{c}(s) \geq (h)c_x^w\right]$ plateaus at approximately ninety-one percent for females and eighty-four percent for males, independently of h. From a probabilistic point of view, this can be interpreted to mean that the affordable annuity consumption rate $\tilde{c}(s)$, is unlikely to be in the region between $(0.75)c_x^w$ and $(1.25)c_x^w$. It will either be much larger or much smaller than the initial consumption rate c_x^w. (A type of bi-modality.)

Comments and Discussion

The above mentioned probabilities are all conditional on survival, thus, they uniformly over-estimate the unconditional probability of consumption shortfall. For example, although the sixty-five year old female in Table 1 may experience an 11.84 percent probability of consumption shortfall if she waits for fifteen years before she annuitizes, (Canadian) population mortality tables indicate that she only has a seventy percent chance of reaching age eighty. Thus, the "true" probability of shortfall is closer to (0.7)(11.84) = 8.2 percent. The authors decided to report conditional probabilities of shortfall so as to (a) avoid the problem of estimating subjective probabilities of survival which may be unknown to the individual and (b) isolate the investment component of the decision to defer.

Table 2

Probability of Success

65 Year Old Female: Probability of Successful deferral with Aggressive Portfolio. Current Annuity Price: $a_{65} = \$10.6088$, Internal Rate of Return: $r_0 = 8.5\%$ Insurance Load: $L_x = 0.10 + (0.005)s$; Portfolio Parameters: $\mu = 13\%$, $\sigma = 17\%$

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
S	Prob.	Prob.	Prob.
5 yrs.	0.9766	0.7132	0.0348
10 yrs.	0.8454	0.7247	0.5400
15 yrs.	0.7809	0.7312	0.6723
20 yrs.	0.7494	0.7298	0.7088
25 yrs.	0.7282	0.7216	0.7148
30 yrs.	0.7105	0.7089	0.7072

65 Year Old Male: Probability of Successful deferral with Aggressive Portfolio. Current Annuity Price: $a_{65} = \$9.50813$, Internal Rate of Return: $r_0 = 8.5\%$; Insurance Load: $L_x = 0.10 + (0.005)s$; Portfolio Parameters: $\mu = 13\%, \sigma = 17\%$

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
s	Prob.	Prob.	Prob.
5 yrs.	0.9594	0.6893	0.0773
10 yrs.	0.7905	0.6801	0.5319
15 yrs.	0.7084	0.6674	0.6221
20 yrs.	0.6619	0.6480	0.6335
25 yrs.	0.6284	0.6246	0.6207
30 yrs.	0.6028	0.6021	0.6014

Table 3

Probability of Success

65 Year Old Female: Probability of Successful deferral with Conservative Port. Current Annuity Price: $a_{65} = \$15.2020$, Internal Rate of Return: $r_0 = 5\%$ Insurance Load: $L_x = 0.15 + (0.005)s$; Portfolio Parameters: $\mu = 10\%, \sigma = 12\%$

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
s	Prob.	Prob.	Prob.
5 yrs.	0.9999	0.8858	0.0
10 yrs.	0.9834	0.9024	0.5834
15 yrs.	0.9563	0.9149	0.8395
20 yrs.	0.9389	0.9202	0.8962
25 yrs.	0.9256	0.9183	0.9102
30 yrs.	0.9123	0.9102	0.9080

65 Year Old Male: Probability of Successful deferral with Conservative Port. Current Annuity Price: $a_{65} = \$13.1260$, Internal Rate of Return: $r_0 = 5\%$; Insurance Load: $L_x = 0.15 + (0.005)s$; Portfolio Parameters: $\mu = 10\%, \sigma = 12\%$

Wait	A: 75% of c_x^w	B: 100% of c_x^w	C: 125% of c_x^w
s	Prob.	Prob.	Prob.
5 yrs.	0.9996	0.8706	0.0011
10 yrs.	0.9628	0.8697	0.6253
15 yrs.	0.9122	0.8669	0.8022
20 yrs.	0.8743	0.8554	0.8340
25 yrs.	0.8408	0.8645	0.8279
30 yrs.	0.8094	0.8079	0.8065

Table 4

Values per Premium Dollar of Annuity Policies in Canada, 1984-1996

	Government of Canada 30-Year Bond	Government of Canada Term Structure	Corporate Term Structure
Annuity Table			
Male			
Average	0.9618	0.9588	0.9030
High	1.0488	0.9911	0.9421
Low	0.8808	0.8965	0.8415
Female			
Average	0.9612	0.9535	0.8948
High	1.0589	0.9988	0.9364
Low	0.8804	0.8911	0.8948
Population Table			
Male			
Average	0.9118	0.9126	0.8668
High	0.9905	0.9540	0.9067
Low	0.8320	0.8406	0.7951
Female			
Average	0.9298	0.9245	0.8727
High	1.0205	0.9669	0.9139
Low	0.8474	0.8620	0.8114

Source: CANNEX Financial Exchange (Median Quotes), 10-Year Guarantee RRSP Funds. C$50,000 purchase.
Annuity Mortality: IAM Tables, 1983-1996, Schedule "G" Adjustments
Population Mortality: Statistics Canada Life Tables
Interest Rates: Globe & Mail

The authors conducted extensive simulations with alternative parameter values. Table 2 and Table 3 display the results of some other simulation runs.[17] In general it appears that the probability of a successful deferral is most sensitive to (a) the current level of interest rates in the market, *vis a vis* the risk premium $\mu - r_0$, and (b) the annuity insurance load L_x. In contrast, the parameters of the interest rate process $(\bar{r}, \gamma, \sigma_r)$ and the annual increase in the annuity load have very little influence on the probability of a successful deferral. The authors believe this to be a direct manifestation of the above mentioned "plateau" effect. In other words, either the individual will have many times the amount of money needed to purchase the same exact life annuity, $\tilde{W}_s / \tilde{a}_{x+s} >> c_x^w$, or the individual will have very little funds with which to purchase the life annuity, $\tilde{W}_s / \tilde{a}_{x+s} << c_x^w$. Consequently, the uncertainty surrounding the future load and interest rate will have little effect on the probability of a successful deferral. In fact, *ceteris paribus*, a higher initial load

[17]The results of many simulations with alternative parameter values are available from the authors upon request.

L_{65} will result in a higher probability of a successful deferral because of the lower consumption rate the do-it-yourself strategy implies.

It also appears that when interest rates are low ($r_0 = 5\%$), the probability of a successful deferral is somewhat invariant to the composition of the investors portfolio.

In conclusion, simulations indicate that in the current low interest rate environment, a sixty-five year old female (male) has a ninety percent (eighty-five percent) chance of being able to beat the rate of return from a life annuity until age eighty.

The above table(s) display the results from a simulation with n = 25,000 runs in which the probability of a successful deferral is computed for various parameter values. The initial level of wealth is normalized to W_0 = 1, and the baseline consumption rate is set equal to $c_{65}^{w} = 1/a_{65}$. Column A displays the probability that the stochastic wealth $\tilde{W}_s$, at time s, will buy at least 75 percent of c_{65}^{w}, in annuities. Column B displays the probability that the stochastic wealth will buy *at least* c_{65}^{w} in annuities. (An exact deferral.) Column C displays the probability that the stochastic wealth will buy *at least* 125 percent of c_{65}^{w} in annuities. Note: We model pure annuity interest rates (IRR's) as mean reverting with a long-term value of $\bar{r} = 8.5\%$, volatility of $\sigma_r = 8\%$, and speed of adjustment parameter of $\gamma = 25\%$. We assume $\rho = 0$ correlation between IRR's and portfolio returns and we assume a proportional insurance load L_x that increases $\frac{1}{2}\%$ annually with age. Mortality: 1996 Individual Annuity Basic Table with dynamic Scale G projection.

ANALYSIS OF CANADIAN LIFE ANNUITY QUOTES

In this section we analyze Canadian data on life annuity quotes in order to (a) obtain an estimate of the embedded load factor L_X and (b) derive parameter ($\bar{r}, \gamma, \sigma_r$) estimates for the stochastic process "driving" the internal rate of return of the life annuity.

To compute loads, we must compare the pure actuarial values to the indicative annuity quotes. Therefore, we must first use the term structure of interest rates in the Canadian economy in order to compute the pure actuarial values. Once this is achieved, we can solve for the mortality adjusted IRR from the annuity – net of loads.[18] The data was supplied by CANNEX Financial Exchanges Limited, an intermediary that compiles payout rates from the most competitive insurance providers in Canada and sells the information to brokers and financial planners.

Description of the Database

The actual data is in the form of a twelve-year time series of monthly payout rates for a single life annuity for a sixty-five year old with a ten year guarantee, per

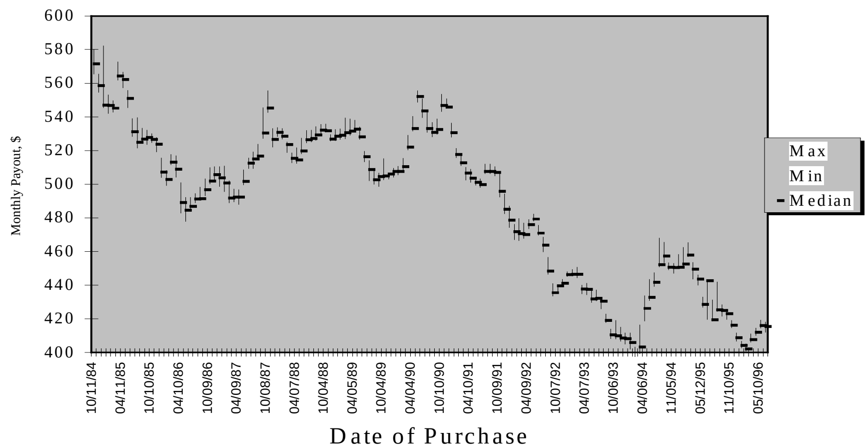
[18]We want to compute the "pure" IRR dynamics without any contamination from the insurance loads.

C$50,000 tax sheltered funds.[19] In other words, the data tracks how much lifetime consumption C$50,000 would buy.

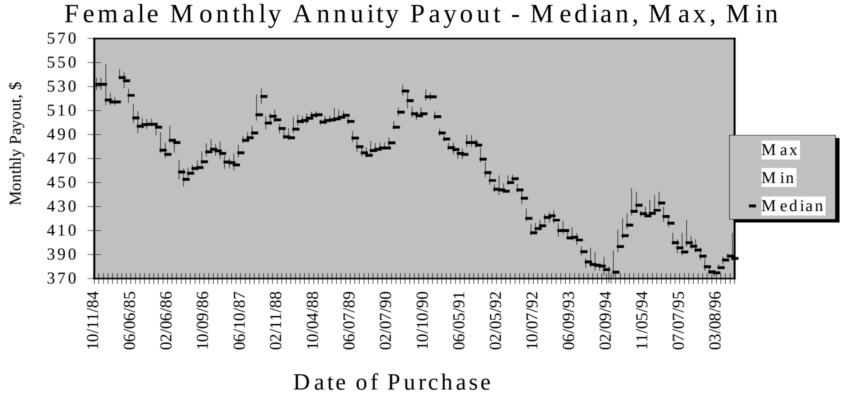
Figure 2 is a graphical display of the time series evolution of the top-ten (most competitive) payout rates from 1984 to 1996. Some stylized, perhaps obvious, facts emerge from a casual examination of the numbers:

Figure 2

Male Monthly Annuity Payout - Median, Max, Min



Female Monthly Annuity Payout - Median, Max, Min



(1) Payout rates fluctuate from month-to-month and are highly correlated with prevailing interest rates in the market. Over the 1984-1997 time frame, the highest payout rate was $571 for males and $544 for females, both in October

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[19]The funds must originate from a Registered Retirement Savings Plan. Indeed, one can obtain slightly more favorable payout rates from tax sheltered (known as qualified) funds due to the possible mitigation of adverse selection.

1984. Likewise, the lowest payout rate was \$374 for males and \$345 for females, both in December 1996.

(2) Males, sixty-five years old, obtain approximately six percent more consumption (monthly payout) than females. In other words, females pay six percent more for the same consumption stream. This "markup" is purely a function of female longevity. The "markup" has ranged from 4.7 percent to 8.4 percent.

(3) There are very substantial benefits to searching. In any particular month, the best and worst payout quotes varied by as much as 6.7 percent, per monthly payout, per C\$50,000. Identical financial products, in a competitive market, do not usually display such wide pricing variations. Indeed, we suspect that the non-competitive quotes are simply an indication of an insurance company that is trying to "turn off" the annuity "tap."

From here on we will focus on the time-series of median payout rates from the top ten firms. Our intention is to compute the Canadian value per premium dollar, using the methodology of Warshawsky (1988) and Mitchell, Poterba and Warshawsky (1997).

Methodology

We use closing quotes, for both corporate and government of Canada bonds, as reported in the business section of the *Globe and Mail*, to extract the term structure of interest rates. In particular, we partition the period into monthly segments and then extracted an appropriate monthly interest rate R_i by solving for an interest rate that would equate the cash flow from the bonds to its present value.[20]

With a monthly vector of interest rates R_i, we "solved" for the implied load factor L_{65} using the present value definition:

$$\$50{,}000 = (\text{quote})_t \times (1 + (L_{65})_t)\left(\sum_{i=1}^{120} \frac{1}{(1+R_i)^i} + \sum_{i=121}^{540} \frac{{}_i p_{65}}{(1+R_i)^i}\right), \tag{20}$$

which leads to,

$$\frac{\$50{,}000}{(\text{quote})_t}\left(\sum_{i=1}^{120} \frac{1}{(1+R_i)^i} + \sum_{i=121}^{540} \frac{{}_i p_{65}}{(1+R_i)^i}\right)^i - 1 = (L_{65})_t. \tag{21}$$

The term $(\text{quote})_t$ denotes the monthly annuity payout rate quoted at time t. The term ${}_i p_{65}$ denotes (with slight abuse of notation) the probability that a sixty-five year old (male and/or female) will survive i more months. The term $(L_{65})_t$ denotes the insurance load that is extracted from market prices at time t. The rationale behind the two summations in equation (20) and (21) is that the first ten years are

[20]On a technical note, the Canadian yield curve extends up to only thirty years, thus the remaining rates, needed for the forty-five years of possible annuity consumption, were taken by extrapolation.

guaranteed, thus there is no mortality contingent component in the numerator. If indeed we were dealing with a straight life annuity, the factor in the right hand side of equation (20) and (21) would be: $\sum_{i=1}^{540} \frac{{}_i p_{65}}{(1+R_i)^i}$.

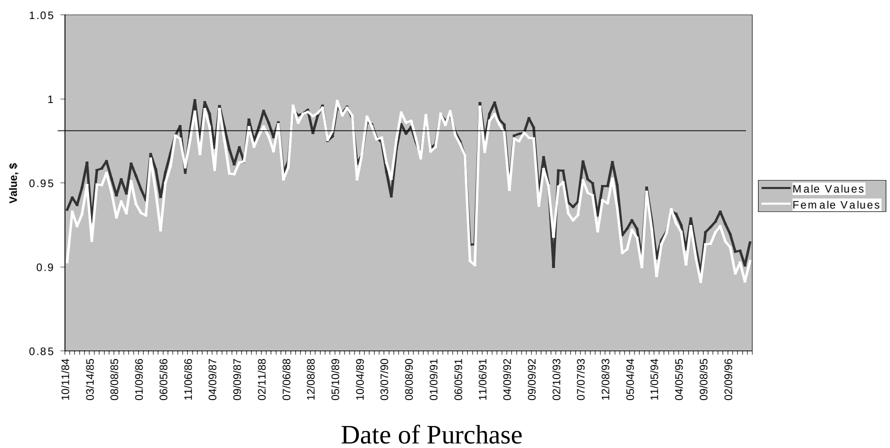
We performed our analysis with two sets of mortality tables. We used the 1983 Johansen Individual Annuity Basic Table[21] and projected forward, using Scale G, to the date at which the annuity was issued. The 1983 Table is considered the standard for individual annuity pricing in both Canada and the United States. For the purpose of comparison, we also used the population mortality tables as provided by the federal agency Statistics Canada,[22] covering the period in question.

Results

The reciprocal of one-plus the insurance load is known as the Value per Premium Dollar (VPD) in the annuity pricing literature. An actuarially fair annuity has VPD equal to one. The higher the insurance load, the lower is the VPD.

Figure 3

Value per Premium Dollar of Annuity Policies in Canada 1984- 1996
Using Government of Canada Term Structure



Each observation has an associated VPD, which is extracted using equation (21). Figure 3 illustrates the time series behavior of the VPD over the sample period, using the government of Canada bond term structure of interest rates together with the annuity mortality table. The (government) VPD's range from 0.93 in the early part of the sample, to approximately 1.00 in the late eighties and early nineties, and then back to 0.90 towards the end of the sample. Being that private annuities are

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[21]We used the 1983 Basic Table (Johansen, Table 10) and not the 1983 Table "a" (Johansen, Table 16), which is the former table with a ten percent loading subtracted and regraduated. The reason being that the authors want to actually estimate the embedded load factor.

[22]Health Reports Supplement No. 13, Life Tables, Canada and the Provinces 1985-1987. Statistics Canada: 85 82-003S.

offered by non-government entities, it is probably more appropriate to use the corporate term structure for discounting cash flows in equation (21).

Figure 4
Value per Premium Dollar of Annuity Policies in Canada, 1984-1996
Using Corporate Term Structure

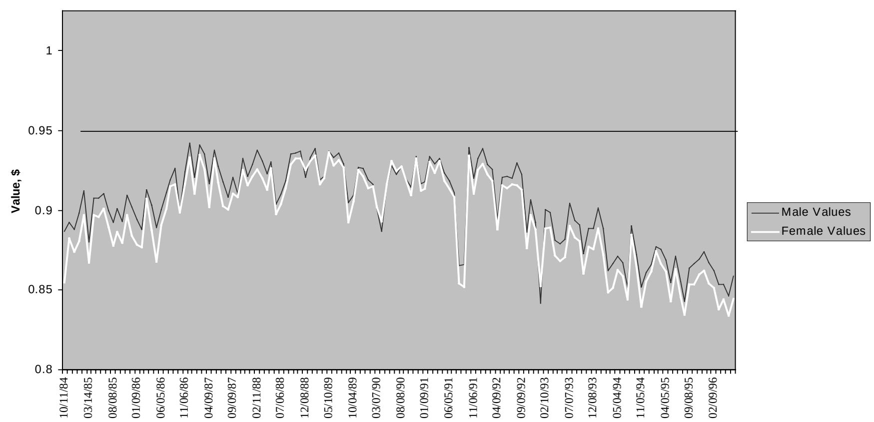

Figure 4 illustrates the time series behavior of the VPD over the sample period, using the Canadian corporate bond term structure of interest rates. These VPD's range from 0.88 in the early part of the sample, to approximately 0.93 in the late eighties and early nineties, and then back to 0.85 towards the end of the sample.

Table 4 tabulates the results from our analysis, in the same format as Mitchell, Poterba and Warshawsky (1997), by displaying the average Value per Premium Dollar using the alternative discounting and mortality methodologies, over the 1984-1996 period. As one would expect from the adverse selection implicit in annuity tables, the VPD's are lower when computed using population tables.

Internal Rate of Return Dynamics

Given the estimates of the insurance load, we can extract it from the annuity quote and then "back out" the internal rate of return (IRR) net of loads, using equation (4). Figure 5 illustrates the time series of derived IRR's from the life annuity and compares it to the yield on a long-term government bond.

The IRR's range from a high of 11.55 percent in the early eighties, to a low of about 5.95 percent in the early nineties. As one would expect, the numbers are virtually identical for males and females.[23] The sample average is 8.49 percent with a standard deviation of 7.91 percent. Finally, using the non-central chi square distribution from equation (17), we obtain a point estimate of 25.1 percent for the speed of adjustment parameter.

[23]This serves as a "check" on the mortality table methodology.

Figure 5
Canada Long Bond Yield
Vs.
Implied Rates of Return (Male & Female)

Date of Purchase

Summary of Annuity Data

The Value per Premium Dollar during the 1984-1996 period, using the annuity mortality table, implied an average insurance load of approximately $L_{65} = 12\%$ when discounting at the corporate rate and about $L_{65} = 6\%$ when discounting at the risk free government rate. Likewise, the Internal Rate of Return - net of loads - from the life annuity was "fitted" to the diffusion process in equation (16) to yield the parameter estimates of $\bar{r} = 8.49\%$, $\gamma = 25.1\%$, and $\sigma_r = 7.91\%$ over the sample period. These estimates were used, amongst others, in the simulation model described in the section "Stochastic Model."[24]

CONCLUSION AND DIRECTIONS FOR RESEARCH

Purchasing a life annuity implies paying a non-refundable lump sum to an insurance company in exchange for a guaranteed constant life-long consumption stream that can not be outlived. This purchase is nonreversible, eliminates the possibility of a bequest and should logically be deferred as long as possible, provided that the consumption stream generated by the annuity can be maintained to within an acceptable probability-of-shortfall tolerance level.

[24]However, as mentioned earlier, the simulation results were highly invariant to the exact specifications of the parameters $\left(\bar{r}, \gamma, \sigma_r\right)$.

This paper developed a stochastic simulation model that computes the probability of a successful deferral - one minus the probability of shortfall - as a function of initial age, waiting time, investment returns, insurance loads and current interest rates. Our stochastic model is predicated on mean reverting interest rate dynamics, normally distributed investment returns together with standard mortality table assumptions. In addition, we use Canadian annuity payout data to estimate (a) the insurance loads and (b) the IRR's implicit in the contract. Consequently, given the current low interest rate environment and the long-run propensity for equities to outperform fixed income investments,[25] we apply our model to estimate that a sixty-five year old female (male) has a ninety percent (eighty-five percent) chance of being able to beat the rate of return from a life annuity until age eighty. Of course, those who consider a ten percent (fifteen percent) probability of shortfall unacceptable may choose to immediately lock-in the current annuity rate.

Further Questions

There are some unanswered issues that we must ponder:

(a) In the presence of income taxes there are benefits in receiving life annuity payments because of the favorable mortality tables used by the tax authorities to amortize capital. We therefore ask: Perhaps it becomes harder to "beat" the annuity on an after-tax basis?
(b) Nominal annuities are not protected against inflation. How does inflation affect the decision to annuitize? The are some insurance companies that offer pseudo-indexed (increasing) annuities. How does their cost structure compare with those of regular annuities? Can one beat the *real* rate of return from an annuity?
(c) Variable Immediate Annuities (VIA) provide some form of market participation in conjunction with lifetime protection at the expense of fluctuating payouts[26] for an excellent description of these products. At first glance, VIA's seem to have the best of both worlds, aside from the bequest issue. Can one "beat" the return from a Variable Immediate Annuity? What are the probabilities of shortfall?

In the time-honored tradition, we leave this for further research.

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[25]Although the interaction between equity markets, interest rates and inflation in the short run is still debated in the academic literature, we refer the interested reader to the comprehensive work by Jeremy Seigel (1995), *Stocks for the Long Run*, in which he documents the 200 year history of North American financial markets, and concludes that equities consistently outperform inflation and fixed income securities over the long run.
[26]See Daily (1994).

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