

## INVESTMENT STRATEGIES IN DEFINED-CONTRIBUTION PENSION SCHEMES

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**ABSTRACT.** We present a dynamic model for optimal investment decisions in defined contribution pension plans. The model determines an optimal fraction of pensioner's savings to be invested in an equity fund, with the rest invested in a bond fund. Since it is difficult to estimate the model parameters exactly, we present sensitivity analysis with respect to various relevant parameters and stress-testing of optimal investment decisions under different equity return scenarios. The model is applied to the actual Slovak DC scheme.

### 1. INTRODUCTION

In recent decades numerous OECD countries introduced privately managed defined contribution (DC) pension plans into their pension systems to complement or replace already existing public schemes. Privately managed DC schemes work on a basis of regular contributions of individual workers to their own pension accounts. The wealth accumulated via these contributions is continually managed by pension funds, which invest in the financial assets such as equities, bonds or cash. Some countries such as Slovakia, Poland or Hungary have actually cut contribution rates in DC schemes or in some way disadvantaged the DC plans as a response to the crisis in 2008. The main goal of this paper is to analyze the level of pensions from the second pillar of the Slovak pension system according to the last legislative changes. Especially, the decrease of the contributions to the funded pillar in Slovakia from 9% to 4% induced a necessity of new calculations. We use the dynamic stochastic accumulation model introduced firstly in [7] and later generalized in [9]. The model determines the optimal fraction of savings to be invested in the equity fund (with the rest in the bond fund), given specific time to retirement, level of accumulated wealth and actual short-term interest rate. Authors in [9] assumed existence of two funds – the bond fund, represented by 1-year zero coupon bonds and the equity fund whose risk-return characteristics corresponded to the US stock index S&P500 during 1996–2002. We generalize the model from [9] to account for any duration of the bond fund. Next, we conduct

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a sensitivity analysis of the model outcomes to all relevant parameters. Most importantly, we perform stress-testing with respect to the most sensitive as well as the most unpredictable parameter – equity returns. To achieve this we utilize real historical stock index scenarios as well as artificially created ones. We present our results on the current Slovak DC scheme and calibrate all our models by latest available data. The achieved levels of savings are recalculated to the replacement rates using non-indexed annuities.

## 2. MODEL

Suppose that a future pensioner deposits once a year a  $\tau_t$ -part of his/her yearly salary  $w_t$  to a pension fund with a  $\delta$ -part of assets in stocks and a  $(1 - \delta)$ -part of assets in bonds, where  $\delta \in [0, 1]$ . Denote by  $s_t$ ,  $t = 1, 2, \dots, T$ , the accumulated sum at time  $t$ , where  $T$  is the expected retirement time. Then the budget-constraint equations read as follows:

$$(1) \quad s_{t+1} = \delta s_t \exp(R^s(t, t+1)) + (1 - \delta) s_t \exp(R^b(t, t+1)) + w_{t+1} \tau_{t+1}.$$

For practical reasons, the quantity  $d_t = s_t/w_t$  is more appropriate (cf. [9]). It can be easily recalculated to the replacement ratio (pension payment/salary), which is the most important value for pensioners. Using  $d_t$  instead of  $s_t$ , one can reformulate the budget-constraint equation (1) as follows:

$$d_{t+1} = d_t \frac{\delta \exp(R^s(t, t+1)) + (1 - \delta) \exp(R^b(t, t+1))}{1 + \beta_t} + \tau_{t+1}, \quad t = 1, 2, \dots, T - 1,$$

where  $d_1 = \tau_1$  and  $\beta_t$  denotes the wage growth  $w_{t+1} = w_t(1 + \beta_t)$ .

The term structure development is driven by one factor Cox-Ingersoll-Ross (CIR) short-rate model presented in [3]

$$(2) \quad dr_t = \kappa(\theta - r_t)dt + \sigma^b \sqrt{r_t} dZ_t, \quad \kappa, \theta, \sigma^b > 0,$$

where  $r_t$  stands for a short rate,  $Z_t$  is the Wiener process,  $\theta$  is the long term interest rate,  $\kappa$  is the rate of reversion and  $\sigma^b$  is the volatility of the process. Suppose that the bond part of the portfolio has duration  $T_b$ . The corresponding return can be modeled using zero coupon bonds. Denote by  $P(t, T_b)$  the price (at time  $t$ ) of zero coupon bond with face value 1 and time to maturity  $T_b$ . In CIR model (see [3]) the term structure of zero coupon bonds can be expressed by explicit formula

$$P(t, T_b) = P(r_t, t, T_b) = A(T_b) e^{-B(T_b)r_t},$$

where

$$A(T_b) = \left( \frac{2\gamma e^{\frac{(\kappa + \lambda + \gamma)T_b}{2}}}{(\kappa + \lambda + \gamma)(e^{\gamma T_b} - 1) + 2\gamma} \right)^{\frac{2\kappa\theta}{\sigma^2}},$$

$$B(T_b) = \frac{2(e^{\gamma T_b} - 1)}{(\kappa + \lambda + \gamma)(e^{\gamma T_b} - 1) + 2\gamma},$$

$$\gamma = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}.$$

Hence

$$R^b(t, t+1) = r_t B(T_b) - \ln(A(T_b)) - r_{t+1} B(T_b - 1) + \ln(A(T_b - 1)).$$

Using a discretization of the short rate process (2), we have (see e.g., [15] or [1])

$$(3) \quad r_{t+1} = g(r_t, \Phi) = \theta + e^{-\kappa}(r_t - \theta) + \left( \sigma^b \sqrt{\frac{r_t}{2\kappa}(1 - e^{-2\kappa})} \right) \Phi,$$

where  $\Phi \sim N(0, 1)$ .

We shall assume the stock prices  $S_t$  are driven by geometric Brownian motion. The annual stock return  $R^s(t, t+1) = \ln(S_{t+1}/S_t)$  can be therefore expressed as  $R^s(t, t+1) = \mu_t^s + \sigma_t^s \Psi$ , where  $\mu_t^s$  and  $\sigma_t^s$  are the mean value and volatility of annual stock returns in the time interval  $[t, t+1)$ ,  $\Psi \sim N(0, 1)$  is a normally distributed random variable. The random variables  $\Phi, \Psi$  are assumed to have 2-dimensional normal distribution with the correlation coefficient  $\rho = E(\Phi\Psi) \in (-1, 1)$ .

Suppose that each year the saver has the possibility to choose a level of stocks included in the portfolio  $\delta_t(I_t)$ , where  $I_t$  denotes the information set consisting of the accumulated wealth  $d_t$ , the history of bond and stock returns and wage growths up to time  $t$ . We suppose that the forecast of the wage growth is deterministic, the stock returns are assumed to be random, independent for different times and the interest rates are driven by the Markov process (2). Then the quantities  $d_t$  and the short rate  $r_t$  are the only relevant information. Hence  $\delta_t(I_t) \equiv \delta_t(d_t, r_t)$ . One can formulate a problem of stochastic dynamic programming

$$(4) \quad \max_{\delta} E(U(d_T))$$

subject to the following recurrent budget constraints

$$(5) \quad d_{t+1} = F_t(d_t, r_t, \delta_t(d_t, r_t), \Phi, \Psi), \quad t = 1, 2, \dots, T-1,$$

where  $d_1 = \tau_1$ ,

$$(6) \quad F_t(d, r, \delta, x, y) = d \frac{\delta e^{\mu_t^s + \sigma_t^s y} + (1 - \delta) e^{rB(T_b) - \ln(A(T_b)) - g(r, x)B(T_b - 1) + \ln(A(T_b - 1))}}{1 + \beta_t} + \tau_{t+1}$$

and the short rate process is driven by (2) and (3) with a given initial short rate  $r_1$ . We assume the stock part of the portfolio is bounded by a given upper barrier function  $\Delta_t: 0 \leq \delta_t(d_t, r_t) \leq \Delta_t$ . The function  $\Delta_t: \{1, \dots, T-1\} \mapsto [0, 1]$  is subject to governmental regulations. In our modeling we use the constant relative risk aversion (CRRA) utility function  $U(d) = -d^{1-a}$ ,  $d > 0$ , where  $a > 1$  is the constant coefficient of relative risk aversion. The model is a generalization of the one presented in [9], where the bond part of the portfolio was represented by zero coupon bonds with time to maturity  $T_b = 1$ .

Let us denote by  $V_t(d, r)$  saver's intermediate utility function at time  $t$  defined as

$$(7) \quad V_t(d, r) = \max_{0 \leq \delta \leq \Delta_t} E(U(d_T) | d_t = d, r_t = r).$$

Then, by using the law of iterated expectations, we obtain the Bellman equation

$$(8) \quad V_t(d, r) = \max_{0 \leq \delta \leq \Delta_t} E[V_{t+1}(F_t(d, r, \delta, \Phi, \Psi), g(r, \Phi))]$$

for every  $d, r > 0$  and  $t = 1, 2, \dots, T - 1$ . Using  $V_T(d, r) = U(d)$ , the optimal strategy can be calculated backwards. One can prove (using the same arguments as [9]) that there exists a unique argument of the maximum in (8)  $\hat{\delta}_t = \hat{\delta}_t(d_t, r_t)$ , i.e.,

$$(9) \quad V_t(d, r) = E[V_{t+1}(F_t(d, r, \hat{\delta}_t(d, r), \Phi, \Psi), g(r, \Phi))].$$

Since the distribution of the random vector  $(\Psi, \Phi)^\top$  is known, the equation (9) can be solved numerically. To be more specific, the density function parametrized by the correlation coefficient  $\rho$  is given by

$$f_\rho(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}$$

and from the definition of expected value, we may rewrite (9) as

$$\max_{0 \leq \delta \leq \Delta_t} \int_{\mathbb{R}^2} V_{t+1}(F_t(d, r, \delta, x, y), g(r, x)) f_\rho(x, y) dx dy$$

and subsequently simplify by a substitution ( $x = \xi\sqrt{1-\rho^2} + \rho y$ ) to

$$(10) \quad \max_{0 \leq \delta \leq \Delta_t} \int_{\mathbb{R}^2} V_{t+1}(F_t(d, r, \delta, \xi\sqrt{1-\rho^2} + \rho y, y), g(r, \xi\sqrt{1-\rho^2} + \rho y)) \times f_0(\xi, y) d\xi dy.$$

For the sake of brevity, we do not discuss a numerical procedure for solving equation (10) here and refer the reader to [9].

### 2.1. Model without future contributions

A simplified model with positive initial investment  $d_1 > 0$  and no future contributions (i.e.,  $\tau_t = 0$  for  $t > 1$ ) was considered by Samuelson in [12], and more explicitly Hakanson [5]. Furthermore, they supposed independent (in time) and identically distributed returns of risky assets and one risk-free asset with deterministic returns. Using the CRRA class of utility functions, they concluded that optimal portfolio composition is independent of time and level of savings.

In our model, bonds are not riskless assets and their returns are not independent in time. However, in the case of no future contributions, the optimal proportions of assets are independent of the level of savings.

**Proposition 2.1.** *Consider the problem (4)–(6) with  $d_1 > 0$ ,  $\tau_t = 0$  for  $t > 1$  (i.e., there are no future contributions) and CRRA utility function  $U = -d^{1-a}$ . Then the value function (7) has the form*

$$V_t(d_t, r_t) = L_t(r_t)U(d_t).$$

Moreover, the optimal strategy  $\hat{\delta}_t$  does not depend on the level of savings  $d_t$ , i.e.,

$$\hat{\delta}_t(d_t, r_t) \equiv \hat{\delta}_t(r_t).$$

*Proof.* We use the backward mathematical induction for  $t = T, T-1, \dots, 1$ . By definition of the value function, we have

$$V_T(d_T, r_T) = U(d_T).$$

Hence

$$V_T(d_T, r_T) = L_T(r_T)U(d_T)$$

with  $L_T(r_T) \equiv 1$ , the statement is obvious for  $t = T$ .

Suppose that  $V_{t+1}(d_{t+1}, r_{t+1}) = L_{t+1}(r_{t+1})U(d_{t+1})$ . Using (6) with  $\tau_{t+1} = 0$  and (8), we have

$$\begin{aligned} V_t(d_t, r_t) &= \max_{0 \leq \delta_t \leq \Delta_t} \mathbb{E} [V_{t+1}(F_t(d_t, r_t, \Phi, \Psi, \delta_t), g(r_t, \Phi))] \\ &= \max_{0 \leq \delta_t \leq \Delta_t} \mathbb{E} [L_{t+1}(g(r_t, \Phi))U(F_t(d_t, r_t, \Phi, \Psi, \delta_t))] \\ &= \max_{0 \leq \delta_t \leq \Delta_t} \mathbb{E} \left[ L_{t+1}(g(r_t, \Phi))U \left( d_t \frac{\delta_t e^{\xi_t} + (1 - \delta_t) e^{\eta_t}}{1 + \beta_t} \right) \right] \\ &= -d_t^{1-a} \min_{0 \leq \delta_t \leq \Delta_t} \mathbb{E} \left[ L_{t+1}(g(r_t, \Phi)) \left( \frac{\delta_t e^{\xi_t} + (1 - \delta_t) e^{\eta_t}}{1 + \beta_t} \right)^{1-a} \right] \\ &= U(d_t)L_t(r_t), \end{aligned}$$

where

$$\begin{aligned} \xi_t &:= \mu_t^s + \sigma_t^s y, \\ \eta_t &:= B(T_b)r_t - \ln A(T_b) - g(r_t, \Phi)B(T_b - 1) + \ln A(T_b - 1), \end{aligned} \tag{11}$$

$$L_t(r_t) := \min_{0 \leq \delta_t \leq \Delta_t} \mathbb{E} \left[ L_{t+1}(g(r_t, \Phi)) \left( \frac{\delta_t e^{\xi_t} + (1 - \delta_t) e^{\eta_t}}{1 + \beta_t} \right)^{1-a} \right].$$

By (11) the optimal strategy  $\hat{\delta}_t$  does not depend on the level of savings  $d_t$ .  $\square$

### 3. BASELINE SCENARIO

#### 3.1. The Slovak pension system

Pensions in Slovakia are operated by a three-pillar system:

1. the public, compulsory, non-funded first pillar (pay-as-you-go),
2. the private, voluntary, fully funded second pillar,
3. the private, voluntary, fully funded third pillar.

The contribution rate is currently set at 18% for the first pillar (in case a pensioner decides to stay only in the public scheme) or 14% for the first pillar and 4% for the second pillar (in case a pensioner decides to save in both pillars)<sup>1</sup>. The savings in the funded pillar are managed by pension asset managers. Each asset manager operating in the second pillar is obliged to manage two funds – a *Guaranteed*

<sup>1</sup>The contribution rate to the private pillar has been recently cut from 9% to 4% with future planned increase to 6%. The development of the contribution rate according to the latest legislative changes is presented in Tab. 1.

*Bond Fund*<sup>2</sup> and a *Non-guaranteed Equity fund* plus any number of additional funds. Savers have a possibility of holding all assets in any fund of their choice or to split the assets into two funds (one of which has to be a Guaranteed fund) by any ratio they choose. This ratio can be changed in time and is subject to the governmental regulations during the last years of a savings process. When approaching retirement, the fraction of savings in a Guaranteed fund has to be gradually increased (see Tab. 2) and is required to reach 100% 3 years before retirement.

**Table 1.** Left: Forecast of interannual gross wage growth in Slovakia. Specific values for years 2013–2015 are the average forecasts of the National Bank of Slovakia, Institute of Financial Policy and Slovenska Sporitelna. Data for years 2016–2051 are from [8]. Right: The contribution rate as a percentage of a gross wage. Source: Law on pension savings no. 43/2004 (as of June 1, 2014).

Year	Wage growth	Year	Contributions
2013	4.37%	2013–2016	4.0 %
2014	4.75 %	2017	4.25 %
2015	5.2 %	2018	4.5 %
2016–2020	6.4 %	2019	4.75 %
2021–2025	5.9 %	2020	5.0 %
2026–2030	5.6 %	2021	5.25 %
2031–2035	5.2 %	2022	5.5 %
2036–2040	4.9 %	2023	5.75 %
2041–2051	4.5 %	2024–2051	6.0 %

**Table 2.** Legislative restrictions on the proportion of savings in equity funds. Source: Law on pension savings no. 43/2004 (as of June 1, 2014).

Age of saver	Year of saving	Maximum % of stocks	$\Delta_t$
$\leq 49$	1.–28.	100 %	1
50–58	29.–37.	$10 \times (59 - \text{age}) \%$	$0.1 \times (59 - \text{age})$
$\geq 59$	38.–40.	0 %	0

### 3.2. Parameters and Data

We have supposed a saving period of  $T = 40$  years. Parameters of the CIR model were estimated<sup>3</sup> using maximum likelihood method published in [2]. The specific values of the parameters are  $\kappa = 0.8993$ ,  $\theta = 0.0226$ ,  $\sigma^b = 0.148$ . It is worth to note that estimated parameters are close to ones used in [9], which were taken from [14]. The maturity of zero coupon bonds representing the duration of the

<sup>2</sup>Guaranteed fund is obliged to deliver a non-negative performance, net of costs, during any rolling 10-year period.

<sup>3</sup>Daily data from period 1999-2012, source: <http://www.euribor-info.com/en/eonia>.

guaranteed fund was set to  $T_b = 3$ . We have used the coefficient of the relative risk aversion  $a = 9$  (the same as [9]). Nominal wage growth in Slovakia (Tab. 1) over the next 40 years was obtained (nominal wage growth) from the most recent available forecasts.

The basic value of the drift  $\mu^s$  and the volatility of the stock part of the portfolio  $\sigma^s$  were estimated from historical annualized monthly returns of the U.S. stock market index S&P 500 including reinvested dividends (total return)<sup>4</sup>. In our calculations we have used value  $\mu^s = 8.44\%$  p.a.,  $\sigma^s = 14.17\%$  p.a. (estimates from the period 1871–2012). The correlation of the stock and bond parts of the portfolio was estimated using historical data<sup>5</sup>. In our calculations we have used the estimate from the period between 1962–2012,  $\rho = -0.01082$ .

### 3.3. Results for the baseline scenario

The output of the model is the function  $\hat{\delta}_t(d_t, r_t)$  representing the optimal proportion of savings invested in equity funds, provided that we are in the  $t$ -th year of saving, the current short rate is  $r_t$  and we have already saved  $d_t$  yearly salaries. The development of the average level of savings and average proportion of the stock investment with standard deviations for 100 000 Monte Carlo simulations can be found in Fig 1. Using the basic model parameters, the average terminal level of savings is relatively low (around 2.5 times of the yearly salary, see also Tab. 3). This is mainly due to low contributions and relatively high wage growth. The right graph shows that at the beginning of saving, the model recommends to invest all savings in the stock fund. The reason is simple. Possible negative return of the stock fund has a small impact on future pension since essential part of the contributions is expected in the future. Later on, return of the stock fund has higher impact on the final level of savings (the ratio of future contributions to the level of savings is lower). Therefore, the decreasing tendency of stock investments is natural. The linear decrease in the last years is due to governmental regulations. The governmental regulations supplemented with high wage growth are the reasons of stagnant level of savings in the last years before retirement.

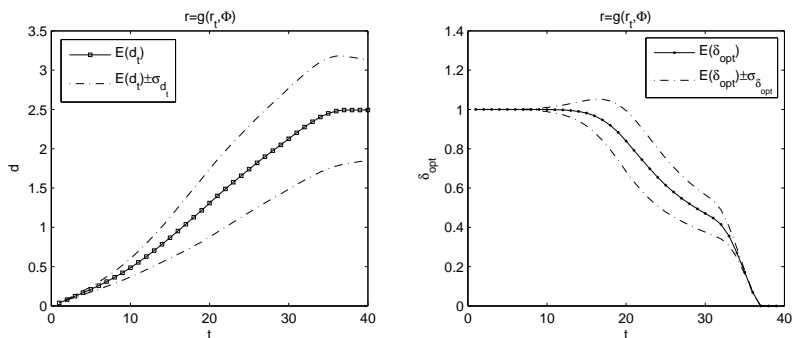
### 3.4. Sensitivity analysis

It is difficult to forecast the model parameters exactly. Therefore, we have performed simulations for the following modifications (for  $t = 1, 2, \dots, 39$ ) of the baseline scenario:

- (M0) Baseline scenario.
- (M1) Contributions  $\tau_t = 4\%$ .
- (M2) Contributions  $\tau_t = 9\%$ .
- (M3) No governmental regulations for the stock fund, i.e.,  $\Delta_t = 1$ .
- (M4) Lower aversion to risk  $a = 5$ .
- (M5) Higher duration of the bond fund  $T_b = 5$ .

<sup>4</sup>Monthly data, source: <http://www.econ.yale.edu/~shiller/data.htm>

<sup>5</sup>S&P 500, daily data, source: <http://finance.yahoo.com/>. US short rate (Effective Federal Funds Rate), source: <http://research.stlouisfed.org/fred2/>.



**Figure 1.** The development of the average level of savings (left) and average proportion of the stock investment with standard deviations (right).

(M6) Lower wage growth  $\tilde{\beta}_{4-39} = \beta_{4-39} - 1\%$ .

(M7) Lower drift of the stock returns  $\mu_t^s = 5\%$ .

(M8) Linear growth of the drift of the stock returns  $\mu_t^s = 2\% + 0.25(t-1)\%$ .

(M9) Higher volatility of the stock returns  $\sigma_t^s = 20\%$ .

(M10) Forbidden mixing of stock and bond funds, i.e.,  $\delta_t \in \{0, 1\}$ .

Expected values of the final level of savings  $E(d_T)$ , standard deviations  $\sigma(d_T)$ , catastrophic scenarios represented by 5% quantiles  $Q_{5\%}(d_T)$  and certainty equivalents (CE) defined as  $U^{-1}[E(U(d_T))]$  (i.e., a certain value having the same utility as the random result of the strategy) can be found in Tab. 3. One can observe that the final level of savings is most of all sensitive to the contribution rate and the drift of the stock returns.

**Table 3.** Sensitivity analysis – comparison with the baseline scenario.

Modification	$E(d_T)$	$\sigma(d_T)$	$Q_{5\%}(d_T)$	CE
(M0)	2.4947	0.6441	1.6226	1.9304
(M1)	1.7922	0.4747	1.1454	1.3591
(M2)	4.0357	1.0757	2.5808	3.0676
(M3)	2.8063	0.8028	1.7302	2.0361
(M4)	2.9284	1.1535	1.5875	2.2103
(M5)	2.4984	0.6487	1.6195	1.9266
(M6)	2.9597	0.7774	1.8997	2.2569
(M7)	1.6873	0.2326	1.3415	1.5550
(M8)	2.2122	0.5093	1.5049	1.7900
(M9)	2.1803	0.4912	1.4893	1.7719
(M10)	2.0326	0.4924	1.4054	1.6857



## 4. STRESS-TESTING

## 4.1. Scenarios and strategies

It is very difficult to forecast the model parameters associated with asset returns. Especially, the estimates of the drifts of the stock returns are usually unreliable. Hence, we tested selected strategies against a set of different models for the equity fund returns. The model for the bond fund was the same as the one used in the previous section. We considered the following drift scenarios  $\mu_t^s$ :

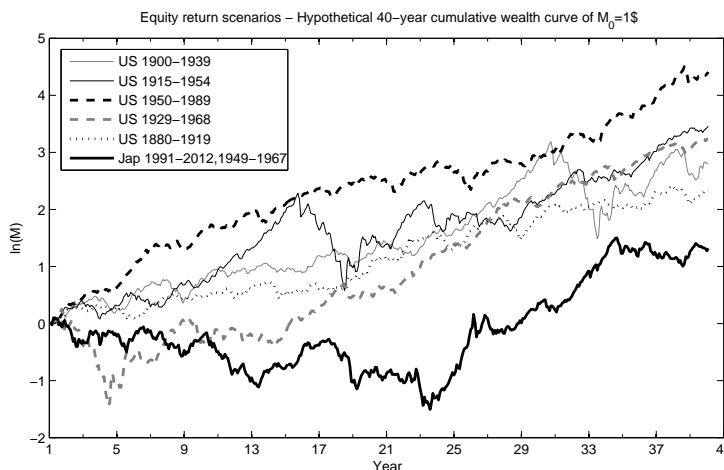
- (SC1)  $\mu_t^s = 11\%$  during the entire saving period.
- (SC2)  $\mu_t^s = 9\%$  during the entire saving period.
- (SC3)  $\mu_t^s = 7\%$  during the entire saving period.
- (SC4)  $\mu_t^s = 5\%$  during the entire saving period.
- (SC5) Linear growth of the drift from 2% to 11.5%  $\mu_t^s = 2\% + 0.25(t - 1)\%$ .
- (SC6) S&P 500 (1900–1939): growth scenario with depression at the end.
- (SC7) S&P 500 (1915–1954): stagnation, boom, recession and recovery.
- (SC8) S&P 500 (1950–1989): long-term healthy growth scenario.
- (SC9) S&P 500 (1929–1968): recession, recovery and growth.
- (SC10) S&P 500 (1880–1919): stagnation and modest growth.
- (SC11) Nikkei 225 (1991–2012, 1949–1967): long-term recession and recovery.

Scenarios (SC6)–(SC11) based on historical returns of the stock indices are summarized in Fig. 2. We tested 15 strategies (ST1)–(ST15) against the set of 11 scenarios (SC1)–(SC11). Strategies (ST1)–(ST11) are the optimal ones according to our dynamic model, i.e., the optimal value  $\hat{\delta}_t(d_t, r_t)$  for the strategy (ST $i$ ) is calculated supposing that the scenario (SC $i$ ) takes place. (ST12) and (ST13) invest all the savings to bond and equity funds, respectively. Strategy (ST14) begins with the investment in the equity fund and each year linearly moves the savings into the bond fund, i.e.,  $\delta_t = \max\{0, 1 - \frac{t-1}{36}\}$ . The last one follows a popular rule “invest (100-age)% to stocks”. Assuming the savings period from the age 22 to the age 62, the ratio of equity investments under this strategy is  $\delta_t = \min\{\Delta_t, 1 - \frac{t+22}{100}\}$ .

## 4.2. Stress-testing: The outcome

For each pair (strategy  $i$ , scenario  $j$ ) 100 000 Monte Carlo simulations were performed supposing that strategy  $i$  is applied and scenario  $j$  takes place. Using the simulations, values of certainty equivalent (CE) indicator were calculated. Results are presented in Tab. 4. One can observe that strategies (ST6)–(ST11) achieve high values in the case when the corresponding scenarios (SC6)–(SC11) take place. On the other hand, they are not so flexible as the other strategies in the case a different scenario occurs.

A natural question which strategy can be regarded as the best under all circumstances arises. The answer to this question obviously depends on how we define an evaluation criterion for the strategies. For some savers it could be, e.g., a strategy that has the highest mean value of the final level of savings averaged from all scenarios, i.e., *Max-Mean* approach. Risk-takers would prefer the strategy with



**Figure 2.** Scenarios (SC6)–(SC11) used in stress-testing. S&P 500, daily data, source: <http://finance.yahoo.com/>. Historical annual returns of the Japanese stock index NIKKEI225, source: <http://indexes.nikkei.co.jp/en/nkave/archives/data>.

**Table 4.** Certainty equivalents CE using various strategies and scenarios.

	(SC1)	(SC2)	(SC3)	(SC4)	(SC5)	(SC6)	(SC7)	(SC8)	(SC9)	(SC10)	(SC11)
(ST1)	2.40	2.00	1.71	1.48	1.76	1.87	1.81	1.94	2.85	1.72	2.64
(ST2)	2.37	2.01	1.73	1.50	1.78	1.83	1.87	2.01	2.72	1.77	2.58
(ST3)	2.28	1.99	1.74	1.54	1.77	1.82	1.94	2.05	2.48	1.77	2.30
(ST4)	2.05	1.87	1.70	1.56	1.70	1.76	1.90	1.95	2.13	1.70	1.92
(ST5)	2.29	1.99	1.73	1.52	1.79	1.80	1.97	2.02	2.62	1.76	2.58
(ST6)	1.91	1.72	1.57	1.44	1.60	3.78	1.33	1.61	2.03	1.74	1.86
(ST7)	1.90	1.73	1.57	1.45	1.61	1.73	4.67	1.68	1.92	1.43	2.49
(ST8)	2.04	1.81	1.62	1.46	1.66	1.83	1.85	3.11	2.51	2.00	1.72
(ST9)	2.01	1.79	1.61	1.46	1.67	1.45	1.45	1.83	4.39	1.41	3.11
(ST10)	1.89	1.72	1.57	1.45	1.60	1.88	1.81	1.93	2.09	2.69	1.38
(ST11)	1.82	1.66	1.53	1.42	1.59	1.45	1.70	1.63	2.26	1.25	6.98
(ST12)	1.40	1.40	1.40	1.40	1.40	1.40	1.40	1.40	1.40	1.40	1.40
(ST13)	2.38	1.99	1.69	1.46	1.76	1.90	1.76	1.91	2.89	1.68	2.58
(ST14)	1.95	1.78	1.64	1.53	1.63	1.73	1.77	1.82	2.05	1.67	1.87
(ST15)	2.16	1.90	1.70	1.53	1.76	1.73	1.88	1.96	2.44	1.68	2.40

the highest value of the CE indicator for the best scenario, i.e., *Max-Max* criterion. Risk averse investors would probably use *Max-Min* approach (maximizing the value for the worst scenario). The best strategies using mentioned criteria are following: for the *Max-Min* criterion, strategy (ST4) is optimal for the *Max-Mean* approach, (ST11) should be used. This is mainly due to high value of the indicator in the case scenario (SC11) occurs. It is in accord with the fact that (ST11) is the winning strategy for the *Max-Max* criterion as well.

5. ANNUITIES FROM THE SECOND PILLAR

Consider a person of age  $x$  years. Denote the probability that this person dies within the next year by  $q_x$ . One-year probabilities of death  $q_x$  are usually given in life tables. By  ${}_k p_x$  denote the probability that the person of age  $x$  will survive at least  $k$  consecutive years. Then  ${}_k p_x = \prod_{h=0}^{k-1} (1 - q_{x+h})$  for  $k = 1, 2, \dots$ . Consider a life annuity-due which provides for annual payments of 1 unit as long as the beneficiary lives (payments are made at the end of each year). Denote by  $a_x$  the net present value of the annuity payments. Then  $a_x = \sum_{k=1}^{\infty} {}_k p_x (1+i)^{-k}$ , where  $i$  represents the annual technical interest rate. In reality pension benefits are not paid annually, but usually with a monthly frequency. In this case one has

$$a_x^{(12)} \sim \left( \sum_{k=1}^{\infty} {}_k p_x (1+i)^{-k} \right) + \frac{11}{24},$$

where  $a_x^{(12)}$  represents the net present value of an annuity of 1 unit per year payable 12 times per year (1/12 unit per month) until the policyholders death (cf. [4]).

Denote by  $M$  the annual annuity payment payed monthly expressed as a fraction of the last yearly salary  $w_T$  before retirement. This value is called replacement rate. Based on the assumption of net premium principle, we have

$$d_T = M a_x^{(12)} \rightarrow M = \frac{d_T}{a_x^{(12)}} \sim \frac{d_T}{\left( \sum_{k=1}^{\infty} {}_k p_x (1+i)^{-k} \right) + \frac{11}{24}}.$$

In Tab. 5, we present replacement rates  $M$  for different levels of savings and technical interest rate. The calculations were performed for  $x = 62$ . We applied current probabilities of death drawn from [13] (static unisex life tables, year 2012). To illustrate the calculated levels of replacement rates, let us consider a person

**Table 5.** Replacement rates  $M$  for different levels of savings and technical interest rate without the longevity assumption.

$d_T/i$	0.00 %	0.50 %	1.00 %	1.50 %	2.00 %	2.50 %	3.00 %
2.0	0.11	0.11	0.12	0.13	0.13	0.14	0.15
2.5	0.13	0.14	0.15	0.16	0.17	0.17	0.18
3.0	0.16	0.17	0.18	0.19	0.20	0.21	0.22
3.5	0.19	0.20	0.21	0.22	0.23	0.24	0.26
4.0	0.21	0.23	0.24	0.25	0.27	0.28	0.29

contributing to the second pillar 6% of the gross wage<sup>6</sup> (i.e., 1/3 of old-age contributions). This saver will receive 2/3 of the pension from the first pillar designed for 50% replacement rate. Therefore, the saving pillar is efficient for this person if it delivers at least 17% replacement rate. Using Tab. 5, one can see that such a replacement rate needs at least 2.5–3 yearly salaries saved (depending on the

<sup>6</sup>The average contribution rate from Tab. 1 is 5.63%. We used close value of 6% for clearer illustration.

technical interest rate). Recall that the average level of savings using the baseline scenario was 2.5 yearly salaries. Considering the risk associated with saving, one can conclude that reaching the first pillar (50%) replacement rate is quite questionable.

When calculating the replacement rates in Tab. 5, we used current probabilities of death, not taking into account the potential longevity of pensioners. Clearly, if the pensioners live longer than we expect, the replacement rates will be lower. To model the effect of longevity on replacement rates, we have applied the Lee-Carter (LC) approach published in [10]. We prepared a database of mortality rates based on previous life tables of Statistical Office of Slovak Republic from the years 1996 to 2013 [13]. Using the `demography` package [6] in statistical software R [11], we estimated the age-specific parameters and time-varying index of LC model. Then by using ARIMA time series, we forecasted future mortality rates for the next 40 years.

**Table 6.** Replacement rates  $M'$  for different levels of savings and technical interest rate with respect to the Lee-Carter model of longevity.

$d_T/i$	0.00 %	0.50 %	1.00 %	1.50 %	2.00 %	2.50 %	3.00 %
2.0	0.10	0.10	0.11	0.12	0.12	0.13	0.14
2.5	0.12	0.13	0.14	0.15	0.15	0.16	0.17
3.0	0.15	0.16	0.17	0.18	0.19	0.20	0.21
3.5	0.17	0.18	0.19	0.21	0.22	0.23	0.24
4.0	0.20	0.21	0.22	0.23	0.25	0.26	0.27

Denote by  $q'_{x+t-1}(t)$ ,  $t = 1, 2, \dots, 40$  the LC-forecasted one-year probability of death in the  $t$ -th year of the predicted time period for an aging pensioner. Analogously, let  ${}_t p'_x$  be the LC-forecasted survival factor defined by  ${}_t p'_x = \prod_{h=0}^{t-1} (1 - q'_{x+h}(h+1))$  for  $t = 1, 2, \dots, 40$ . Then the net replacement rate  $M'$  with respect to the Lee-Carter model of longevity with survival factors  ${}_1 p'_x, {}_2 p'_x, \dots, {}_{40} p'_x$  can be approximated as

$$M' \sim \frac{d_T}{\left(\sum_{k=1}^{\infty} {}_k p'_x (1+i)^{-k}\right) + \frac{11}{24}}.$$

In Tab. 6, we present the replacement rates  $M'$  for different levels of savings and technical interest rate. The calculations were performed (as in the previous case) for  $x = 62$ . In comparison with Tab. 5, we received slightly lower replacement rates.

## 6. CONCLUSIONS

We extended the dynamic stochastic model introduced firstly in [7] and later generalized in [9]. The last legislative changes in Slovakia allow the pension asset managers to increase the duration of the bond fund. Therefore, we generalized the model to account for any duration. The model can be utilized in any other DC

scheme. For better understanding of the results, we recalculated the final levels of savings to the replacement rates. The achievement of the first pillar (50%) replacement rate is not certain. Since it is very difficult to estimate the parameters of the model, we performed a sensitivity analysis for various parameters settings. The final level of savings is most of all sensitive to the contribution rate and the drifts of the stock returns. The estimates of the drifts of the stock returns are usually unreliable. Therefore, we considered several strategies which were tested against a set of scenarios of the drifts. For a particular investor, the optimal strategy depends on the preferred criterion.

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