Sign-changing solutions of the nonlinear heat equation with positive initial value

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We consider the nonlinear heat equation $u_t - \Delta u = |u|^\alpha u$ on $\mathbb{R}^N$, where $\alpha > 0$. It is well known that the Cauchy problem is locally well-posed in a variety of spaces. For instance, for every $\alpha > 0$, it is well-posed in the space $C_0(\mathbb{R}^N)$ of continuous functions that converge to 0 at infinity. It is also well-posed in $L^p(\mathbb{R}^N)$ for $p \geq 1$, $p > \frac{N\alpha}{2}$, but not well-posed in L^p for $1 \leq p < \frac{N\alpha}{2}$ if $\alpha > \frac{2}{N}$. In particular, for such p there exist positive initial values $u_0 \in L^p$ for which there is no local in time positive solution. Also, if one considers the initial value $u_0(x) = c|x|^{-\frac{2}{\alpha}}$ for all $x \in \mathbb{R}^N \setminus \{0\}$, with $c > 0$, it is known that if c is small, there exists a global in time (positive) solution with u_0 as initial value, and in fact this solution is self-similar. On the other hand, if c is large, there is no local in time positive solution, self-similar or otherwise. We prove that in the range $\frac{2}{N-2} < \alpha < \frac{4}{N-2}$, for every $c > 0$, there exist infinitely many self-similar solutions to the Cauchy problem with initial value $u_0(x) = c|x|^{-\frac{2}{\alpha}}$. Of course, these solutions are sign-changing if c is sufficiently large. Also, for the same range of α, we prove the existence of local in time sign-changing solutions for a class of nonnegative initial values $u_0 \in L^p$, for $1 \leq p < \frac{N\alpha}{2}$, for which no local in time positive solution exists.

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