solutions for those parameters p, q, a, λ or N for which the results in the above mentioned papers are not satisfactory.

In the case of (NBC) (the problem with **N**onlinear **B**oundary **C**onditions), the study was almost completely done in [CFQ] for $N = 1$. Particularly, it was shown that the exponent $p = 2q - 1$ is critical for the blow up in the following sense:

(i) if $p < 2q - 1$ (or $p = 2q - 1$ and $a < q$) then there exist solutions of (NBC) which blow up in finite time,
(ii) if $p > 2q - 1$ (or $p = 2q - 1$ and $a > q$) then all solutions of (NBC) exist globally and are globally bounded,
(iii) if $p = 2q - 1$ and $a = q$ then all nontrivial solutions of (NBC) exist globally but they are unbounded; they tend pointwise to a singular stationary solution.

The assertions (i) and (ii) were shown also for Ω being a ball in $\mathbb{R}^N$, $N > 1$. However, if Ω is a general bounded domain in $\mathbb{R}^N$, $N > 1$, then [CFQ] or [E] imply blow up of suitable solutions of (NBC) only for $p \le q$ and the global existence and boundedness is shown in [CFQ] only for $p > c(q)$, $q < \frac{N+1}{N-1}$, where

$$c(q) := \frac{N - q(N-2)}{N + 1 - q(N-1)}(q+1) - 1 > 2q - 1.$$

The main result of this paper concerning the global existence for (NBC) in the case of a general domain Ω is the following:

(a) if $p > 2q - 1$ then all solutions of (NBC) exist globally and are globally bounded,
(b) if $p < 2q - 1$ (or $p = 2q - 1$ and a is sufficiently small) then there exist initial functions u_0 such that the corresponding solutions of (NBC) blow up in $L^\infty(\Omega)$-norm.

It has to be mentioned that in the case (b) we do not know whether the blow up occurs in finite or infinite time. We find only a subsolution u^+ such that any positive stationary solution has to intersect u^+ so that the solution of (NBC) starting at u^+ cannot be bounded. In the case (a) we show that a simple substitution leads to the case $p > c(q)$ which was already solved in [CFQ]. Hence, for $p > 2q - 1$ we obtain global existence, boundedness and also the existence of a positive stationary solution of (NBC).

Considering the (positive) stationary solutions of (NBC) we are mainly interested in the case $q < p < 2q - 1$, $N > 1$. The results of [CFQ] imply that in this case there exists $a_0 \ge 0$ such that the stationary problem corresponding to (NBC) has

(j) no positive solutions for $a < a_0$,
(jj) at least one positive solution for $a > a_0$,
(jjj) in the subcritical case ($q < \frac{N}{N-2}$) at least two positive solutions for $a \in \{a_1, a_2, \cdots\}$, where $a_k \to \infty$.

Moreover, if Ω is a ball then $a_0 > 0$ and (NBC) has at least one positive stationary symmetric solution for $a = a_0$ and at least two positive stationary symmetric solutions for $a > a_0$ (see [CFQ] for a more precise information for $N = 1$). The main

difficulty in proving this additional property for a general domain is the absence of apriori estimates for stationary solutions. In this paper we show that for a general domain Ω

(α) $a_0 > 0$,
(β) in the subcritical case, (NBC) has at least two positive stationary solutions for almost all $a > a_0$.

Moreover, for Ω being a ball in $\mathbb{R}^2$ we find apriori estimates for all positive stationary solutions (note that the apriori estimates in [CFQ] for $N > 1$ concern only symmetric solutions).

The proof of (α) is based on the apriori estimate of $\min\{u(x)\,;\, x \in \partial\Omega\}$, where u is any positive stationary solution. The proof of (β) is based on a trick of M. Struwe [S1].

Concerning the problem (DGT) (the problem with **D**amping **G**radient **T**erm), it is known that for $p > q > 1$ blow-up of solutions in L^∞-norm in finite time can occur (see [CW1], [KP], [F], [Q1]) while for $p \le q$ any solution u is bounded in $[0,T) \times \overline{\Omega}$, where T is the maximal existence time for u (see [F]). In this paper we show that if the existence time T of a solution u of (DGT) is finite then $\lim_{t\to T-} \|u(t,\cdot)\|_{L^\infty(\Omega)} = +\infty$. Consequently, the solution exists globally if $p \le q$.

Our main results concerning the stationary solutions of (DGT) are the following:

(k) if $q \ge p$ then there exists $\lambda_0 > 0$ such that the stationary problem corresponding to (DGT) has
  (k1) no positive solutions for $\lambda < \lambda_0$,
  (k2) at least one positive solution for $\lambda = \lambda_0$,
  (k3) at least two positive solutions for $\lambda > \lambda_0$,
(kk) if $p < (N+2)/(N-2)$ (in the case $N > 2$) and $q < \min\{2, (N+2)/N\}$ then there exists $\lambda_0 \ge 0$ such that (DGT) has at least one positive stationary solution for any $\lambda > \lambda_0$ (see Theorem 6.2 and Remark 6.2).

The stationary problem for (DGT) was studied also in [AW], [C], [CW1], [CW2], [FQ2], [SZ]. However, these studies concern mostly the case where Ω is a ball in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$ when one can make use of the symmetry of the solution and apply time map technique (shooting method). Let us also emphasize that in the case (k) we do not need any subcriticality condition for p or q since we work with dynamical methods in this case. The proof of (kk) is based on the use of Leray-Schauder degree and the apriori estimates from [BT] and [FLN]. Finally, let us mention that in the case (k), i.e. $q \ge p$,

(kα) if $p = q$ then $\lambda_0 \ge p\,\mathrm{diam}\,(\Omega)^{-p}$ (see [C, Theorem 4]),
(kβ) if $q = 2$ and $N = 1$ then there are exactly two positive solutions for $\lambda > \lambda_0$ (see [S, Example 3.2.2])

and in the case $p < (N+2)/(N-2)$ and Ω being a ball in $\mathbb{R}^N$

(kkα) if $q < 2p/(p+1)$ ($\Rightarrow \quad q < \min(2, (N+2)/N)$) then there exists a positive solution for any $\lambda > 0$. Moreover, this solution is unique if $N = 1$ (see [C, Theorem 3]).
(kkβ) If $q = 2p/(p+1)$ and $\lambda \le (2p)^p/(p+1)^{2p+1}$ then (DGT) does not have

positive stationary solutions. The estimate on λ is precise if and only if $N=1$ (see [C, Theorem 3], [FQ2]).

(kkγ) If $q > 2p/(p+1)$ and $\lambda > 0$ is small then (DGT) does not have positive stationary solutions (see the proof of Theorem 3 (iii) in [C]).

2. Global existence for (NBC).

In this section we show that the assumption $p > 2q-1$ implies the global existence and boundedness of solutions of (NBC). Our results also imply the existence of a positive stationary solution for (NBC), since the zero solution is unstable.

Due to the results of [CFQ], it is sufficient to consider the case $N > 1$. As shown in [CFQ], (NBC) generates a local semiflow in $\{u \in W^{1,r}(\Omega)\,;\, u \geq 0\}$ for any $r > N$. Hence, we shall suppose $u_0 \in W^{1,r}(\Omega)$ for some $r > N$, $u_0 \geq 0$.

In [CFQ], the estimates from [FK] were used to get the global existence and boundedness results for (NBC) under the assumption

$$q < \frac{N+1}{N-1}, \qquad p > c(q) := \frac{N-q(N-2)}{N+1-q(N-1)}(q+1) - 1. \tag{2.1}$$

We use this information and a simple substitution to get the desired result.

Theorem 2.1. *If $p > 2q-1 > 1$ then any solution of* (NBC) *exists globally and stays uniformly bounded.*

Proof: Let u be a maximal solution of (NBC), $m \geq 1$. Then $v := u^m$ solves the problem

$$\begin{cases} v_t = \triangle v - \dfrac{m-1}{m}\dfrac{1}{v}|\nabla v|^2 - mav^{p^*} & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial v}{\partial n} = mv^{q^*} & \text{on } (0,T)\times\partial\Omega, \\ v(0,x) = u_0^m(x) & x \in \overline{\Omega}, \end{cases} \tag{2.2}$$

where $p^* = (p+m-1)/m$, $q^* = (q+m-1)/m$ and T is the maximal existence time for u. Using the comparison principle one simply gets $v \leq w$, where w solves the problem

$$\begin{cases} w_t = \triangle w - maw^{p^*} & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial w}{\partial n} = mw^{q^*} & \text{on } (0,T)\times\partial\Omega, \\ w(0,x) = u_0^m(x) & x \in \overline{\Omega}. \end{cases} \tag{2.3}$$

Now it is sufficient to verify that the couple (p^*, q^*) fulfils the condition (2.1) iff

$$2m > (q-1)(N-1) \quad \text{and} \quad 2m(p+1-2q) > (q-1)\big((p-1)(N-1)-(q-1)(N-2)\big),$$

which is clearly true if m is sufficiently large. Hence, for m large one can apply the results of [CFQ] for the solution w to get its global existence and boundedness and, consequently, also the global existence and boundedness for v and u (it is obvious that the linear factor m in (2.3) does not play any significant role in [CFQ, Theorem 4.6]). $\square$

Corollary 2.1. *If $p > 2q-1$ then* (NBC) *has a positive stationary solution.*

PROOF: Put $u_0 \equiv \varepsilon$, where $\varepsilon > 0$ is small enough. Since (NBC) possesses a Lyapunov function Φ (see [CFQ]) and $\Phi(u_0) < 0 = \Phi(0)$, the ω-limit set of the solution starting at u_0 consists of (nonnegative) equilibria which are different from 0. Due to the maximum principle, these equilibria are positive. □

3. Blow up for (NBC).

In this section we shall suppose that $p \le 2q-1$ (and a is small enough if $p = 2q-1$) and we shall show that there exists a solution of (NBC) which blows up (in finite or infinite time). As a by-product of our considerations we obtain also an apriori bound for $\min\limits_{x\in\partial\Omega} u(x)$, where u is any positive stationary solution of (NBC).

Lemma 3.1. *Let $\alpha > 2$ be fixed and $u_\delta(x) := \left[\frac{1}{\varepsilon}\big(\delta - dist(x,\partial\Omega)\big)^+\right]^\alpha$, where $\delta > 0$, $\alpha\varepsilon^{\alpha(q-1)} = \delta^{\alpha(q-1)+1}$ and $v^+ := \max(v,0)$. If δ is sufficiently small then u_δ is a subsolution for* (NBC) *and any positive stationary solution u of* (NBC) *fulfils $\min\limits_{\partial\Omega}(u - u_\delta) < 0$.*

PROOF: One can easily verify that u_δ fulfils the boundary condition in (NBC) for any $\delta > 0$. Further suppose that $\operatorname{dist}(x,\partial\Omega) \le \delta$ and δ is sufficiently small. Denoting $d(x) := \operatorname{dist}(x,\partial\Omega)$ and $\varphi(d) := \left[\frac{1}{\varepsilon}(\delta - d)^+\right]^\alpha$ one has $u_\delta(x) = \varphi\big(d(x)\big)$ and

$$\triangle u_\delta = (\varphi'' \circ d)|\nabla d|^2 + (\varphi' \circ d)\triangle d.$$

Let $y = y(x) \in \partial\Omega$ be the closest point to x in $\partial\Omega$ and let $n = n(x)$ be the unit (outward) normal to $\partial\Omega$ at $y(x)$. Then we have

$$|\nabla d|^2 = \Big(\frac{\partial d}{\partial n}\Big)^2 = 1, \quad \frac{\partial^2 d}{\partial n^2} = 0, \quad |\triangle d| \le C,$$

where C is some constant depending only on the curvature of $\partial\Omega$ (cf. [GT, Lemmas 14.16 and 14.17]). Using these estimates and the inequality $p \le 2q-1$ (and $a \ll 1$ if $p = 2q-1$) one can easily check that

$$\triangle u_\delta \ge \frac{1}{2}\varphi'' \circ d \ge a\varphi^p \circ d = au_\delta^p$$

for δ sufficiently small, where the inequalities are strict if $d(x) < \delta$. Hence u_δ is a (strict) subsolution for $\delta \le \delta_0$ and $u_\delta\big|\partial\Omega = \big(\frac{\delta}{\varepsilon}\big)^\alpha \to +\infty$ as $\delta \to 0+$.

Now suppose that u is a positive stationary solution, $u \ge u_{\delta_0}$ on $\partial\Omega$. Put $\Omega^- := \{x \in \Omega\,;\, u(x) < u_{\delta_0}(x)\}$. If $\Omega^- \ne \emptyset$ then the function $w := u - u_{\delta_0}$ fulfils $w = 0$ on $\partial\Omega^-$ and $\triangle w = \triangle u - \triangle u_{\delta_0} \le au^p - au_{\delta_0}^p < 0$ in Ω^-, i.e. $w > 0$ in Ω^- which is a contradiction. Hence $\Omega^- = \emptyset$ and $u \ge u_{\delta_0}$ in $\overline{\Omega}$.
Choose $\delta \le \delta_0$ such that $u \ge u_\delta$ in $\overline{\Omega}$ and $u(x_0) = u_\delta(x_0)$ at some $x_0 \in \overline{\Omega}$. This choice leads to a contradiction with the maximum principle:

if $x_0 \in \Omega$ then $u_\delta(x_0) \ne 0$ and $\triangle u(x_0) \ge \triangle u_\delta(x_0) > au_\delta^p(x_0) = au^p(x_0)$;

if $x_0 \in \partial\Omega$ then $\triangle u_\delta(x) \ge au_\delta^p(x) + \eta > au^p(x) = \triangle u(x)$ for some $\eta > 0$ and all $x \in \Omega$ close to x_0 which gives a contradiction with $\frac{\partial(u_\delta - u)}{\partial n}(x_0) = 0$, $(u_\delta - u)(x_0) = 0$, $u_\delta \le u$. □

Corollary 3.1. *The solution of* (NBC) *starting at* u_δ *blows up.*

Proof: Let u be the solution starting at u_δ. Then $u_t \geq 0$ due to the maximum principle. If u is bounded, then $u(t,\cdot)$ has to converge to a stationary solution $w \geq u_\delta$ since the orbit $\{u(t,\cdot)\,;\, t\geq 0\}$ is relatively compact in the appropriate Sobolev space (see [CFQ]). However, this gives us a contradiction with $\min\limits_{\partial\Omega}(w-u_\delta)<0$. $\square$

4. Stationary solutions for (NBC).

Suppose $q<p<2q-1$, $N>1$ and put

$$a_0 := \inf\{a>0\,;\text{ there exists a positive stationary solution of (NBC)}\}.$$

It follows from [CFQ] that $a_0<\infty$. First we prove the assertion (α) from the introduction.

Theorem 4.1. *If* $a>0$ *is small enough then* (NBC) *does not have positive stationary solutions.*

Proof: By contradiction. Suppose that for $a_m\downarrow 0$ there exist positive stationary solutions u_m. By Lemma 3.1 we have $\min_{\partial\Omega} u_m = u_m(x_m)\leq K$ for some $x_m\in\partial\Omega$ and a positive constant K. Let Ω_m be the component of the set $\{x\in\Omega\,;\, u_m(x)<2u_m(x_m)\}$ containing x_m in its closure. Let v_m be the solution of the problem $\triangle v_m=0$ in Ω_m, $v_m=u_m$ on $\partial\Omega_m$. Then $\frac{\partial v_m}{\partial n}(x_m)\leq 0$ since v_m attains its minimum at x_m (and $\partial\Omega_m\cap U_m=\partial\Omega\cap U_m$ for some neighbourhood U_m of x_m). On the other hand, putting $w_m:=u_m-v_m$ we have $w_m=0$ on $\partial\Omega_m$, $0\leq\triangle w_m = a_m u_m^p\leq a_m 2^p u_m^p(x_m)$ in $\Omega_m\subset\Omega$. The standard regularity theory implies now

$$u_m^q(x_m)=\frac{\partial u_m}{\partial n}(x_m)\leq\frac{\partial w_m}{\partial n}(x_m)\leq Ca_m2^pu_m^p(x_m)\leq Ca_m2^pK^p \tag{4.1}$$

for suitable $C>0$, hence $u_m(x_m)\to 0$ as $m\to\infty$. Using (4.1) again, we get $1\leq Ca_m2^pu_m^{p-q}(x_m)\to 0$, which is a contradiction. $\square$

Now suppose $q<p<2q-1$, $q<\frac{N}{N-2}$ if $N>2$, and $a>a_0$. Then it follows from [CFQ] that there exists a positive stationary solution u of (NBC) which is a local minimizer of the corresponding functional

$$\Phi(u)=\Phi_a(u)=\frac12\int_\Omega|\nabla u|^2\,dx+\frac{a}{p+1}\int_\Omega|u|^{p+1}\,dx-\frac{1}{q+1}\int_{\partial\Omega}|u|^{q+1}\,dS$$

in the Sobolev space $W^{1,2}(\Omega)$.

Putting $w_\varepsilon(x):=\varepsilon^{-\frac{q+\delta}{q-1}}\big(\varepsilon-\operatorname{dist}(x,\partial\Omega)\big)^+$, where $0<\delta<\frac{2q-1-p}{p-q}$, one can straightforwardly check that $\Phi(w_\varepsilon+u)\to-\infty$ as $\varepsilon\to 0+$. Hence to obtain a second critical point of Φ (lying above u) one can use the mountain pass theorem for Φ with respect to the convex set $\{w\in W^{1,2}(\Omega)\,;\, w\geq u\}$ similarly as in the proof of Theorem 2.1 (i) in [CFQ]. The difficulty consists in verifying the corresponding Palais-Smale condition (cf. also Remark 2.4 in [CFQ]). Using a trick of M. Struwe we are able to do this only for almost all $a\geq a_0$.

Theorem 4.2. *Let $q < p < 2q-1$, $q < \frac{N}{N-2}$. Then for a.a. $a \geq a_0$, the problem* (NBC) *has at least two positive stationary solutions.*

Proof: Fix $a_2 > a_1 > a_0$ and let u_{a_1} be a positive solution corresponding to a_1. As shown in [CFQ], choosing u_{a_2} a global minimizer of Φ_{a_2} in $K_1 := \{u \in W^{1,2}(\Omega)\,;\ 0 \leq u \leq u_{a_1}\}$ we get a stationary solution of (NBC) with $0 < u_{a_2} < u_{a_1}$ in $\overline{\Omega}$, $\Phi_{a_2}(u_{a_2}) < \Phi_{a_2}(0) = 0$. Put $S := \{u \in K_1\,;\ \Phi_{a_2}(u) = \Phi_{a_2}(u_{a_2})\}$. Then $\Phi'_{a_2}(u) = 0$ for any $u \in S$ and the set S is compact since Φ'_{a_2} has the form identity $+\,\mathcal{F}$, where $\mathcal{F}$ maps K_1 into a compact set. Moreover, $\nu_0 := \frac{1}{2}\operatorname{dist}(S, \{u\,;\ u \geq u_{a_1}\}) > 0$. Next we show by contradiction that there exists $\nu > 0$ $(\nu \leq \nu_0)$ such that

$$\delta := \delta(\nu) := \inf\{\Phi_{a_2}(u)\,;\ \operatorname{dist}(u,S) = \nu\} - \Phi_{a_2}(u_{a_2}) > 0.$$

Hence assume that $\delta(\nu_n) \leq 0$ for some $\nu_n \downarrow 0$. Let n be fixed and $\nu := \nu_n$. Then there exist u_m such that $\operatorname{dist}(u_m, S) = \nu$ and $\limsup_{m\to\infty} \Phi_{a_2}(u_m) \leq \Phi_{a_2}(u_{a_2})$. Consequently, $u_m = u_m^S + v_m$, where $u_m^S \in S$ and $\|v_m\| = \nu$. We may suppose $u_m^S \to u^S \in S$ and $v_m \rightharpoonup v$, $\|v\| \leq \nu$.

If $v_m \to v$ then $\operatorname{dist}(u^S + v, S) = \nu$, $\Phi_{a_2}(u^S + v) \leq \Phi_{a_2}(u_{a_2})$.

If $v_m \not\to v$ then $\Phi_{a_2}(u^S + v) < \limsup_{m\to\infty} \Phi_{a_2}(u_m^S + v_m) \leq \Phi_{a_2}(u_{a_2})$ so that $u^S + v \notin S$, $0 < \operatorname{dist}(u^S + v, S) \leq \|v\| \leq \nu$.

Let w^S be a local minimizer of Φ_{a_2} in $\{u\,;\ \|u - u^S\| \leq \operatorname{dist}(u^S + v, S)\}$ such that $w^S \notin S$. By the definition of S we have $w^S \notin K_1$. By the same way as in the end of the proof of [CFQ, Lemma 2.4] one gets $w^S \in C^1(\overline{\Omega})$, $\|w^S - u^S\|_{C^1(\overline{\Omega})} \to 0$ for $\nu = \nu_n \to 0$. Since $\operatorname{dist}_{C^1(\overline{\Omega})}(S, C^1(\overline{\Omega})\backslash K_1) > 0$ by the maximum principle and $w^S \notin S$, we get a contradiction.

Now choose ν and $\delta = \delta(\nu)$ with the properties above and fix $\varepsilon > 0$ such that $\Phi_{a_2}(u_{a_1} + w_\varepsilon) < \Phi_{a_2}(u_{a_2})$. Further fix $\alpha \in (0, a_2 - a_1)$ such that $\frac{\alpha}{p+1}\int_\Omega u^{p+1}\,dx \leq \frac{\delta}{3}$ for any $u \in \{v\,;\ \operatorname{dist}(v,S) \leq \nu\} \cup \{u_{a_1} + w_\varepsilon\}$ and let $u_{a_2+\alpha}$ be a fixed positive stationary solution for $a = a_2 + \alpha$ lying below u_{a_2}. Put

$$\begin{aligned}
K_2 &:= \{u \in W^{1,2}(\Omega)\,;\ u \geq u_{a_2+\alpha}\}\\
P &:= \{\tilde{p} \in C([0,1], K_2)\,;\ \tilde{p}(0) = u_{a_2}, \tilde{p}(1) = u_{a_1} + w_\varepsilon\}\\
\gamma_a &:= \inf_{\tilde{p}\in P} \sup_{u\in\tilde{p}([0,1])} \Phi_a(u) \qquad \text{for } |a - a_2| < \alpha.
\end{aligned}$$

Then, obviously, $\gamma : (a_2 - \alpha, a_2 + \alpha) \to \mathbb{R}$ is a nondecreasing function so that γ is differentiable almost everywhere. Choose $a \in (a_2 - \alpha, a_2 + \alpha)$ such that there exists γ'_a. We shall show that there exists a positive stationary solution u of (NBC) with $\Phi_a(u) = \gamma_a$. Since any global minimizer u_a of Φ_a in K_1 fulfils

$$\begin{aligned}
\Phi_a(u_a) &\leq \Phi_a(u_{a_2}) \leq \Phi_{a_2}(u_{a_2}) + \frac{\delta}{3} = \inf\{\Phi_{a_2}(u)\,;\ \operatorname{dist}(u,S) = \nu\} - \frac{2\delta}{3}\\
&< \inf\{\Phi_a(u)\,;\ \operatorname{dist}(u,S) = \nu\} \leq \gamma_a,
\end{aligned}$$

we find two positive solutions for (NBC) and we are done.

We shall proceed similarly as in [S1, Lemma 6.3]. Let $a_m \in (a_2-\alpha, a)$, $a_m \uparrow a$, and let $p_m \in P$ be such that $\sup_{u\in p_m} \Phi_a(u) \le \gamma_a + (a-a_m)$ (where $p_m = p_m([0,1])$). The definition of γ_{a_m} implies now that $S_m := \{u \in p_m\,;\, \Phi_{a_m}(u) \ge \gamma_{a_m} - (a-a_m)\} \neq \emptyset$. Since $\Phi_a(u) \ge \Phi_{a_m}(u)$ we get also that

$$W_{m_0} := \{u \in K_2\,;\, \gamma_{a_m} - (a-a_m) \le \Phi_{a_m}(u) \le \Phi_a(u) \le \gamma_a + (a-a_m) \quad \text{for some } m \ge m_0\}$$

is nonempty, $W_{m+1} \subset W_m$. It is easy to see that for $u \in W_{m_0}$ we have

$$\frac{1}{p+1}\int_\Omega u^{p+1}\,dx \le \frac{\gamma_a - \gamma_{a_m}}{a-a_m} + 2 \qquad \text{for suitable } m \ge m_0,$$

so that W_{m_0} is bounded in $L^{p+1}(\Omega)$.

For $u \in K_2$, put

$$g(u) := \sup_{\substack{v\in K_2\\ \|u-v\|\le 1}} \langle \Phi_a'(u), u-v\rangle, \qquad g_m(u) := \sup_{\substack{v\in K_2\\ \|u-v\|\le 1}} \langle \Phi_{a_m}'(u), u-v\rangle.$$

Let $K(u) := u - \Phi_a'(u)$ and let P_2 be the orthogonal projection in $W^{1,2}(\Omega)$ onto K_2. Then K is a compact map and

$$\langle u-K(u), u-P_2K(u)\rangle \le g(u)\max(1, \|u-P_2K(u)\|).$$

Using the characterization of the projection P_2 we get

$$\langle K(u)-P_2K(u), u-P_2K(u)\rangle \le 0 \qquad \text{for any } u \in K_2$$

and adding the last two inequalities we obtain

$$\|u-P_2K(u)\| \le \max(g(u), \sqrt{g(u)}) \qquad \text{for any } u \in K_2. \tag{4.2}$$

Suppose that there exist $u_m \in W_m$ such that $g(u_m) \to 0$. Choosing $v = u_m + \frac{u_m}{\|u_m\|}$ in the definition of $g(u_m)$ we get $-\langle \Phi_a'(u_m), u_m\rangle \le g(u_m)\|u_m\|$. Adding this inequality to the inequality $(q+1)\Phi_a(u_m) \le C$ and using the boundedness of W_m in $L^{p+1}(\Omega)$ we get

$$\int_\Omega |\nabla u_m|^2\,dx \le \frac{1}{q-1} g(u_m)\|u_m\| + \tilde{C},$$

which gives the boundedness of $\{u_m\}$ in $W^{1,2}(\Omega)$. Hence we may suppose that (a subsequence of) $\{u_m\}$ converges weakly to some $u \in K_2$. Now the compactness of K and (4.2) give us $u_m \to u = P_2K(u)$, $\Phi_a(u) = \gamma_a$. Since $u_{a_2+\alpha}$ is a strict subsolution for (NBC) we get $\Phi_a'(u) = 0$ (cf. the proof of Lemma 2.4 in [CFQ]).

Now assume that the sequence $\{u_m\}$ above does not exist, i.e. $g(u) \geq 4\kappa$ for some $\kappa > 0$ and any $u \in W_{m_1}$. We may suppose that $u_{a_2}, u_{a_1} + w_\varepsilon \notin W_{m_1}$ and that $g(u) \geq 3\kappa$, $g(u) - g(u_m) \leq \kappa$ for some neighbourhood $\tilde{W}$ of W_{m_1} in K_2 such that $u_{a_2}, u_{a_1} + w_\varepsilon \notin \tilde{W}$ and $\tilde{W}$ is bounded in $L^{p+1}(\Omega)$. By [S2, Lemma 1.6], there exists a Lipschitz continuous vector field $\tilde{e} : \tilde{W} \to W^{1,2}(\Omega)$ such that

$$\begin{aligned} \tilde{e}(u) + u \ &\in K_2, \\ \|\tilde{e}(u)\| \ &< 1, \\ \langle \Phi'_a(u), \tilde{e}(u) \rangle \ &< -\min\Big\{ \frac{g(u)^2}{C}, 1 \Big\} \end{aligned}$$

for any $u \in \tilde{W}$, where $C > 0$ is a fixed constant. Consequently, if m is sufficiently large then $\langle \Phi'_{a_m}(u), \tilde{e}(u) \rangle < -\beta$ for some $\beta > 0$ and any $u \in \tilde{W}$.
Now let $\eta : W^{1,2}(\Omega) \to [0,1]$ be a Lipschitz function such that $\eta = 1$ on W_{m_1} and $\eta = 0$ outside $\tilde{W}$. Extend $\tilde{e}$ to K_2 by letting $e(u) := \eta(u)\tilde{e}(u)$ for $u \in \tilde{W}$, $e(u) := 0$ for $u \notin \tilde{W}$. The function e is Lipschitz and

$$\langle \Phi'_{a_m}(u), e(u) \rangle \begin{cases} < -\beta & \text{for } u \in W_{m_1}, \\ \leq 0 & \text{for } u \in K_2, \\ = 0 & \text{for } u \notin \tilde{W}. \end{cases}$$

Let $\psi : [0, \infty) \times K_2 \to K_2$ be the solution of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t}\psi(t,u) &= e\big(\psi(t,u)\big), \\ \psi(0,u) &= u. \end{cases}$$

Let $p_m^t := \psi(t, p_m)$, $q_m^t := \{u \in p_m^t \,;\, \Phi_{a_m}(u) \geq \gamma_{a_m} - (a - a_m)\}$. Since $\frac{d}{dt}\Phi_{a_m}\big(\psi(t,u)\big) \leq 0$ for any u and $\frac{d}{dt}\Phi_{a_m}\big(\psi(t,u)\big)\Big|_{t=0} \leq -\beta$ for $u \in q_m^t$, we get $\inf_{u \in p_m^t} \Phi_{a_m}(u) < \gamma_{a_m}$ for t large enough which gives us a contradiction with the definition of γ_{a_m}. □

In the rest of this section suppose that $N = 2$, $q < p < 2q - 1$.

Lemma 4.1. *Let u_n be positive stationary solutions of* (NBC) *with $a = a_n \leq A < \infty$ such that $U_n := \max_{\overline{\Omega}} u_n \to +\infty$ as $n \to \infty$. Put $V_n := \max_{\overline{\Omega}} |\nabla u_n|$ and let $\varepsilon > 0$. Then*

$$\lim_{n\to\infty} \frac{U_n^q}{V_n} = \lim_{n\to\infty} \frac{V_n}{U_n^{q+\varepsilon}} = 0.$$

Proof: If u is a positive stationary solution of (NBC) then $w := |\nabla u|^2$ fulfils

$$\triangle w = 2pu^{p-1}w + 2\sum_{i,j}\Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 > 0 \qquad \text{in } \Omega,$$

hence w attains its maximum on the boundary $\partial\Omega$. Consequently, $V_n = |\nabla u_n(\tilde{x}_n)|$ and $U_n = u_n(x_n)$ for some $x_n, \tilde{x}_n \in \partial\Omega$. Put $\alpha := \dfrac{q-1}{q-1+\varepsilon/2}$ and choose a unit vector ν_n such that ν_n is not tangential to $\partial\Omega$ at $\tilde{x}_n$ and $\Big|\dfrac{\partial u_n}{\partial \nu_n}(\tilde{x}_n)\Big| \geq \dfrac{1}{2}V_n$. We may suppose that $\tilde{x}_n + t\nu_n \in \Omega$ for $t > 0$ small ($t < 4U_n^{1-q}$). The estimate $0 < u_n \leq U_n$ implies that there exist $t_n \in \Big(0, \dfrac{4U_n}{V_n}\Big]$ such that $\Big|\dfrac{\partial u}{\partial \nu_n}(\tilde{x}_n + t_n\nu_n)\Big| \leq \dfrac{1}{4}V_n$ so that the $C^{1,\alpha}(\overline{\Omega})$-norm of u_n can be estimated below by

$$\|u_n\|_{C^{1,\alpha}} \geq \frac{1}{4^{\alpha+1}} V_n^{1+\alpha} U_n^{-\alpha}. \tag{4.3}$$

On the other hand, the L^r-estimates (with $r > \frac{N}{1-\alpha}$) imply

$$\begin{aligned} \|u_n\|_{C^{1,\alpha}} &\leq C_1 \|u_n\|_{W^{2,r}} \leq C_2 \left(\|a_n u_n^p\|_{L^r} + \|u_n^q\|_{W^{1,r}}\right) \\ &\leq C_3\,(U_n^p + U_n^{q-1}V_n) \leq C_4 U_n^{q-1} V_n, \end{aligned} \tag{4.4}$$

since $V_n \geq U_n^q$ and $p < 2q-1$. Using (4.3) and (4.4) we get $V_n \leq C\,U_n^{(q-1+\alpha)/\alpha} = C\,U_n^{q+\varepsilon/2}$, so that $\lim_{n\to\infty} \dfrac{V_n}{U_n^{q+\varepsilon}} = 0$.

To show $U_n^q/V_n \to 0$, suppose the contrary, i.e. $V_n \leq C\,U_n^q$ for suitable $C > 0$ (and a suitable subsequence of $\{V_n\}$). Put $y := A_n(x - x_n)U_n^{q-1}$ and $v_n = v_n(y) := u_n(x)/U_n$, where A_n is an orthogonal 2×2 matrix such that the transformation $x \mapsto y$ maps the tangent to $\partial\Omega$ at x_n to the line $\{y = (y_1, y_2) \in \mathbb{R}^2\,;\, y_2 = 0\}$ and the point $x_n - \nu_n$ (where ν_n is the unit outward normal to $\partial\Omega$ at x_n) to the point $(0,1)$. Then v_n fulfils

$$\begin{aligned} \triangle v_n &= \frac{a_n}{U_n^{2q-p-1}} v_n^p \qquad \text{in } \Omega^n, \\ \frac{\partial v_n}{\partial y_2} &= -v_n^q \qquad \text{on } \partial\Omega^n, \end{aligned}$$

where $\Omega^n := \{y = y(x)\,;\, x \in \Omega\}$. Moreover, $v_n > 0$, $\max_{\overline{\Omega^n}} v_n = v_n(0) = 1$ and $|\nabla v_n| \leq C$. Passing to the limit we get $v_n \to v$, where v is a nonnegative harmonic function in the halfspace $[y_2 > 0]$ fulfilling the boundary condition $\partial v/\partial y_2 = -v^q$. Moreover, $v(0) = 1$, $v \leq 1$ and $|\nabla v| \leq C$. Hence, $w := -\dfrac{\partial v}{\partial y_2}$ is harmonic, bounded by C and $w = v^q$ on $[y_2 = 0]$. The Poisson's formula ([SW, Theorem II.2.1]) gives us

$$w(0,\lambda) = c \int_{[y_2=0]} \frac{v^q(y)\lambda}{\lambda^2 + |y|^2}\,dy \geq c \int_{[y_2=0]\cap[|y_1|\leq 1/(2C)]} \Big(\frac{1}{2}\Big)^q \frac{\lambda}{\lambda^2+|y|^2}\,dy \geq \tilde{c}/\lambda,$$

since $v(y) \geq \frac{1}{2}$ for $|y| \leq 1/(2C)$. This estimate gives us a contradiction, since $\varphi(\cdot) := v(0,\cdot) : \mathbb{R}^+ \to [0,1]$ fulfils $\varphi'(\lambda) = -w(0,\lambda) \leq -\tilde{c}/\lambda$. ☐

Theorem 4.3. *Let $\Omega = \{x \in \mathbb{R}^2\,;\, |x| < 1\}$ and $q < p < 2q-1$. Then all positive stationary solutions of* (NBC) *are uniformly bounded for a varying in a bounded subset of $\mathbb{R}^+$.*

Proof: Suppose the contrary and let $u = u_n$ be as in Lemma 4.1 (we shall fix n and omit the index n). Let (r,φ) be the polar coordinates in $\mathbb{R}^2$ and let $\tilde u$ be the solution of the problem

$$\begin{aligned} \triangle \tilde u &= 0 \qquad \text{in } \Omega,\\ \tilde u &= u \qquad \text{on } \partial\Omega. \end{aligned}$$

Then $\tilde u \ge u$, hence $\tilde u_r := \dfrac{\partial \tilde u}{\partial r} \le \dfrac{\partial u}{\partial r} = u^q$ on $\partial\Omega$.

Put $w := r\tilde u_r$. Then w is a harmonic function in Ω, $w \le u^q \le U^q$ on $\partial\Omega$ (where $U := \max_{\overline\Omega} u$). Hence $w \le U^q$ in $\overline\Omega$ and $\tilde u_r = w/r \le 2U^q$ in $\{x \in \mathbb{R}^2\,;\, \frac12 \le |x| \le 1\}$. Since $\tilde u$ is harmonic in Ω, we have $|\nabla \tilde u(x)| \le U/\operatorname{dist}(x,\partial\Omega) \le 2U \le 2U^q$ for $|x| \le \frac12$. Hence,

$$\tilde u_r \le 2U^q \qquad \text{in } \overline\Omega. \tag{4.5}$$

Choose $\alpha \in (0,1)$ and $\varepsilon > 0$ such that

$$p < 2q-1-(1-\alpha)(q-1)-\alpha\varepsilon. \tag{4.6}$$

Since

$$\begin{aligned} \triangle(u-\tilde u) &= au^p \le AU^p \qquad \text{in } \Omega,\\ u-\tilde u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

the L^r-estimates imply

$$\|u-\tilde u\|_{C^{1,\alpha}} \le C\,\|u-\tilde u\|_{W^{2,r}} \le \tilde C\, U^p \tag{4.7}$$

for any $r > 2/(1-\alpha)$. Using (4.5)–(4.7) we obtain the estimate

$$\frac{\partial u}{\partial r}(x) \le C_1 U^q \qquad \text{if } |x| > 1 - U^{1-q+\varepsilon}.$$

Now our assumptions and Lemma 4.1 imply $U_n^q/V_n \to 0$, hence $V = |\nabla u(\tilde x)| = (K+1)U^q$ for some $\tilde x = \tilde x_n \in \partial\Omega$ and $K = K_n \to \infty$. Consequently, denoting $u_\varphi := \frac{\partial u}{\partial \varphi}$ we have $|u_\varphi(\tilde x)| \ge KU^q$ and we may suppose $u_\varphi(\tilde x) \ge KU^q$. Let $(1,\tilde\varphi)$ be the polar coordinates of $\tilde x$ and choose $\hat\varphi := \sup\{\varphi < \tilde\varphi\,;\, u_\varphi(1,\varphi) \le \frac K4 U^q\}$ (using obvious identification $0 \equiv 2\pi$). Then $u_\varphi(1,\hat\varphi) = \frac K4 U^q$ and $\omega := |\tilde\varphi - \hat\varphi| < \frac 4K U^{1-q}$ since u is bounded by U. Now the Schauder estimates imply

$$\begin{aligned} \|u-\tilde u\|_{C^{2,\mu}} \le C\|au^p\|_{C^{0,\mu}} &\le CaU^p + Ca(2U^p)^{1-\mu}\big(pU^{p-1}(K+1)U^q\big)^\mu \\ &\le q(K+1)U^{2q-1} \end{aligned}$$

for μ sufficiently small and U large. Since $|u_{\varphi r}| = |qu^{q-1}u_\varphi| \le q(K+1)U^{2q-1}$ on $\partial\Omega$, we have also $|\tilde u_{\varphi r}| \le 2q(K+1)U^{2q-1}$ on $\partial\Omega$. Now $\tilde u_\varphi$ is harmonic and, similarly as in the case of $\tilde u$, the last estimate implies $|\tilde u_{\varphi r}| \le 4q(K+1)U^{2q-1}$ in $\overline{\Omega}$. Consequently, $|u_{\varphi r}| \le 5q(K+1)U^{2q-1}$ in $\overline{\Omega}$.

Put $S := \{(r,\varphi) \in \Omega\,;\, \hat\varphi < \varphi < \tilde\varphi,\ 1-\kappa < r < 1\}$, where $\kappa := \frac{K}{20q(K+1)}U^{1-q}$. Then

$$
\begin{aligned}
u_\varphi(r,\tilde\varphi)\ &\ge KU^q - (1-r)5q(K+1)U^{2q-1} \ge \frac{3}{4}KU^q \qquad \text{for } r \ge 1-\kappa,\\
u_\varphi(r,\hat\varphi)\ &\le \frac{K}{4}U^q + (1-r)5q(K+1)U^{2q-1} \le \frac{1}{2}KU^q \qquad \text{for } r \ge 1-\kappa.
\end{aligned}
$$

Hence,

$$
\int_S \frac{1}{r^2}u_{\varphi\varphi}d\varphi\,dr = \int_{1-\kappa}^1 \frac{1}{r^2}\big(u_\varphi(r,\tilde\varphi) - u_\varphi(r,\hat\varphi)\big)\,dr \ge \frac{\kappa KU^q}{4(1-\kappa)^2} \ge 4U \tag{4.8}
$$

if K (or U) is sufficiently large. On the other hand, we know that $u_r \le C_1U^q$ in S, hence

$$
\int_S u_{rr}dr\,d\varphi \ge -\int_{\hat\varphi}^{\tilde\varphi} C_1U^q\,d\varphi = -\omega C_1U^q \ge -\frac{4C_1}{K}U \ge -U \tag{4.9}
$$

for K sufficiently large. By Lemma 4.1 we have $|u_r| \le |\nabla u| \le U^{2q-1}$ for n sufficiently large so that

$$
\Big|\int_S \frac{1}{r}u_r dr\,d\varphi\Big| \le \frac{1}{1-\kappa}U^{2q-1}\omega\kappa = \frac{1}{2q(1-\kappa)K^2}U \le U \tag{4.10}
$$

for K large enough. Using (4.8)–(4.10) we get $\int_S \triangle u\,dx \ge 2U$. However,

$$
\int_S \triangle u\,dx = a\int_S u^p\,dx \le aU^p\kappa\omega \le U
$$

for U and/or K large enough, which gives a contradiction. $\square$

5. Global existence for (DGT).

In this section we shall suppose that Ω is a smoothly bounded domain in $\mathbb{R}^N$, $N \ge 1$, $p,q > 1$, $r > N\max(1,q-1)$ and

$$
u_0 \in W_0^{1,r}(\Omega)^+ := \{u \in W^{1,r}(\Omega)\,;\, u \ge 0 \ \text{ in } \Omega \ \text{ and } u = 0 \text{ on } \partial\Omega\}.
$$

It is known (see e.g. [A1]) that (DGT) generates a local semiflow on $W_0^{1,r}(\Omega)^+$ and that for any $u_0 \in W_0^{1,r}(\Omega)^+$ there exists a unique maximal solution $u \in C\big([0,T), W_0^{1,r}(\Omega)^+\big)$, where $T = T(u_0)$ is the maximal existence time for u. Moreover, this semiflow is order-preserving.

By $\|\cdot\|_\infty$ we shall denote the norm in $L^\infty(\Omega)$. The main result of this section is the following

Theorem 5.1. (i) *If* $T<\infty$ *then* $\limsup_{t\to T-}\|u(t,\cdot)\|_\infty=+\infty$.
(ii) *If* $q\ge p$ *then* $T=+\infty$ *and* $\sup_{t\ge 0}\|u(t,\cdot)\|_\infty<\infty$.
(iii) *If* $q\ge p$ *and* $u_t\ge 0$ *then* $\sup_{t\ge t_0}\|\nabla u(t,\cdot)\|_\infty<\infty$ *for any* $t_0>0$.

PROOF: To prove (i) it is sufficient to show that an L^∞-estimate for u implies also an L^∞-estimate for ∇u. More precisely, let $0<t_0<T_0<T<\infty$, $C_1:=\max_{t\le T_0}\|u(t,\cdot)\|_\infty<\infty$ and $C_0:=\|\nabla u(t_0,\cdot)\|_\infty$. Then we shall show that $C_0<\infty$ and that there exists a constant $C_2=C_2(C_0,C_1,T)$ such that $\|\nabla u(t,\cdot)\|_\infty\le C_2$ for any $t\in[t_0,T_0]$.
By [A1, Theorem 14.6] we have $u\in C\big((0,t_0],W^{1,rq}(\Omega)\big)$ hence $|\nabla u|^q\in C\big((0,t_0],L^r(\Omega)\big)$. Since $W^{1,r}(\Omega)\hookrightarrow C(\overline{\Omega})$, we have also $u^p\in C\big([0,t_0],L^r(\Omega)\big)$ and the variation of constants formula for u on the interval $[t_0/2,t_0]$ gives us $u(t_0,\cdot)\in W^{2-\varepsilon,r}(\Omega)$ for any $\varepsilon>0$. Since $W^{2-\varepsilon,r}(\Omega)\hookrightarrow C^1(\overline{\Omega})$ for $\varepsilon>0$ small enough, we have $C_0<\infty$.

Now put $f(y):=y^q$, $g(y):=\lambda y^p$ and choose C^∞-functions f_k,g_k $(k=1,2,\dots)$ such that

- $f_k=f$ and $g_k=g$ on $[1,\infty)$,
- $f_k\ge f$ and $g_k\le g$ on $[0,1]$, $f_k'(0)=0$,
- $f_k\to f$ and $g_k\to g$ in $C^1\big([0,\infty)\big)$ as $k\to\infty$.

Let u_k be the solution of the problem

$$(\mathrm{DGT})_k\qquad\begin{cases} v_t=\Delta v-f_k(|\nabla v|)+g_k(v) & \text{in } (t_0,\infty)\times\Omega,\\ v=0 & \text{on } (t_0,\infty)\times\partial\Omega,\\ v(t_0,x)=u(t_0,x) & x\in\overline{\Omega}.\end{cases}$$

Recall that $u(t_0,\cdot)\in W^{2-\varepsilon,r}(\Omega)$ for any $\varepsilon>0$. By [A2, Theorem 7.3 and Corollary 9.4], the problem $(\mathrm{DGT})_k$ generates a local semiflow in $W_0^{1+\delta,r}(\Omega)^+$ for $0<\delta<\min(\frac{1}{r},1-\frac{N}{r},1-(q-1)\frac{N}{r})$ and, denoting by T_k the maximal existence time of u_k in this space, we have $u_k\in C^\infty\big((t_0,T_k)\times\overline{\Omega}\big)$. We shall show that $T_k>T_0$ and $\|\nabla u_k(t,\cdot)\|_\infty\le C_2$ for any $t\in[t_0,T_0]$ where $C_2=C_2(C_0,C_1,T)$ is independent of k. Then the variation of constants formula for $z_k:=u-u_k$, the Gronwall's inequality for $\|z_k(t,\cdot)\|_{W^{2-\varepsilon,r}(\Omega)}$ and a pass to the limit for $k\to\infty$ gives us $|\nabla u|\le C_2$.

First notice that $u_k\le u$ by the maximum principle and that it is sufficient to find the estimate $\|\nabla u_k(t,\cdot)\|_\infty\le C_2$ for any $t\in[t_0,\min(T_k,T_0))$ since then the variation of constants formula gives an apriori bound also in $W^{1+\delta,r}(\Omega)$, hence $T_k>T_0$.

Fix k and let $\tilde T<T_k$, $\tilde T\le T_0$. The function $w:=\frac12|\nabla u_k|^2$ fulfils the equation

$$w_t=\Delta w-\sum_{i,j}(u_k)^2_{x_ix_j}-\sum_j\frac{f_k'(\nabla u_k)}{|\nabla u_k|}(u_k)_{x_j}w_{x_j}+2g_k'(u_k)w \tag{5.1}$$

Since $\sup_{t\in[t_0,\tilde T]}2g_k'(u_k)\le 2\lambda p\max(2,C_1^{p-1})=:\hat C$ if k is large enough, the maximum principle implies that the function $z:=we^{-\hat C(t-t_0)}$ attains its maximum Z in $Q:=[t_0,\tilde T]\times\overline{\Omega}$ on the parabolic boundary $(\{t_0\}\times\overline{\Omega})\cup([t_0,\tilde T]\times\partial\Omega)$.

If $Z \le C_0^2/2$ then $\frac{1}{2}|\nabla u_k|^2 = w \le \frac{1}{2}C_0^2 e^{\hat{C}T}$ in Q and we are done.
If $Z > C_0^2/2$ then $Z = z(t, x_0)$ for some $t \in (t_0, \tilde{T}]$ and $x_0 \in \partial\Omega$. Consequently,

$$\Big|\frac{\partial u_k}{\partial n}(t, x_0)\Big| = |\nabla u_k(t, x_0)| = \max_{x\in\overline{\Omega}} |\nabla u_k(t, x)| = \sqrt{2Z}e^{\hat{C}(t-t_0)/2}.$$

Since u_k is smooth at (t, x_0), we have

$$0 = (u_k)_t(t, x_0) = \triangle u_k(t, x_0) - |\nabla u_k(t, x_0)|^q. \tag{5.2}$$

If ν is any unit tangential vector to $\partial\Omega$ at x_0 then, obviously,

$$\Big|\frac{\partial u_k}{\partial \nu}(t, x)\Big| \le \tilde{C}\Big|\frac{\partial u_k}{\partial n}(t, x)\Big|\,|x - x_0| \qquad \text{for } x \in \partial\Omega,\ x \to x_0,$$

where $\tilde{C}$ is some constant depending only on the curvature of $\partial\Omega$ at x_0. Consequently,

$$\Big|\Big(\triangle u_k - \frac{\partial^2 u_k}{\partial n^2}\Big)(t, x_0)\Big| \le \tilde{C}\Big|\frac{\partial u_k}{\partial n}(t, x_0)\Big|. \tag{5.3}$$

Since $|\nabla u_k(t, \cdot)|$ attains its maximum at x_0, we have $\dfrac{\partial^2 u_k}{\partial n^2}(t, x_0) \le 0$. This inequality together with (5.2) and (5.3) imply $|\nabla u_k(t, x_0)|^{q-1} \le \tilde{C}$, which gives the desired estimate.

(ii) If $q \ge p$ then it follows from [F] that the function $\psi(x) := \alpha^{2/(p-1)}e^{\alpha(\sum_i x_i + C)}$ is a supersolution for u if α and C are large enough. Hence, $u(t, x) \le \max_{\overline{\Omega}} \psi$ for any $t < T$ and $x \in \overline{\Omega}$. Now the assertion (ii) follows from (i).

Note that choosing $\varphi(x) := \min\{\psi(x), K\mathrm{dist}\,(x, \partial\Omega)\}$ with K sufficiently large we obtain a supersolution φ for $u(t, \cdot)$, $t \ge t_0$, which gives us an apriori bound $|\nabla u| = \Big|\dfrac{\partial u}{\partial n}\Big| \le K$ on the boundary $\partial\Omega$.

(iii) Let $q \ge p$ and $u_t \ge 0$. Then

$$\triangle u = u_t + |\nabla u|^q - \lambda u^p \ge |\nabla u|^q - C_1 \qquad \text{for some } C_1 > 0,$$

and, consequently,

$$\sum_{i,j} u_{x_i x_j}^2 \ge C_2(\triangle u)^2 \ge C_3|\nabla u|^{2q} - C_4 \qquad \text{for some } C_2, C_3, C_4 > 0.$$

By the note in the proof of (ii), the function $w := \frac{1}{2}|\nabla u|^2$ is bounded on $\partial\Omega$ so that the last inequality together with (5.1), the boundedness of u and the maximum principle imply the boundedness of w in $[0, \infty) \times \overline{\Omega}$. $\square$

6. Stationary solutions for (DGT).

Throughout this section we suppose that Ω is a smoothly bounded domain in $\mathbb{R}^N$, $N \geq 1$. By a (stationary) solution we mean always a classical positive stationary solution.

Lemma 6.1. *Let $q \geq p$, $\lambda_1 > 0$. Then there exists $K = K(\lambda_1) > 0$ such that any positive stationary solution u of* (DGT) *with $\lambda \leq \lambda_1$ fulfils $\|u\|_{C^1(\overline{\Omega})} \leq K$.*

Proof: We shall use similar arguments as in the proof of Theorem 5.1 (ii), (iii). One can easily find a function

$$\varphi(x) = \varphi_\alpha(x) = \min\{\psi_\alpha(x), K \operatorname{dist}(x, \partial\Omega)\},$$

where $\psi_\alpha(x) = \alpha^{2/(p-1)} e^{\alpha(\sum_i x_i + C)}$ and $K = K(\alpha)$ is a continuous nondecreasing function of α, $\lim_{\alpha\to\infty} K(\alpha) = +\infty$, such that for $\alpha \geq \alpha_0$, φ is a strict supersolution for (DGT) with any $\lambda \leq \lambda_1$. Now suppose that u is a positive stationary solution of (DGT) with $\lambda \leq \lambda_1$ which does not lie below φ_{α_0}. Choosing $\alpha_1 := \inf\{\alpha\,;\ \varphi_\alpha \geq u\}$ we have $\varphi_{\alpha_1} \geq u$ and either $\dfrac{\partial \varphi_{\alpha_1}}{\partial n}(x_1) = u(x_1)$ for some $x_1 \in \partial\Omega$ or $\varphi_{\alpha_1}(x_2) = u(x_2)$ for some $x_2 \in \Omega$. Since both possibilities lead to the contradiction with the maximum principle, we have $u \leq \varphi_{\alpha_0}$, i.e. we have an apriori bound (say C_1) for u in $L^\infty(\Omega)$ and an apriori bound for $\left|\dfrac{\partial u}{\partial n}\right| = |\nabla u|$ on $\partial\Omega$.

Putting $w := \frac{1}{2}|\nabla u|^2$ and assuming that w attains its maximum at some $x_0 \in \Omega$, we get by (5.1) (with $w_t = 0$, $\triangle w(x_0) \leq 0$, $w_{x_j}(x_0) = 0$) and (DGT)

$$\begin{aligned} 2\lambda_1 p C_1^{p-1} w(x_0) &\geq 2\lambda p u^{p-1}(x_0) w(x_0) \geq \sum_{i,j} u^2_{x_i x_j}(x_0) \\ &\geq C_2\big(\triangle u(x_0)\big)^2 = C_2\big(|\nabla u(x_0)|^q - \lambda u^p(x_0)\big)^2 \\ &\geq C_3 w^q(x_0) - C_4, \end{aligned}$$

which gives an apriori bound for $w(x_0)$. $\square$

Remark 6.1. The apriori bound in $C^1(\overline{\Omega})$ and standard regularity results for the stationary problem related to (DGT) imply also an apriori bound in $W^{2,r}(\Omega)$ for any $r > 1$ so that the set of positive stationary solutions for $\lambda \leq \lambda_1$ is relatively compact in $C^1(\overline{\Omega})$.

Theorem 6.1. *Let $q \geq p$. Then there exists $\lambda_0 > 0$ such that the stationary problem corresponding to* (DGT)

(i) *does not have positive solutions for $\lambda < \lambda_0$,*
(ii) *has at least one positive solution for $\lambda = \lambda_0$ and at least two positive solutions for $\lambda > \lambda_0$.*

Proof: To prove (i) suppose the contrary, i.e. there exist solutions u_n with $\lambda = \lambda_n \downarrow 0$. By Lemma 6.1, these solutions are uniformly bounded in $C(\overline{\Omega})$ by some constant C_1. Denoting by ν_n the norm of u_n in $W^{1,2}(\Omega)$, multiplying the (stationary)

equation in (DGT) by u_n and integrating by parts we get

$$
\begin{aligned}
(6.1) \qquad c\nu_n^2 &\leq \int_\Omega |\nabla u_n|^2\,dx = -\int_\Omega |\nabla u_n|^q u_n\,dx + \lambda_n \int_\Omega u_n^{p+1}\,dx \\
&\leq \lambda_n C_1^{p-1} \int_\Omega u_n^2\,dx \leq \lambda_n C_1^{p-1} \nu_n^2
\end{aligned}
$$

for suitable $c > 0$, which gives us a contradiction.

(ii) Suppose that (DGT) has a positive stationary solution u_0 for some $\lambda_0 > 0$ and let $\lambda > \lambda_0$. Then $u_0 \in W^{2,r}(\Omega)$ by Remark 6.1 and u_0 is a (strict) subsolution for (DGT). By the maximum principle, $u_t \geq 0$ for the solution u of (DGT) starting at u_0. Consequently, the function $u(t,\cdot)$ is bounded in $W^{1,\infty}(\Omega)$ by Theorem 5.1 (iii). Standard parabolic regularity results imply now the boundedness of $u(t,\cdot)$ in $W^{2-\varepsilon,r}(\Omega)$ for any $r > 1$ and $\varepsilon > 0$ so that the orbit $\{u(t,\cdot)\}_{t\geq 0}$ is relatively compact in $C^1(\overline{\Omega})$. Since $u_t \geq 0$, we have $u(t,\cdot) \to \tilde{u}$ as $t \to +\infty$, where $\tilde{u}$ is a positive stationary solution of (DGT).

To see that (DGT) has a positive stationary solution at least for some λ, let u_0 be a nonnegative $C^2(\overline{\Omega})$-function such that $u_0 = 0$ on $\partial\Omega$, $\triangle u_0 \geq |\nabla u|^q$ in a neighbourhood U of $\partial\Omega$ and $u_0 \geq \varepsilon > 0$ in $\Omega\backslash U$ (It is sufficient to choose $u_0(x) := w(\operatorname{dist}(x,\partial\Omega))$ for x close to $\partial\Omega$, where w is the solution of O.D.E. $w(0) = 0$, $w'(0) = C \gg 1$, $w''(y) = 2w'^q(y)$ for $y \in (0,\delta]$, $u_0(x) := w(\delta)$ for $\operatorname{dist}(x,\partial\Omega) > \delta$, and then regularize u_0 in the $\delta/2$-neighbourhood of $\{x\,;\,\operatorname{dist}(x,\partial\Omega) = \delta\}$.). Then u_0 is a subsolution for (DGT) if λ is sufficiently large, hence (similarly as above) we get the existence of a positive stationary solution.

Until now, we have shown the existence of a $\lambda_0 > 0$ such that the stationary problem corresponding to (DGT) has

(j) no solutions for $\lambda < \lambda_0$,
(jj) at least one solution for $\lambda > \lambda_0$.

To prove the existence of a solution for $\lambda = \lambda_0$, let u_n be solutions corresponding to $\lambda_n \downarrow \lambda_0$. Due to the apriori bounds (Lemma 6.1 and Remark 6.1) we know that u_n converge to some nonnegative stationary solution of (DGT) with $\lambda = \lambda_0$. To show $u \not\equiv 0$, suppose the contrary. Then similarly as in (6.1) we get

$$
\begin{aligned}
c\nu_n^2 &\leq \int_\Omega |\nabla u_n|^2\,dx \leq \lambda_n \int_\Omega u_n^{p+1}\,dx \leq \lambda_n \Big(\int_\Omega u_n^2\,dx\Big) \max_{\overline{\Omega}} u_n^{p-1} \\
&\leq \lambda_n \nu_n^2 \max_{\overline{\Omega}} u_n^{p-1}
\end{aligned}
$$

and since $u_n \to 0$ (even in $C^1(\overline{\Omega})$), we get a contradiction.

Now let $\lambda > \lambda_0$ and let u_λ be the positive stationary solution which we have got as the limit of the solution $\hat{u}$ of (DGT) starting at u_0 (= positive stationary solution corresponding to λ_0). Choose $K > 0$, $K > \sup_{t\geq 0} \|\hat{u}(t,\cdot)\|_{C^1(\overline{\Omega})}$ and let f_K be a smooth cut-off function for the function $y \mapsto y^q$; more precisely, $f_K(y) = y^q$

for $y \in [0, K]$, $f_K(y) = K^q + 1$ for $y \geq K+1$, $f_K' > 0$ on $[K, K+1)$, $f_K(y) \leq y^q$ for any $y \geq 0$. Consider the problem

$$(\mathrm{DGT})_K \qquad \begin{cases} u_t = \triangle u - f_K(|\nabla u|) + \lambda u^p & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_1(x) & x \in \overline{\Omega}, \end{cases}$$

where $0 \leq u_1 \leq u_\lambda$, $u_1 \in W^{2-\varepsilon,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for some $r > N/(1-2\varepsilon)$. Since $f_K(y) = y^q$ for $y \leq \|\hat{u}(t,\cdot)\|_{C^1(\overline{\Omega})}$, the function u_λ is a positive stationary solution of $(\mathrm{DGT})_K$ and $\hat{u}(t,\cdot)$ is a nondecreasing solution of $(\mathrm{DGT})_K$ connecting u_0 to u_λ. Moreover, 0 is a stable stationary solution of $(\mathrm{DGT})_K$ and one can easily find a positive function $\tilde{u}_0$ such that $\dfrac{\partial \tilde{u}_0}{\partial n} > 0$ on $\partial\Omega$ and the solution of $(\mathrm{DGT})_K$ starting at $\tilde{u}_0$ tends (in a monotone way) to 0 as $t \to \infty$. Denoting $S^\tau u_1 := u(\tau,\cdot)$ where u is the solution of $(\mathrm{DGT})_K$ starting at u_1, we get that S^τ is (for any $\tau > 0$) an order-preserving discrete semigroup which maps the order interval $[0, u_\lambda] \subset W^{2-\varepsilon,r}(\Omega)$ into a relatively compact subset of $[0, u_\lambda]$. Moreover, 0 or u_λ is an equilibrium of S^τ which is stable from above or from below, respectively. Due to [AH, Lemma 5], there exists another equilibrium u^τ of S^τ which lies between 0 and u_λ. Since u^τ lies neither above u_0 nor below $\tilde{u}_0$, we have

$$\min\{\|u^\tau - u_\lambda\|_{C^1(\overline{\Omega})}, \|u^\tau\|_{C^1(\overline{\Omega})}\} \geq c_0 > 0 \tag{6.2}$$

for some c_0 which is independent of τ. The variation of constants formula and a straightforward estimate imply that the set $\{u^\tau\}_{\tau\in(0,\tau_0)}$ is bounded in $W^{2-\varepsilon,r}(\Omega)$ and hence we may find a sequence $\tau_k \downarrow 0$ such that $u^{\tau_k} \to u_K$ in $W^{2-2\varepsilon,r}(\Omega)$. Due to (6.2), u_K is a positive stationary solution of $(\mathrm{DGT})_K$ which lies in $[0, u_\lambda]\backslash\{0, u_\lambda\}$. Put $w_K := \frac{1}{2}|\nabla u_K|^2$. We show that $w_K \leq \frac{1}{2}K^2$ for K sufficiently large so that u_K is also a stationary solution of (DGT).

Since $w_K \leq \frac{1}{2}|\nabla u_\lambda|^2 \leq \frac{1}{2}K^2$ on the boundary $\partial\Omega$, we may assume that w_K attains its maximum at some $x_0 \in \Omega$. Suppose $w_K(x_0) > \frac{1}{2}K^2$. Using an analogue to (5.1) we get, similarly as in the proof of Lemma 6.1,

$$\begin{aligned} C_1 w_K(x_0) &\geq 2\lambda p u_K^{p-1}(x_0) w_K(x_0) \geq \sum (u_K)^2_{x_i x_j}(x_0) \geq C_2\big(\triangle u_K(x_0)\big)^2 \\ &= C_2\big(f_K(|\nabla u_K(x_0)|) - \lambda u_K^p(x_0)\big)^2 \geq C_3 K^{2q} - C_4 \end{aligned} \tag{6.3}$$

On the other hand, due to the L^r-estimates for the stationary problem corresponding to $(\mathrm{DGT})_K$ we have

$$\begin{aligned} \|u_K\|_{W^{2,r}(\Omega)} &\leq C_5 + C_6 \|f(|\nabla u_K|)\|_{L^r(\Omega)} \\ &\leq C_5 + C_6 (K^q + 1)^{(r-1)/r} \Big(\int_\Omega f(|\nabla u_K|)\,dx\Big)^{1/r} \\ &\leq C_7 + C_8 K^{q(r-1)/r}, \end{aligned}$$

since

$$\begin{aligned}\int_\Omega f(|\nabla u_K|)\,dx &= \int_\Omega \triangle u_K\,dx + \lambda\int_\Omega u_K^p\,dx = \int_{\partial\Omega}\frac{\partial u_K}{\partial n}\,dS + \lambda\int_\Omega u_K^p\,dx\\ &\le \lambda\int_\Omega u_K^p\,dx \le C_9.\end{aligned}$$

Consequently,

$$w_K(x_0) \le \frac{1}{2}\|u_K\|^2_{C^1(\overline{\Omega})} \le C_{10}\|u_K\|^2_{W^{2,r}(\Omega)} \le C_{11} + C_{12}K^{2q(r-1)/r}. \tag{6.4}$$

Now (6.3) and (6.4) yield a contradiction if K is sufficiently large. □

Theorem 6.2. *Let $q < \min(2,(N+2)/N)$ and let $p < (N+1)/(N-1)$ if $N > 1$. Then there exists $\lambda_0 \ge 0$ such that* (DGT) *has at least one positive stationary solution for any $\lambda > \lambda_0$.*

Proof: Let $X := W_0^{1,2}(\Omega)$ considered with the scalar product $\langle u,v\rangle := \int_\Omega \nabla u\nabla v\,dx$ and the norm $\|u\| := \sqrt{\langle u,u\rangle}$. Let the operators $F, G : X \to X$ be defined by

$$\langle F(u),v\rangle := \int_\Omega |u|^p v\,dx, \qquad \langle G(u),v\rangle := \int_\Omega |\nabla u|^q v\,dx.$$

By the corresponding Sobolev imbedding theorems, F and G are well defined and compact. Put $K^+ := \{u \in X\,;\, u \ge 0\}$ and let P^+ be the orthogonal projection in X onto K^+. Consider the inequality

$$u \in K^+ : \qquad \int_\Omega \big(-\triangle u - \lambda u^p + |\nabla u|^q\big)(v-u) \ge 0 \qquad \text{for any } v \in K^+,$$

which is equivalent to the operator equation

$$u - P^+(\lambda F(u) - G(u)) = 0. \tag{6.5}$$

In the same way as in [S3, Theorem I.2.4] (cf. also [CFQ, Lemma 2.4]) one can easily show that any solution of (6.5) is also a stationary solution of (DGT). Now the proof of Theorem 1 in [Q2, p. 582] (based on the apriori estimates of Brézis and Turner [BT]) implies that the Leray-Schauder degree $\deg(\mathrm{id} - P^+F, 0, B_C)$ or $\deg(\mathrm{id} - P^+F, 0, B_\varepsilon)$ is well defined and equals 0 or 1, respectively, where $B_\eta := \{u \in X\,;\, \|u\| < \eta\}$, C is large and ε small. Fix C and ε. Then $\deg(\mathrm{id} - P^+(F-aG), 0, B_C) = 0$ and $\deg(\mathrm{id} - P^+(F-aG), 0, B_\varepsilon) = 1$ for $a \in (0,a_0]$ so that there exists a nontrivial solution $u = u(a)$ of the equation $u = P^+(F(u) - aG(u))$ for any $a \in (0,a_0]$. By our considerations above, $u(a)$ solves also the equation $0 = \triangle u - a|\nabla u|^q + u^p$. Put $v := a^{1/(q-1)}u$. It is easily seen that v is a positive stationary solution of (DGT) with $\lambda = \left(\frac{1}{a}\right)^{(p-1)/(q-1)}$. □

Remark 6.2. The condition $p < (N+1)/(N-1)$ in Theorem 6.2 can be weakened to $p < (N+2)/(N-2)$ since then one can use [FLN, Theorem 1.2 and Remark 1.5] in order to get apriori estimates for the solutions of the equation $0 = \triangle u + u^p + s\Phi$ (where $s \geq 0$ and Φ is the first eigenfunction of the operator $\triangle$ in X) which are sufficient for the determination of $\deg(\mathrm{id} - P^+F, 0, B_C)$.

Remark 6.3. If the assumptions of Theorem 6.2 are fulfilled and $q > p$ then one can use also the Leray-Schauder degree to get 2 positive stationary solutions of (DGT) for λ large enough (cf. Theorem 6.1). Using the notation from the proof of Theorem 6.2, it is sufficient to use the homotopy

$$H(t,u) := u - P^+(tF(u) - a_0G(u)), \qquad t \in [0,1]$$

to show that $\deg(\mathrm{id} - P^+(F - a_0G), 0, B_K) = 1$ if K is large enough ($K \gg C$). The admissibility of H follows from the following contradiction argument.

Suppose that $H(t_n, u_n) = 0$ and $\|u_n\| \to \infty$. Put $v_n := u_n/\|u_n\|$. We may suppose that $v_n \rightharpoonup v \in X$ weakly. Multiplying the differential equation corresponding to $H(t_n, u_n) = 0$ by $u_n/\|u_n\|^{q+1}$ gives $\int_\Omega v_n|\nabla v_n|^q\,dx \to 0$, which implies $v \equiv 0$. Integrating the equation corresponding to $H(t_n, u_n) = 0$ we get

$$C_1\|u_n\|^q_{W^{1,q}(\Omega)} \leq a_0 \int_\Omega |\nabla u_n|^q\,dx \leq t_n \int_\Omega u_n^p\,dx \leq C_2\|u_n\|^p_{W^{1,q}(\Omega)},$$

which gives an apriori bound for $\|u_n\|_{W^{1,q}(\Omega)}$. Finally, multiplying the equation corresponding to $H(t_n, u_n) = 0$ by $u_n/\|u_n\|^2$ we get

$$1 \leq t_n \int_\Omega v_n^2 u_n^{p-1}\,dx \leq \|v_n\|^{1/r}_{L^{2r}(\Omega)} \|u_n\|^{1/r'}_{L^{(p-1)r'}(\Omega)}, \tag{6.6}$$

where we choose $1 < r < N/(N-2)$ and $1 < r' < Nq/\big((N-q)(p-1)\big)$ (if $N > 1$), $\frac{1}{r} + \frac{1}{r'} = 1$, so that the right hand side in (6.6) can be estimated by $\|v_n\|^{1/r}_{L^{2r}(\Omega)} \|u_n\|^{1/r'}_{W^{1,q}(\Omega)} \to 0$, which gives a contradiction.

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On global existence and stationary solutions for two classes of semilinear parabolic problems

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Abstract. We investigate stationary solutions and asymptotic behaviour of solutions of two boundary value problems for semilinear parabolic equations. These equations involve both blow up and damping terms and they were studied by several authors. Our main goal is to fill some gaps in these studies.

Keywords: global existence, blow up, semilinear parabolic equation, stationary solution

Classification: 35K60, 35J65, 35B40

1. Introduction.

Consider the following two problems

$$\text{(NBC)}\qquad \begin{cases} u_t = \triangle u - au^p & \text{in } (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial n} = u^q & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & x\in\overline{\Omega}, \end{cases}$$

$$\text{(DGT)}\qquad \begin{cases} u_t = \triangle u - |\nabla u|^q + \lambda u^p & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) & x\in\overline{\Omega}, \end{cases}$$

where Ω is a smoothly bounded domain in $\mathbb{R}^N$, $p,q>1$, $a,\lambda>0$ and $u_0\in W^{1,\infty}(\Omega)$ is a non-negative function. These problems were studied by many authors (see e.g. [CFQ], [E], [FQ1], [LGMW] in the case of (NBC) and [AW], [C], [CW1], [CW2], [F], [KP], [Q1] in the case of (DGT)). In both problems there is a blow-up term (u^q and λu^p) and a damping term ($-au^p$ and $-|\nabla u|^q$). These terms cause that the corresponding solutions admit an interesting asymptotic behaviour which strongly depends on the parameters p,q,a,λ. The main purpose of this paper is to fill some gaps in the studies of these problems; i.e. to investigate the behaviour of the

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