The Keller-Segel system as a gradient flow

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GENERAL MODEL

$$\frac{\partial \rho}{\partial t} = \Delta(\rho^m) - \mathrm{div}\left(\rho \nabla \mathcal{K} * \rho\right) \quad \text{in } (0, +\infty) \times \mathbb{R}^d , \tag{1}$$

where $\mathcal{K}$ is a given attractive interaction potential.

Remark:

$$\int_{\mathbb{R}^d} \rho(x,t)\ \mathrm{d}x = \int_{\mathbb{R}^d} \rho_0(x)\ \mathrm{d}x =: M$$

In dimension 2, take $\mathcal{K} := -\dfrac{1}{2\pi} \log|\cdot|$, the Poisson kernel:

THE CLASSICAL PATLAK-KELLER-SEGEL SYSTEM

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta \rho - \mathrm{div}(\rho \nabla \Phi) & \text{in } (0, +\infty) \times \mathbb{R}^2 \\ \Delta \Phi = -\rho & \text{in } (0, +\infty) \times \mathbb{R}^2 , \\ \rho(t=0) = \rho_0 \geq 0 & \text{in } \mathbb{R}^2 . \end{cases} \tag{KS}$$

KNOWN RESULTS [B., BILER, DOLBEAULT, CARLEN, CARRILLO, FIGALLI, KARCH, JAGER, LUCKHAUS, MASMOUDI, NAGAI, NAITO, NADZIEJA, PERTHAME, SENBA, VÉLAZQUEZ, ...]

Under the assumptions

$$\rho_0 \geq 0\,, \quad \rho_0 \in L^1(\mathbb{R}^2)\,, \quad |x|^2 \rho_0 \in L^1(\mathbb{R}^2) \quad \text{and} \quad \rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)\,. \tag{H}$$

- If $M < 8\pi$, solutions to (KS) exist globally in time and converge exponentially fast to the self-similar profile.
- If $M = 8\pi$, solutions to (KS) exist globally in time and blowup as a Dirac mass of mass $8\,\pi$ centred at the centre of mass in infinite time.
- If $M > 8\pi$, solutions to (KS) blowup in finite time.

Open questions:

- How does the solution blowup? [Herrero & Vélazquez, Kavallaris & Souplet, Suzuki, Raphael & Schweyer, ...],
- What happens after blowup [Vélazquez, Dolbeault & Schmeiser, ...],
- Can this model be useful for (more) realistic biological problems [Calvez, Meunier, Perthame, ...].

THE FREE ENERGY FUNCTIONAL

$$\mathcal{F}_{\mathrm{PKS}}[\rho] = \int_{\mathbb{R}^2} \rho(x) \log \rho(x)\, \mathrm{d}x + \frac{1}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x) \log|x-y| \rho(y)\, \mathrm{d}x\, \mathrm{d}y \ .$$

If ρ is a smooth solution to (KS) then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{PKS}}[\rho(t)] = -\int_{\mathbb{R}^2} \rho \left|\nabla \left(\log \rho - c\right)\right|^2 \, \mathrm{d}x \leq 0 \ .$$

LOGARITHMIC HARDY-LITTLEWOOD-SOBOLEV'S INEQUALITY [CARLEN-LOSS, 1992]

Let $f \in L^1_+(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ are bounded in $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f \, \mathrm{d}x = M$, then

$$\int_{\mathbb{R}^2} f \log f + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) \log|x-y| f(y)\, \mathrm{d}x\, \mathrm{d}y \geq -C(M) \ . \qquad \text{(logHLS)}$$

Let $\lambda \geq 0$, the minimisers of (logHLS) are the translations of

$$\bar{\rho}_\lambda(x) := \frac{M}{\pi} \frac{\lambda}{\left(\lambda + |x|^2\right)^2} \ .$$

Key estimate:

$$\left(1 - \frac{M}{8\pi}\right) \int_{\mathbb{R}^2} \rho \log \rho \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0] + C(M) \frac{M}{8\pi} < \infty \quad \text{if } M < 8\,\pi.$$

In the Wasserstein metric the solution to the system (KS) is a gradient flow of the free energy:

$$\rho_t = -\nabla_{W} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] \ .$$

THE JORDAN-KINDERLEHRER-OTTO (JKO) SCHEME

Given a time step τ, we define the solution by the minimising scheme:

$$\rho_\tau^{k+1} \in \operatorname{argmin}_{\rho \in \mathcal{K}} \left[\frac{\mathcal{W}_2^2(\rho, \rho_\tau^k)}{2\tau} + \mathcal{F}_{\mathrm{PKS}}[\rho] \right] \ ,$$

where $\mathcal{S} := \{\rho \ : \ \int_{\mathbb{R}^2} \rho = M, \quad \int_{\mathbb{R}^2} \rho \log \rho < \infty \quad \text{and} \quad \int_{\mathbb{R}^2} |x|^2 \rho(x)\,\mathrm{d}x < \infty\}$.

Push-forward: T transports μ onto ν and denote $T\#\mu = \nu$ if

$$\int_{\mathbb{R}^2} \zeta[T(x)]\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^2} \zeta(x)\,\mathrm{d}\nu(x) \qquad \forall \zeta \in \mathcal{C}_0^b(\mathbb{R}^2) \ .$$

WASSERSTEIN DISTANCE

$$\mathcal{W}_2^2(\mu, \nu) := \inf_{T \,:\, \nu = T\#\mu} \int_{\mathbb{R}^2} |x - T(x)|^2 \ \mathrm{d}\mu(x) \ .$$

THE NON-LINEAR PKS SYSTEM IN DIMENSION $d \geq 3$ [BEDROSSIAN, BERTOZZI, B., CARRILLO, CHAVANIS, LAURENÇOT, OGAWA, RODRIGUEZ, SIRE, SUGIYAMA, SUZUKI, TAKAHASHI, YAHAGI, ...]

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \Delta(\rho^m) - \operatorname{div}(\rho \nabla \Phi) & \text{in } (0, +\infty) \times \mathbb{R}^d \\ \Delta \Phi = -\rho & \text{in } (0, +\infty) \times \mathbb{R}^d \, , \end{cases} \tag{NKSd}$$

which corresponds to (1) with the kernel

$$\mathcal{Y}_0 = c_d \, \frac{1}{|x|^{d-2}} \quad \text{where} \quad c_d := \frac{\Gamma(d/2)}{2\,(d-2)\,\pi^{d/2}} \, .$$

The diffusion and interaction term “balance” if

$$m = m_d := 2 - \frac{2}{d} \in (1, 2) \, .$$

KNOWN RESULTS [SUGIYAMA, 2006 & 2007]

- if $m > m_d$ then all the solutions to (NKSd) exist globally in time,
- if $m < m_d$ then there are solutions to (NKSd) blowing-up in finite time and there are global-in-time solutions.

In the case $m = m_d$.

MAIN THEOREM: CRITICAL MASS [B., CARRILLO, LAURENÇOT, 2009]

Under the assumptions

$$\rho_0 \geq 0\,, \quad \rho_0 \in L^1(\mathbb{R}^2, (1+|x|^2)\,\mathrm{d}x) \quad \text{and} \quad \rho_0 \in L^m(\mathbb{R}^2)\,. \tag{H'}$$

There exists a constant M_c such that

- if $M < M_c$, solutions exist globally in time and there is a radially symmetric compactly supported self-similar solution,
- if $M = M_c$, solutions exist globally in time. There existare global in time solutions not blowing-up in infinite time. There are infinitely many compactly supported stationary solutions,
- if $M > M_c$, there are solutions which blowup in finite time and self-similar blowingup solutions.

Open questions:

- $M < M_c$: does the self-similar solution attract all the solution? [Yao for radially symetric solutions, ...]
- $M = M_c$: Are they blowingup solutions? When the solutions do not blowup are they attracted by the stationary solutions? [Bedrossian, ...]
- $M > M_c$: Do all the solution blowup in finite time? Are they blowingup solutions with positive energy [Bedrossian & Kim for radially symetric solutions, ...]?

THE FREE ENERGY

$$\mathcal{G}[\rho] := \int_{\mathbb{R}^d} \frac{\rho^m(t,x)}{m-1}\,\mathrm{d}x - \frac{c_d}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{u(t,x)\,u(t,y)}{|x-y|^{d-2}}\,\mathrm{d}x\,\mathrm{d}y\ .$$

The free energy $t \mapsto \mathcal{G}[\rho(t)]$ is non-increasing along the flow of (NKSd).

VARIANT TO THE HARDY-LITTLEWOOD-SOBOLEV (VHLS) INEQUALITY [$\sim$ LIEB, 1983]

$$C_{\mathrm{HLS}} := \sup\left\{ \frac{\int_{\mathbb{R}^d} h(x)\,(\mathcal{Y}_0 * h)(x)\,\mathrm{d}x}{\|h\|_m^m\,\|h\|_1^{2/d}} \ :\ h \in (\mathrm{L}^1 \cap \mathrm{L}^m)(\mathbb{R}^d), h \neq 0 \right\} < \infty\ . \tag{2}$$

CRITICAL MASS

Define

$$M_c := \left[\frac{2}{(m-1)\,C_{\mathrm{HLS}}\,c_d}\right]^{d/2}\ . \tag{3}$$

Key estimate:

$$\mathcal{G}[\rho] \geq \frac{C_{\mathrm{HLS}}}{2}\left(1 - \frac{M}{M_c}\right)\|\rho\|_m^m\ .$$

Let $m = m_d$. Consider

$$\begin{cases} \partial_t u = \operatorname{div}\left[\nabla u^m - u\nabla v\right] , \\ \tau\partial_t v = \Delta v - \alpha v + u , \end{cases} \quad (t,x) \in (0,\infty) \times \mathbb{R}^d , \tag{4}$$

MAIN RESULTS WHEN $d = 2$

When $d = 2$.

- All the solutions to (4) exists globally in time if $M < 8\pi$ [Calvez & Corrias, 2008],
- For any M, there exists τ such that they are global-in-time solution to (4) [Biler, Corrias & Dolbeault, 2011].

By a change of variable we imposed $M = 1$ and the chemo-sensitivity χ plays the role of M is the previous results.

THE PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM [BILER, B., CALVEZ, CORRIAS, DOLBEAULT, ISHITA, KUNII, LAURENÇOT, SENBA, MONTARU, SUGIYAMA, SUZUKI, YOKOTA...]

$$\begin{cases} \partial_t u = \operatorname{div}\left[\nabla u^m - \chi u\nabla v\right] , \\ \tau\partial_t v = \Delta v - \alpha v + u , \end{cases} \quad (t,x) \in (0,\infty) \times \mathbb{R}^d , \tag{5}$$

We define

$$\chi_c := \frac{2}{(m-1)C_{\mathrm{HLS}}}\,, \tag{6}$$

GLOBAL EXISTENCE [B. & LAURENÇOT, 2012]

Let $\tau > 0$, $\alpha \geq 0$, u_0 be a non-negative function in $\mathrm{L}^1(\mathbb{R}^d, (1+|x|^2)\,\mathrm{d}x) \cap \mathrm{L}^m(\mathbb{R}^d)$ satisfying $\|u_0\|_1 = 1$ and $v_0 \in \mathrm{H}^1(\mathbb{R}^d)$.
If $\chi < \chi_c$ then there exists a weak solution (u,v) to the parabolic-parabolic Keller-Segel system (5), that is, for all $t > 0$

- $u(t) \geq 0$, $\|u(t)\|_1 = 1$,
- $u \in \mathrm{L}^\infty(0,t;\mathrm{L}^1(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x) \cap \mathrm{L}^m(\mathbb{R}^d))$, $u^{m/2} \in \mathrm{L}^2(0,t;\mathrm{H}^1(\mathbb{R}^d))$,
- $v \in \mathrm{L}^\infty(0,t;\mathrm{H}^1(\mathbb{R}^d)) \cap \mathrm{L}^2(0,t;\mathrm{H}^2(\mathbb{R}^d)) \cap \mathrm{W}^{1,2}(0,\mathrm{t};\mathrm{L}^2(\mathbb{R}^{\mathrm{d}}))$, $v(0) = v_0$,

and for all $t > 0$ and $\xi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \xi\,(u(t) - u_0)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} (\nabla(u^m) - \chi\, u\,\nabla v)\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}s = 0\,,$$

$$\tau\,\partial_t v = \Delta v - \alpha\, v + u \quad \text{a.e. in} \quad (0,t)\times\mathbb{R}^d\,.$$

Open questions:

- Are they blowingup solutions?
- Are they global-in-time solutions for any mass?

FREE ENERGY FUNCTIONAL

$$\mathcal{E}_\alpha[u,v] := \int_{\mathbb{R}^d} \left\{ \frac{|u(x)|^m}{\chi(m-1)} - u(x)\,v(x) + \frac{1}{2}\,|\nabla v(x)|^2 + \frac{\alpha}{2}\,v(x)^2 \right\} \mathrm{d}x \ ,$$

Let

$$\mathcal{Y}_\alpha(x) := \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{|x|^2}{4s} - \alpha s\right) \mathrm{d}s\,, \quad x \in \mathbb{R}^d\,,$$

For $u \in \mathrm{L}^1(\mathbb{R}^d)$, $S_\alpha(u) := \mathcal{Y}_\alpha * u$ solves $-\Delta S_\alpha(u) + \alpha S_\alpha(u) = u$ in $\mathbb{R}^d$.

HARDY-LITTLEWOOD-SOBOLEV INEQUALITY FOR THE BESSEL KERNEL

For $\alpha > 0$,

$$\sup \left\{ \frac{\int_{\mathbb{R}^d} h(x)\,(\mathcal{Y}_\alpha * h)(x)\,\mathrm{d}x}{\|h\|_m^m\,\|h\|_1^{2/d}} \ : \ h \in (\mathrm{L}^1 \cap \mathrm{L}^m)(\mathbb{R}^d), h \neq 0 \right\} = C_{\mathrm{HLS}} \ .$$

Key estimate:

$$\mathcal{E}_\alpha[u,v] \geq \frac{C_{\mathrm{HLS}}\,\chi_c}{2\chi} \left(1 - \frac{\chi}{\chi_c}\right) \|u\|_m^m \quad \text{and} \quad \|\nabla v\|_2^2 + \alpha\|v\|_2^2 \leq 4\,\mathcal{E}_\alpha[u,v] + C_1\,\|u\|_1^{2/d}\,\|u\|_m^m\,.$$

Introduce the set

$$\mathcal{K} := (\mathcal{P}_2 \cap \mathrm{L}^m)(\mathbb{R}^d) \times \mathrm{H}^1(\mathbb{R}^d)$$

Given $(u_0, v_0) \in \mathcal{K}$ and $h > 0$, we define:

MINIMISING SCHEME

$$\begin{cases} (u_{h,0}, v_{h,0}) = (u_0, v_0)\,, \\ (u_{h,n+1}, v_{h,n+1}) \in \mathrm{Argmin}_{(u,v)\in\mathcal{K}} \mathcal{F}_{h,n}[u,v]\,, \qquad n \geq 0\,, \end{cases} \tag{7}$$

where

$$\mathcal{F}_{h,n}[u,v] := \frac{1}{2h} \left[\frac{\mathcal{W}_2^2(u, u_{h,n})}{\chi} + \tau\, \|v - v_{h,n}\|_2^2 \right] + \mathcal{E}_\alpha[u,v]\,,$$

Let (u,v) be the minimiser of $\mathcal{F}_{h,n}$ in $\mathcal{K}$.
Let $\zeta \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $w \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. For $\delta \in (0,1)$, define $T_\delta := \mathrm{id} + \delta\,\zeta$ and

$$\begin{aligned} u_\delta &:= T_\delta \# u\ , && \text{(perturbation in the "optimal transport" sense)} \\ v_\delta &:= v + \delta\, w && \text{(perturbation in the usual } \mathrm{L}^2\text{-sense)}\ . \end{aligned}$$

We want to compute

$$\lim_{\delta\to 0} \frac{\mathcal{F}_\tau[u_\delta, v_\delta] - \mathcal{F}_\tau[u,v]}{\delta} \quad (\geq 0)$$

All the term are standard, except

$$\int_{\mathbb{R}^d} \frac{u\,v - u_\delta\,v_\delta}{\delta} = \int_{\mathbb{R}^d} u\ \frac{v - v_\delta(\mathrm{id}+\delta\zeta)}{\delta} = \int_{\mathbb{R}^d} u \left[\frac{v - v(\mathrm{id}+\delta\zeta)}{\delta} - w(\mathrm{id}+\delta\zeta)\right]\ ,$$

since

$$\frac{v - v\circ(\mathrm{id}+\delta\zeta)}{\delta} \rightharpoonup -\zeta\cdot\nabla v \quad \text{in } \mathrm{L}^2(\mathbb{R}^d),$$

whereas u is only in $(\mathrm{L}^1 \cap \mathrm{L}^m)(\mathbb{R}^d)$ and $m < 2$.

MAIN DIFFICULTY

Improve the regularity of u.

Preliminary remark: consider the two ordinary differential equations describing gradient flow:

$$\dot{x}(t) = -\nabla\Phi[x(t)] \quad \text{and} \quad \dot{y}(t) = -\nabla\Psi[y(t)]$$

Differentiate each function along the other's flow:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi[y(t)] &= -\langle\nabla\Phi[y(t)], \nabla\Psi[y(t)]\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\Psi[x(t)] &= -\langle\nabla\Psi[x(t)], \nabla\Phi[x(t)]\rangle\end{aligned}$$

Let us consider the following variational problem:

$$\text{Find } u_{h,n} \text{ which minimises} \quad u \mapsto \frac{1}{2h}\mathcal{W}_2^2(u, u_{h,n-1}) + \mathcal{F}[u] \qquad (\textit{i.e. } u_t = -\text{''}\nabla_{\mathcal{W}}\text{''}\mathcal{F}[u])$$

Imagine now that we can find a displacement convex functional $\mathcal{V}$ such that the dissipation of $\mathcal{F}$ along the flow $S^{\mathcal{V}}$:

$$D^{\mathcal{V}}\mathcal{F}[\mu] := \limsup_{t\to 0} \frac{\mathcal{F}[\mu] - \mathcal{F}[S_t^{\mathcal{V}}\mu]}{t} \qquad (\text{''}= -\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[S_t^{\mathcal{V}}\mu]_{|t=0}\text{''})$$

is non-negative. By the preliminary remark

$$D^{\mathcal{V}}\mathcal{F}[u_{h,n}] = \limsup_{t\to 0} \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{t}$$

And as $\mathcal{V}$ is displacement convex, the above the tangent formulation gives:

$$D^{\mathcal{V}}\mathcal{F}[u_{h,n}] \leq \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{h}\ .$$

By definition of the minimising scheme, for any $u \in \mathcal{K}$

$$\frac{1}{2h}\mathcal{W}_2^2(u_{h,n}, u_{h,n-1}) + \mathcal{F}[u_{h,n}] \leq \frac{1}{2h}\mathcal{W}_2^2(u, u_{h,n-1}) + \mathcal{F}[u] \tag{8}$$

Choosing $u = S_t^{\mathcal{V}}(u_{h,n})$ in (8) we obtain

$$\mathcal{F}[u_{h,n}] - \mathcal{F}[S_t^{\mathcal{V}} u_{h,n}] \leq \frac{1}{2h}\left(\mathcal{W}_2^2(S_t^{\mathcal{V}} u_{h,n}, u_{h,n-1}) - \mathcal{W}_2^2(u_{h,n}, u_{h,n-1})\right)$$

Dividing by t and letting $t \to 0$, (9) with $u = u_{h,n}$ and $v = u_{h,n-1}$ yields

$$D^{\mathcal{V}}\mathcal{F}[u_{h,n}] \leq \frac{\mathcal{V}[u_{h,n-1}] - \mathcal{V}[u_{h,n}]}{h}$$

Because $\mathcal{V}$ is displacement convex and $S^{\mathcal{V}}$ is the associated semigroup means

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\mathcal{W}_2^2(S_t^{\mathcal{V}} u, v) \leq \mathcal{V}[v] - \mathcal{V}[S_t^{\mathcal{V}} u] \tag{9}$$

FRAMEWORK

Consider the classical parabolic-elliptic Patlak-Keller-Segel system when $M = 8\pi$ and the 2-moment is unbounded.

THE "CRITICAL" NONLINEAR FOKKER-PLANCK EQUATION

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\left(\sqrt{u}\right) + \dfrac{1}{\sqrt{2\lambda}}\,\mathrm{div}(x\,u) & t > 0\,,\ x \in \mathbb{R}^2\,, \\ u(0) = u_0 \geq 0 & x \in \mathbb{R}^2\,, \end{cases} \tag{10}$$

Define

FAST DIFFUSION FUNCTIONAL

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left(\sqrt{u} - \sqrt{\bar\rho_\lambda}\right)^2}{\sqrt{\bar\rho_\lambda}}\,\mathrm{d}x$$

It follows that for classical solutions u of (10) ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[u(t)] = -\int_{\mathbb{R}^2} u(t,x)\left|\nabla\left(\frac{1}{\sqrt{\bar\rho_\lambda}} - \frac{1}{\sqrt{u}}\right)\right|^2 \mathrm{d}x \leq 0\,.$$

The $\bar\rho_\lambda$ are stationary solutions of (KS).

If ρ is the smooth solution to (KS) with $M = 8\pi$ we obtain

$$\mathcal{D}[\rho(t)] := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -8\int_{\mathbb{R}^2} |\nabla(\rho^{1/4})|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} \rho^{3/2} \,\mathrm{d}x \;.$$

GAGLIARDO-NIRENBERG-SOBOLEV INEQUALITY, [DEL PINO, DOLBEAULT]

For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,

$$\pi \int_{\mathbb{R}^2} |f|^6 \,\mathrm{d}x \le \int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x \int_{\mathbb{R}^2} |f|^4 \,\mathrm{d}x \;,$$

and there is equality if and only if f is a multiple of a translate of $\bar{\rho}_\lambda^{1/4}$ for some $\lambda > 0$.

DISSIPATION OF $\mathcal{H}_\lambda$

For all solution ρ to (KS) of mass $M = 8\pi$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho] = \mathcal{D}[\rho(t)] \le 0 \;,$$

and moreover, there is equality if and only ρ is a translate of $\bar{\rho}_\lambda$ for some $\lambda > 0$.

The displacement convexity of $\mathcal{H}_\lambda$ is formally obvious from the fact that

$$\mathcal{H}_\lambda[u] = \int_{\mathbb{R}^2} \left(-2\sqrt{u(x)} + \sqrt{\frac{1}{2\lambda}\frac{|x|^2}{2}u(x)} \right) \,\mathrm{d}x + C \ .$$

where $-\sqrt{u(x)}$ and $|x|^2 u(x)$ are displacement convex.

Using the MMS technique gives

ABOVE THE TANGENT FORMULATION

$$\mathcal{H}_\lambda[u_n] - \mathcal{H}_\lambda[u_{n+1}] \geq \frac{1}{2}\int_{\mathbb{R}^2} \left[\sqrt{\frac{1}{2\lambda}}\, x + \frac{\nabla u_n}{u_n^{3/2}} \right] \cdot (\nabla\psi(x) - x)\, u_n \,\mathrm{d}x \ .$$

where $\nabla\psi$ is such that $\nabla\psi \# u_n = u_{n+1}$.

GLOBAL EXISTENCE AND LARGE TIME BEHAVIOUR

Given any density ρ_0 with total mass 8π such that there exists $\lambda > 0$ with

$$\mathcal{H}_\lambda[\rho_0] < \infty.$$

Then there exists $\rho \in \mathcal{AC}^0([0,T], \mathcal{P}_2(\mathbb{R}^2))$, with $\rho(t) \in L^1(\mathbb{R}^2)$ for all $t \geq 0$ being a global-in-time weak solution of (KS). Moreover, the solutions constructed satisfy

$$\mathcal{F}_{\mathrm{PKS}}[\rho(t)] \leq \mathcal{F}_{\mathrm{PKS}}[\rho_0]\,,$$

and

$$\mathcal{H}_\lambda[\rho(t)] + \int_0^t \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \mathcal{H}_\lambda[\rho_0]\,.$$

Furthermore,

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\bar\rho_\lambda] \qquad \lim_{t\to\infty} \|\rho(t) - \bar\rho_\lambda\|_{L^1(\mathbb{R}^2)} = 0\,.$$

And the system satisfies the hypercontractivity property *i.e.* for any $t^* > 0$, the constructed solution ρ is bounded in $L^\infty(t^*, \infty, L^p(\mathbb{R}^2))$, for any $p \in (1, \infty)$.

Talagrand's inequality:

$$\mathcal{W}_2^2(\rho, \bar\rho_\lambda) \leq 2\sqrt{2\lambda}\,\mathcal{H}_\lambda[\rho]\,.$$

Basin on attraction: If $\lambda \neq \mu$ then

$$\mathcal{W}_2(\bar\rho_\mu, \bar\rho_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} \left|\frac{\lambda}{\mu}x - x\right|^2 \bar\rho_\mu = +\infty\,.$$

1. INTRODUCTION
   - The classical Patlak-Keller-Segel system
   - The non-linear Keller-Segel model

2. MAIN RESULTS

3. IDEA OF THE PROOF + REVISITING RESULTS ON KS
   - Main tools
   - Application to the classical Keller-Segel system
   - Main idea of the proof

MAIN DIFFICULTY

Above the choice of the auxiliary gradient flow naturally comes from the existence of another Liapunov functional which is different from the energy. Such a nice structure does not seem to be available for our problem.

Let (u, v) be a minimiser of $\mathcal{F}_h$ in $\mathcal{K}$. Introduce the solutions U and V to the initial value problems

$$\begin{cases} \partial_t U - \Delta U = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, & U(0) = u \quad \text{in } \mathbb{R}^d, \\ \partial_t V - \Delta V + \alpha V = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, & V(0) = v \quad \text{in } \mathbb{R}^d. \end{cases} \tag{11}$$

Classical results ensure that $(U(t), V(t))$ belongs to $\mathcal{K}$ for all $t \geq 0$ and therefore

$$\mathcal{F}_h[u,v] \leq \mathcal{F}_h[U(t), V(t)] , \qquad t \geq 0 .$$

Let us compute

$$\mathcal{F}_h[U(t), V(t)] - \mathcal{F}_h[u,v] .$$

- We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\alpha[U,V] = -\mathcal{D} + \mathcal{R}\,,$$

where

$$\mathcal{D}(t) := \frac{4}{m\chi}\|\nabla\left(U^{m/2}(t)\right)\|_2^2 + \|(\Delta V - \alpha V + U)(t)\|_2^2\,, \quad t > 0\,,$$

and

$$\mathcal{R}(t) := \|U(t)\|_2^2 - \alpha\int_{\mathbb{R}^d}(UV)(t,x)\,\mathrm{d}x\,, \quad t > 0\,.$$

Whence

$$\mathcal{E}_\alpha[U(t),V(t)] - \mathcal{E}_\alpha[u,v] \le -\int_0^t \mathcal{D}(s)\,\mathrm{d}s + \int_0^t \mathcal{R}(s)\,\mathrm{d}s\,, \quad t > 0\,.$$

- As the linear heat equation (11) can be interpreted as the gradient flow of the functional $\mathcal{H} = \int u\log u$ for the Kantorovich-Wasserstein distance $\mathcal{W}_2$ in $\mathcal{P}_2(\mathbb{R}^d)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_2^2(U(t),u_0) \le \mathcal{H}[u_0] - \mathcal{H}[U(t)]\,, \quad t > 0\,.$$

We obtain, by monotonicity of $s \mapsto \mathcal{H}[U(s)]$

$$\mathcal{W}_2^2(U(t),u_0) - \mathcal{W}_2^2(u,u_0) \le 2\int_0^t \left(\mathcal{H}[u_0] - \mathcal{H}[U(s)]\right)\mathrm{d}s \le 2t\left(\mathcal{H}[u_0] - \mathcal{H}[U(t)]\right)$$

- Furthermore, it readily follows

$$\|V(t) - v_0\|_2^2 - \|v - v_0\|_2^2 \le t\left(\|\nabla v_0\|_2^2 + \alpha\|v_0\|_2^2 - \|\nabla V(t)\|_2^2 - \alpha\|V(t)\|_2^2\right) \qquad (12)$$

for all $t > 0$.

Combining the above estimates gives, for $t > 0$,

$$\begin{aligned} 0 &\leq \mathcal{F}_h[U(t), V(t)] - \mathcal{F}_h[u, v] \\ &\leq \frac{t}{h\chi}\left(\mathcal{H}[u_0] - \mathcal{H}[U(t)]\right) - \int_0^t \mathcal{D}(s)\,\mathrm{d}s + \int_0^t \mathcal{R}(s)\,\mathrm{d}s \\ &\qquad + \frac{\tau t}{2h}\left(\|\nabla v_0\|_2^2 + \alpha\,\|v_0\|_2^2 - \|\nabla V(t)\|_2^2 - \alpha\,\|V(t)\|_2^2\right)\ , \end{aligned}$$

which also reads

$$\frac{1}{t}\int_0^t \mathcal{D}(s)\,\mathrm{d}s \leq A_h(t) + \frac{1}{t}\int_0^t \mathcal{R}(s)\,\mathrm{d}s\,, \quad t > 0\,, \tag{13}$$

where

$$A_h(t) := \frac{\mathcal{H}[u_0] - \mathcal{H}[U(t)]}{h\chi} + \frac{\tau}{2h}\left(\|\nabla v_0\|_2^2 + \alpha\,\|v_0\|_2^2 - \|\nabla V(t)\|_2^2 - \alpha\,\|V(t)\|_2^2\right)\ .$$

We can control $\mathbb{R}$ and let $t \to 0$ to obtain

$$\frac{4}{m\chi}\|\nabla(u^{m/2})\|_2^2 + \|\Delta v - \alpha v + u\|_2^2 \leq 2A_h(0) + C_2\left(\mathcal{E}_\alpha[u_0, v_0] + \mathcal{E}_\alpha[u_0, v_0]^{1/(m-1)}\right) \tag{14}$$

FURTHER REGULARITY OF THE MINIMISERS

Let $\chi \in (0, \chi_c)$, $(u_0, v_0) \in \mathcal{K}$, $h \in (0, 1)$, and consider a minimiser (u, v) of $\mathcal{F}_h$ in $\mathcal{K}$. Then $u \in \mathrm{L}^2(\mathbb{R}^d)$.

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Merci pour votre attention