Finite time singularities for the free boundary incompressible Euler equations

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September, 2012

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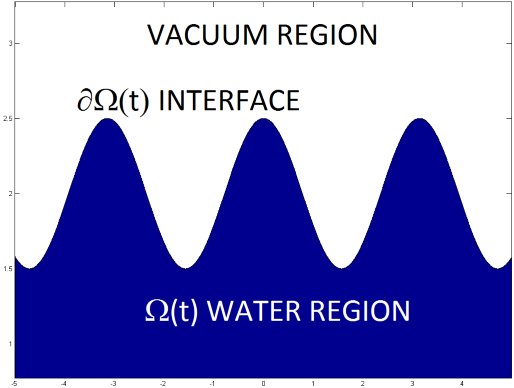
The 2D water wave problem



$$\left.\begin{array}{ll}\text{Velocity:} & u(x,t) \in \mathbb{R}^2 \\ \text{Pressure:} & p(x,t) \in \mathbb{R}\end{array}\right\} \text{defined for } x \in \Omega(t)$$

THE EQUATIONS

- $\nabla \cdot u = 0, \quad \nabla \times u = 0 \quad$ in $\Omega(t)$.
- $u_t + (u \cdot \nabla) u = -\nabla p - \mathrm{g}(0, 1), \quad$ in $\Omega(t)$.
- $p = 0 \quad$ at $\partial\Omega(t)$.
- $\partial\Omega(t)$ moves with the fluid.

Irrotational flows

We assume that the vorticity is zero in the interior of the domain $\Omega(t)$. We can consider that the vorticity is supported on the free boundary curve $z(\alpha, t)$ and it has the form

$$\nabla^{\perp} \cdot u(x, t) = \omega(\alpha, t)\delta(x - z(\alpha, t)).$$

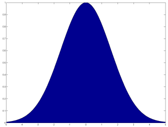
i.e. the vorticity is a Dirac measure on z defined by

$$\langle \nabla^{\perp} \cdot u, \eta \rangle = \int_{\mathbb{R}} \omega(\alpha, t)\eta(z(\alpha, t))d\alpha,$$

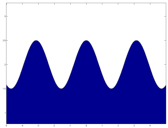
with $\eta(x)$ a test function.

Scenarios

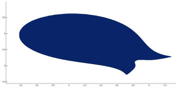
- Asymptotically flat: $\lim_{\alpha \to \infty} (z(\alpha, t) - (\alpha, 0)) = 0,$



- Periodic curves: $z(\alpha + 2k\pi, t) = z(\alpha, t) + 2k\pi(1, 0).$



- Compact: $z(\alpha + 2k\pi, t) = z(\alpha, t).$



The water waves equations

Let us assume that $z(\alpha, t)$ is smooth and satisfies the chord-arc condition. We have to solve

$$\left.\begin{array}{r} \nabla \cdot u = 0 \\ \nabla \times u = 0 \end{array}\right\} \text{in } \Omega(t)$$

We can write

$$u = \nabla\phi \qquad u = \nabla^{\perp}\psi$$

with

$$\begin{array}{cc} \Delta\phi = 0 & \Delta\psi = 0 \\ \phi|_{\partial\Omega(t)} = \Phi & \partial_n\psi|_{\partial\Omega(t)} = -\frac{\partial_\alpha\Phi(\alpha)}{|z(\alpha,t)|} \end{array}$$

"Biot-Savart law"

$$u(x,t) = \frac{1}{2\pi} \int \frac{(x - z(\alpha,t))^{\perp}}{|x - z(\alpha,t)|^2} \omega(\alpha,t) d\alpha \text{ for } x \in \Omega(t) \text{ (interior)}$$

Here, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow v^{\perp} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$

The velocity u, the velocity potential ϕ, and the vorticity amplitude ω, all carry the same information.

However, to obtain ω from u or ϕ, one has to solve an integral equation.

We can write

$$u(z(\alpha,t),t)\cdot z_\alpha(\alpha,t)=\partial_\alpha\Phi(\alpha,t)=BR(z,\omega)\cdot z_\alpha(\alpha,t)+\frac{\omega(\alpha,t)}{2}$$

where the Birkhoff-Rott integral is defined by

$$BR(z,\omega)=\frac{1}{2\pi}PV\int_{\mathbb{R}}\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\omega(\alpha,t)d\alpha$$

Also we have to solve

$$\partial_t u + (u \cdot \nabla) u = -\nabla p - (0,1) \quad \text{in } \Omega(t)$$

that we can write

$$\partial_t \phi + \frac{1}{2}|u|^2 = -p - y.$$

We will take

$$p|_{\partial\Omega(t)} = 0$$

The equations may be rewritten in the form

$$
\begin{aligned}
z_t(\alpha,t) &= BR(z,\omega)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t),\\
\omega_t(\alpha,t) &= -2\partial_t BR(z,\omega)\cdot\partial_\alpha z - \partial_\alpha\Big(\frac{|\omega|^2}{4|\partial_\alpha z|^2}\Big) + \partial_\alpha(c\,\omega)\\
&\quad + 2c\,\partial_\alpha BR(z,\omega)\cdot\partial_\alpha z(\alpha,t) - 2\mathrm{g}\partial_\alpha z_2,
\end{aligned}
$$

where the Birkhoff-Rott integral is defined by

$$
BR(z,\omega) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\,\omega(\alpha,t)\,d\alpha
$$

The linearized equation

A linearization around a flat contour $(\alpha, \epsilon f(\alpha, t))$ and $\omega = \epsilon g(\alpha, t)$, allows us to find

$$f_t(\alpha, t) = H(g)(\alpha, t),$$
$$g_t(\alpha, t) = -\sigma \partial_\alpha f(\alpha, t),$$

It can be written as follows:

$$f_{tt}(\alpha, t) = -\sigma \Lambda(f)(\alpha, t), \qquad \begin{cases} \sigma < 0 \Rightarrow & e^{|\sigma\xi|^{\frac{1}{2}} t}, \quad e^{-|\sigma\xi|^{\frac{1}{2}} t} \\ \sigma > 0 \Rightarrow & \cos(|\sigma\xi|^{\frac{1}{2}} t), \quad \sin(|\sigma\xi|^{\frac{1}{2}} t) \end{cases}$$

and

$$E_L(t) = \frac{1}{2} \int (\sigma |\partial_\alpha f|^2 + |\Lambda^{\frac{1}{2}} g|^2) d\alpha.$$

Rayleigh-Taylor condition

- Rayleigh-Taylor condition:

$$\sigma(\alpha,t) = -(\nabla p(z(\alpha,t),t)) \cdot \partial_\alpha^\perp z(\alpha,t) > 0,$$

By taking the divergence of the Euler equation

$$-\Delta p = |\nabla v|^2 \geq 0$$

which, together with the fact that the pressure is zero on the interface then by Hopf's lemma

$$\sigma(\alpha,t) \equiv -|z_\alpha^\perp(\alpha,t)|\partial_n p(z(\alpha,t),t) > 0,$$

where ∂_n denotes the normal derivative.

Local existence

Theorem (Sijue Wu, 1997)

Local existence for initial data satisfying $z_0(\alpha) \in H^k$ and $\omega_0(\alpha) \in H^{k-1}$ ($k \geq 4$),

$$\mathcal{F}(z_0)(\alpha, \beta) < \infty, \quad \text{and} \quad \sigma(\alpha, 0) > 0.$$

where

$$\mathcal{F}(z)(\alpha, \beta, t) = \frac{|\beta|}{|z(\alpha, t) - z(\alpha - \beta, t)|} \qquad \forall\, \alpha, \beta \in (-\pi, \pi),$$

and

$$\mathcal{F}(z)(\alpha, 0, t) = \frac{1}{|\partial_\alpha z(\alpha, t)|}.$$

Energy of the system

Case $z \in H^4$:

$$E(t) = \|z\|^2_{H^3}(t) + \int_{-\pi}^{\pi} \frac{\sigma(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} |\partial_\alpha^4 z(\alpha,t)|^2 d\alpha$$
$$+ \|\mathcal{F}(z)\|^2_{L^\infty}(t) + \|\omega\|^2_{H^2}(t) + \|\varphi\|^2_{H^{4-\frac{1}{2}}}(t),$$

for $\sigma(\alpha,t) > 0$ and $\varphi(\alpha,t)$ given by

$$\varphi(\alpha,t) = \frac{\omega(\alpha,t)}{2|\partial_\alpha z(\alpha,t)|} - c(\alpha,t)|\partial_\alpha z(\alpha,t)|.$$

- Beale, Hou & Lowengrub (1993)
- Ambrose & Masmoudi (2005)

Previous work

- Solutions exist and stay smooth for short time

  [Sijue Wu (1997); see also Lannes, Christodoulou-Lindblad, Lindblad, Ambrose-Masmoudi, Coutand-Shkoller, Shatah-Zeng, Cordoba-Cordoba-Gancedo, Alazard-Burq-Zuilly,..]

- For small initial data solutions remain smooth for exponentially long time

  [Sijue Wu (2009)]

Results on 3D water waves

- Global existence for small initial data

  [Wu (2011) and Germain-Masmoudi-Shatah (2011)]

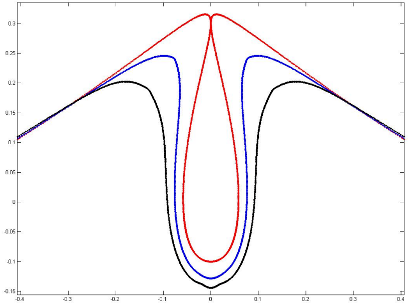
- Also may drop, for local existence, restriction to irrotational flows

  [Christodoulou-Lindblad, Zhang-Zhang]

Numerics

- Global existence
- Turning and ....
- Singularity

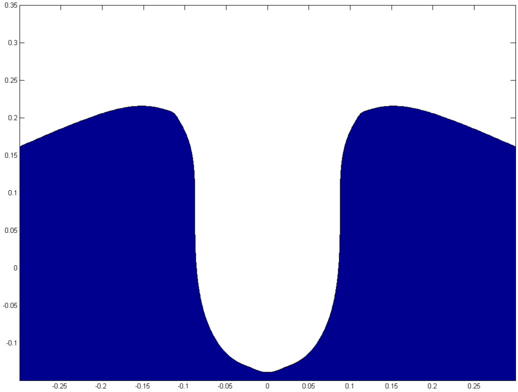
Splash Singularity



Numerics were performed using the method developed by Beale, Hou and Lowengrub with special modifications to keep it reliable as we approach the SPLASH.

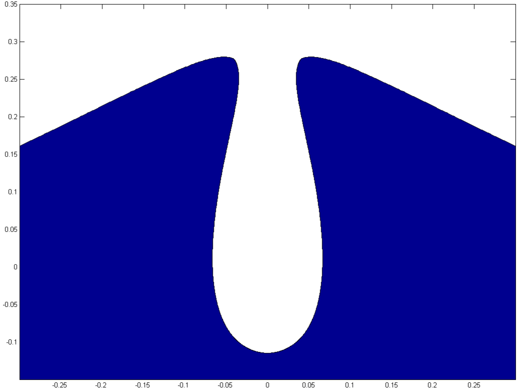
Salient Features

At first, the interface $\partial\Omega(t)$ is the graph of a function $x_2 = F(x_1, t)$.



Salient Features

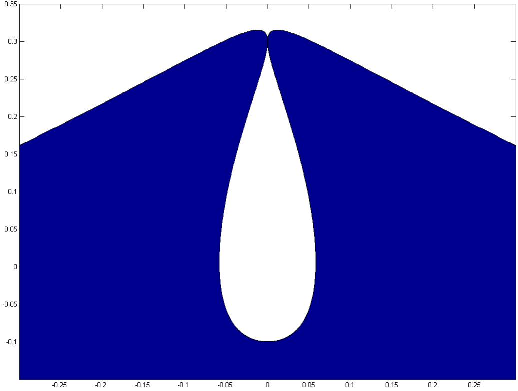
Later, the interface $\partial\Omega(t)$ is no longer the graph of a function



We say that a "turnover" has occurred.

Salient Features

Still later, the interface $\partial\Omega(t)$ touches itself at one point, but is otherwise smooth.



We say that our solution reaches a SPLASH.

Singularities for Water waves: Theorems

Theorem: Turnover

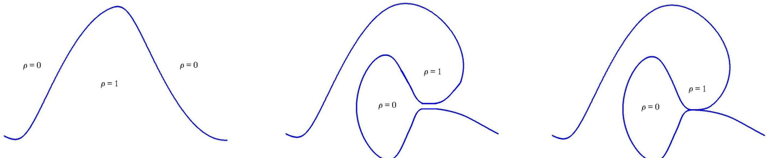
There exists a non-empty open set of smooth initial data for which the solution of water waves may start as a graph at time t_0, then fail to be a graph at a later time t_1. That is, a TURNOVER may occur.

Theorem: Splash singularity

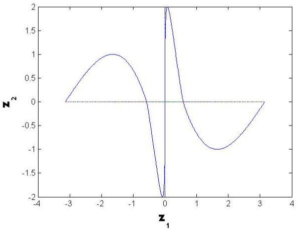
There exists a non-empty open set of smooth initial data for which the solution of water waves develops a splash singularity in finite.

Theorem: Stability from the splash

Given an approximate solution $(x(\alpha, t), \gamma(\alpha, t))$ of water waves (up to the splash) then near $(x(\alpha, t), \gamma(\alpha, t))$ there exists an exact solution $(z(\alpha, t), \omega(\alpha, t))$ of water waves.



Ideas of the proof of Turnover



- There exists a curve $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ and an amplitude of the vorticity $\omega(\alpha)$ with the following properties:

1. $z_1(\alpha) - \alpha$ and $z_2(\alpha)$ are smooth $2\pi -$ *periodic* functions and $z(\alpha)$ satisfies the arc-chord condition,
2. $z(\alpha)$ is odd and
3. $\partial_\alpha z_1(\alpha) > 0$ if $\alpha \neq 0$, $\partial_\alpha z_1(0) = 0$ and $\partial_\alpha z_2(0) > 0$,

such that

$$(\partial_\alpha v_1)(0) < 0.$$

Steps of the proof of splash singularity

- The water wave equations are invariant under time reversal. To obtain a solution that ends in a splash, we can therefore take our initial condition to be a splash, and show that there is a smooth solution for small times $t > 0$.

What is NOT a Splash

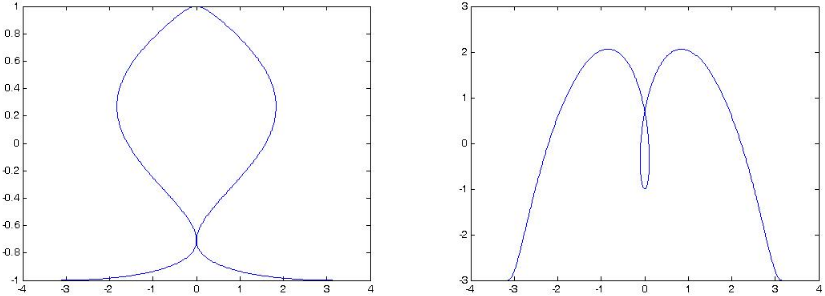
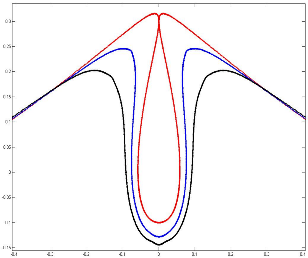
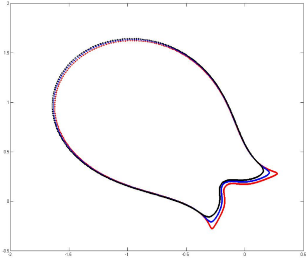


Figure: Two examples of non-splash curves.

For a splash curve we cannot use the amplitude of the vorticity.





$$\Omega(t) \qquad \longrightarrow \qquad \tilde{\Omega}(t)$$

$$P(w) = \left(\tan\left(\frac{w}{2}\right)\right)^{\frac{1}{2}}$$

We define the new potential and a stream function

$$\tilde{\psi}(\tilde{x},\tilde{y},t) \equiv \psi(P^{-1}(\tilde{x},\tilde{y}),t), \qquad \tilde{\phi}(\tilde{x},\tilde{y},t) \equiv \phi(P^{-1}(\tilde{x},\tilde{y}),t)$$

the new velocity,

$$\tilde{v}(\tilde{x},\tilde{y},t) \equiv \nabla\tilde{\phi}(\tilde{x},\tilde{y},t)$$

and the restriction

$$\tilde{\Phi}(\alpha,t) = \tilde{\phi}(\tilde{z}(\alpha,t),t), \qquad \tilde{\Psi}(\alpha,t) = \tilde{\psi}(\tilde{z}(\alpha,t),t).$$

Thus

$$\begin{aligned} \Delta\tilde{\phi}(\tilde{x},\tilde{y},t) &= 0 \quad \text{in } P(\Omega(t)) \\ \tilde{\phi}\Big|_{\tilde{z}(\alpha,t)} &= \tilde{\Phi}(\alpha,t) \\ \tilde{v} &\equiv \nabla\tilde{\phi} \quad \text{in } P(\Omega(t)) \end{aligned}$$

And also

$$\Phi(\alpha,t) = \tilde{\Phi}(\alpha,t) \quad \Psi(\alpha,t) = \tilde{\Psi}(\alpha,t) \quad u_{\text{normal}}(\alpha,t) = \tilde{u}_{\text{normal}}(\alpha,t)$$

IMPORTANT: In $P(\Omega(t)) = \tilde{\Omega}(t)$ we can use the amplitude of the vorticity $\tilde{\omega}$ even if the curve in $\Omega(t)$ is a splash curve.

Take $\tilde{z}(\alpha, t)$ and $\tilde{\omega}(\alpha, t)$ to be the unknowns. Write the water wave equation in the form

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t) BR(\tilde{z}, \tilde{\omega})(\alpha, t) + \tilde{c}(\alpha, t)\tilde{z}_\alpha(\alpha, t)$$

$$\begin{aligned}\tilde{\omega}_t(\alpha, t) = & -2\partial_t BR(\tilde{z}, \tilde{\omega})(\alpha, t) \cdot \tilde{z}_\alpha(\alpha, t) - |BR(\tilde{z}, \tilde{\omega})|^2 \partial_\alpha Q^2(\alpha, t) \\ & -\partial_\alpha \left(\frac{Q^2(\alpha, t)}{4} \frac{\tilde{\omega}(\alpha, t)^2}{|z_\alpha(\alpha, t)|^2} \right) + 2\tilde{c}(\alpha, t)\partial_\alpha BR(\tilde{z}, \tilde{\omega}) \cdot \tilde{z}_\alpha(\alpha, t) \\ & +\partial_\alpha \left(\tilde{c}(\alpha, t)\tilde{\omega}(\alpha, t) \right) - 2\mathrm{g}\partial_\alpha \left(P_2^{-1}(\tilde{z}(\alpha, t)) \right).\end{aligned}$$

where

$$Q^2(\alpha, t) = \left| \frac{dP}{dw}(z(\alpha, t)) \right|^2.$$

Local Existence in the Tilde Domain

Theorem

Let $z^0(\alpha)$ *be a splash curve. Let* $u^0(\alpha) \cdot (z^0_\alpha)^\perp(\alpha) \in H^4(\mathbb{T})$ *satisfying:*

1. $u^0(\alpha_1) \cdot \frac{(z^0_\alpha)^\perp}{|z^0_\alpha|} < 0, u^0(\alpha_2) \cdot \frac{(z^0_\alpha)^\perp}{|z^0_\alpha|} < 0.$
2. $\int_{\partial\Omega} u^0 \cdot \frac{(z^0_\alpha)^\perp}{|z^0_\alpha|} ds = \int_{\mathbb{T}} u^0(\alpha) \cdot (z^0_\alpha)^\perp d\alpha = 0.$

Then there exist a finite time $T > 0$*, a curve* $\tilde{z}(\alpha, t) \in C([0, T]; H^4)$ *satisfying:*

1. $P^{-1}(\tilde{z}_1(\alpha, t)) - \alpha, P^{-1}(\tilde{z}_2(\alpha, t))$ *are* 2π*-periodic,*
2. $P^{-1}(\tilde{z}(\alpha, t))$ *satisfies the arc-chord condition for all* $t \in (0, T]$*,*

and $\tilde{u}(\alpha, t) \in C([0, T]; H^3(\mathbb{T}))$ *which provides a solution of the water waves equations in the new domain* $\tilde{z}^0(\alpha) = P(z^0(\alpha))$*.*

A priori energy estimates

$$E(t) = \|\tilde{z}\|^2_{H^3}(t) + \int_{\mathbb{T}} \frac{Q^2\sigma_{\tilde{z}}}{|\tilde{z}_\alpha|^2}|\partial_\alpha^4\tilde{z}|^2 d\alpha(t) + \|F(\tilde{z})\|^2_{L^\infty}(t)$$
$$+ \|\tilde{\omega}\|^2_{H^2}(t) + \|\varphi\|^2_{H^{3+\frac{1}{2}}}(t) + \frac{|\tilde{z}_\alpha|^2}{m(Q^2\sigma_{\tilde{z}})(t)} + \sum_{l=0}^{4}\frac{1}{m(q_l)(t)}$$

where

$$\varphi(\alpha,t) = \frac{Q^2(\alpha,t)\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|} - \tilde{c}(\alpha,t)|\tilde{z}_\alpha(\alpha,t)|$$
$$c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$
$$-\int_{-\pi}^{\alpha}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$

$\sigma_{\tilde{z}}$ is the R-T function.

The function φ allows two show the following cancelation:

$$\frac{d}{dt}\int_{\mathbb{T}}\frac{Q^2\sigma_{\tilde{z}}}{|\tilde{z}_\alpha|^2}|\partial_\alpha^4\tilde{z}|^2 d\alpha = \text{Controlled Quantities} + S$$
$$\frac{d}{dt}\int \Lambda\partial_\alpha^3\varphi(\alpha,t)\partial_\alpha\varphi(\alpha,t)d\alpha = \text{Controlled Quantities} - S$$

where

$$S = \int_{-\pi}^{\pi} 2Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\Lambda(\partial_\alpha^3\varphi)d\alpha,$$

We prove that

$$\left|\frac{d}{dt}\tilde{E}(t)\right| \leq C + C(\tilde{E}(t))^{100}.$$

The regularization

$$
\begin{aligned}
z_t^{\varepsilon,\delta,\mu}(\alpha,t) &= \phi_\delta * \phi_\delta * \left(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})\right)(\alpha,t) \\
&\quad + \phi_\mu * \left(c^{\varepsilon,\delta,\mu}\left(\phi_\mu * \partial_\alpha z^{\varepsilon,\delta,\mu}\right)\right)(\alpha,t),
\end{aligned}
$$

$$
\begin{aligned}
\omega_t^{\varepsilon,\delta,\mu} = &+ \ldots \\
&+ \ldots \\
&- 2\varepsilon \frac{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})} \Lambda(\phi_\mu * \phi_\mu * \varphi^{\varepsilon,\delta,\mu}),
\end{aligned}
$$

$$
\begin{aligned}
c^{\varepsilon,\delta,\mu}(\alpha) =& \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta))}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2} \\
&\times \phi_\delta * \phi_\delta * (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta \\
&-\int_{-\pi}^{\alpha}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2} \\
&\times \phi_\delta * \phi_\delta * (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta,
\end{aligned}
$$

$$
\varphi^{\varepsilon,\delta,\mu} = \frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|} - \mathcal{C}^{\varepsilon,\delta,\mu},
$$

$$
\begin{aligned}
\mathcal{C}^{\varepsilon,\delta,\mu} =& \phi_\delta * \phi_\delta * \left(\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|}\cdot(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})))(\beta)d\beta\right. \\
&-\phi_\delta * \phi_\delta * \left(\int_{-\pi}^{\alpha}\frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|}\cdot(\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta\right).
\end{aligned}
$$

Stability

(x, γ, ζ) are the solutions of

$$
\left\{
\begin{array}{lll}
x_t & = & Q^2(x)BR(x,\gamma) + bx_\alpha + f \\
b & = & \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}(Q^2BR)_\beta \frac{x_\alpha}{|x_\alpha|^2}d\beta}_{b_s} \\
 & & + \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi} f_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha} f_\beta \frac{x_\beta}{|x_\beta|^2}d\beta}_{b_e} \\
\gamma_t & & +2BR_t(x,\gamma)\cdot x_\alpha = -(Q^2(x))_\alpha |BR(x,\gamma)|^2 + 2bBR_\alpha(x,\gamma)\cdot \\
 & & \qquad +(b\gamma)_\alpha - \left(\frac{Q^2(x)\gamma^2}{4|x_\alpha|^2}\right)_\alpha - 2(P_2^{-1}(x))_\alpha + g \\
\zeta(\alpha,t) & = & \frac{Q_x^2(\alpha,t)\gamma(\alpha,t)}{2|x_\alpha(\alpha,t)|} - b_s(\alpha,t)|x_\alpha(\alpha,t)|
\end{array}
\right.
$$

$$
\begin{aligned}
D(\alpha,t) \equiv& z(\alpha,t) - x(\alpha,t)\\
d(\alpha,t) \equiv& \omega(\alpha,t) - \gamma(\alpha,t)\\
\mathcal{D}(\alpha,t) \equiv& \varphi(\alpha,t) - \zeta(\alpha,t)
\end{aligned}
$$

$$
\mathcal{E}(t) \equiv \left(\|D\|^2_{H^3} + \int_{-\pi}^{\pi} \frac{Q^2 \sigma_z}{|z_\alpha|^2} |\partial_\alpha^4 D|^2 + \|d\|^2_{H^2} + \|\mathcal{D}\|^2_{H^{3+\frac{1}{2}}} \right).
$$

Then we have that

$$
\left| \frac{d}{dt} \mathcal{E}(t) \right| \leq \mathcal{C}(t)(\mathcal{E}(t) + \delta(t))
$$

where

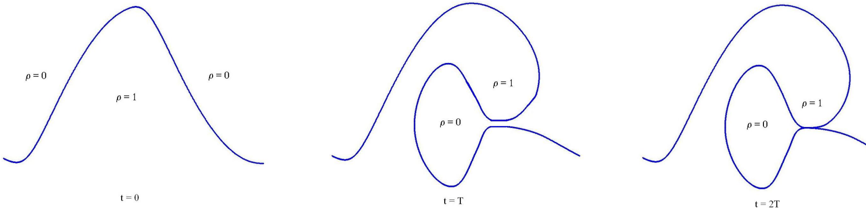
$$
\mathcal{C}(t) = \mathcal{C}(E(t), \|x\|_{H^{5+\frac{1}{2}}}(t), \|\gamma\|_{H^{3+\frac{1}{2}}}(t), \|\zeta\|_{H^{4+\frac{1}{2}}}(t), \|F(x)\|_{L^\infty}(t))
$$

and

$$
\delta(t) = (\|f\|_{H^{5+\frac{1}{2}}}(t) + \|g\|_{H^{3+\frac{1}{2}}}(t))^k
$$

We would like to prove

(III) There exists a solution of the water wave problem,for which the interface starts as a graph, then turns over, and finally forms a SPLASH.



Graph to Splash: Sketch of the proof

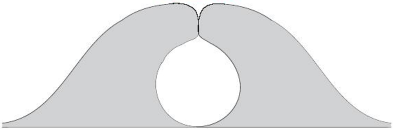
- Compute the constant in the stability theorem, i.e. quantify how fast solutions with near starting conditions separate.
- From a given solution obtained by simulation, calculate (using a computer!!) rigorous bounds in some H^k norm on how well the candidate satisfies the equation.
- By the stability theorem, there should be a function which solves the water waves equation, is a graph at time 0 and a splash at time T which is close enough to the candidate.

Further Results

- Singularities in 3D
- Splat

A Variant of the Splash:

SPLAT!



At time t_2, the interface self-intersects along an arc, but u and $\partial\Omega$ are otherwise smooth.

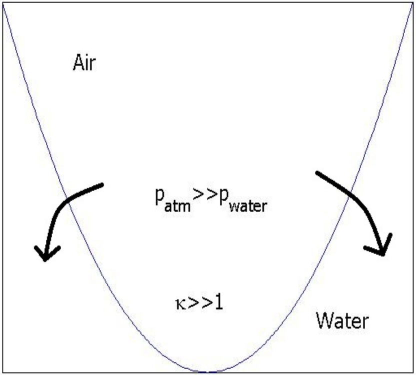
- Surface tension

Splash singularities with surface tension

Laplace-Young law

$$p_{\text{atm}} - p_{\text{fluid}} = \tau\kappa$$

where τ is a constant which depend on the fluid we are considering and κ is the curvature.



Water Waves equations with surface tension

In the physical domain

$$
\begin{aligned}
z_t(\alpha,t) &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t),\\
\varpi_t(\alpha,t) &= -2\partial_t BR(z,\varpi)\cdot\partial_\alpha z - \partial_\alpha\Big(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\Big) + \partial_\alpha(c\,\varpi)\\
&\quad + 2c\,\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z(\alpha,t) - 2\mathrm{g}\partial_\alpha z_2 + \tau\partial_\alpha\kappa(\alpha,t)
\end{aligned}
$$

In the tilda domain

$$
\begin{aligned}
\tilde{z}_t(\alpha,t) =& Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})(\alpha,t) + \tilde{c}(\alpha,t)\tilde{z}_\alpha(\alpha,t)\\
\tilde{\omega}_t(\alpha,t) =& -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)\\
& - \partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|z_\alpha(\alpha,t)|^2}\right) + 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t)\\
& + \partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2\partial_\alpha\left(P_2^{-1}(\tilde{z}(\alpha,t))\right) + \tau NT(\alpha,t).
\end{aligned}
$$

The new term

$$
NT(\alpha,t) = \partial_\alpha\left(Q\tilde{\kappa}(\alpha,t)\right) + \text{extra lower terms}
$$

The new energy

- Energy without the R-T condition.

$$E = \ldots + 2|\tilde{z}_\alpha|^3 \int Q^7 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2 + \frac{1}{\tau} \int Q^8 \partial_\alpha^3 \tilde{\varpi} \Lambda \partial_\alpha^3 \tilde{\varpi} + \frac{1}{2|\tilde{z}_\alpha|\tau^2} \int Q^9 \left(\partial_\alpha^3 \tilde{\varpi}\right)^2 \tilde{\varpi}^2$$

Ambrose (2003)

- Energy with the R-T condition.

$$E = \ldots + \frac{\tau|\tilde{z}_\alpha|^3}{2} \int Q^7 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2 + \int Q^4 \partial_\alpha^3 \tilde{\varphi} \Lambda \tilde{\varphi} + |\tilde{z}_\alpha|^2 \tau \int \left(C||\tilde{\kappa}||_{H^1}(t) + \tilde{\kappa}\right) Q^7 \partial_\alpha^2 \tilde{\kappa} \Lambda \partial_\alpha^2 \tilde{\kappa} + 2|\tilde{z}_\alpha| C||\tilde{\kappa}||_{H^1}(t) \int Q^4 \left(\partial_\alpha^3 \tilde{\varpi}\right)^2 + |\tilde{z}_\alpha| \int \sigma Q^6 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2$$

Ambrose and Masmoudi (2005) and Ambrose and Masmoudi (2009)

Thank you!

References

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