# Finite time blow up in the Nordheim equation for bosons

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# Plan

1. Introduction and main result.
2. Motivation: B-E condensation.
3. Sketch of the proof.

# The Nordheim Equation for bosons

$f \equiv f(p,t)$: density function of particles with momentum $p$ at time $t$ (mass$= 1$).

Homogeneous gas: the density function $f$ is independent of the space variable $x$.

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,p) &= Q(f)(t,p), \qquad t > 0,\ p \in \mathbb{R}^3. \\
Q(f)(t,p) &= \int\int\int_{\mathbb{R}^9} W\left(p, p_2, p_3, p_4\right) q(f) dp_2 dp_3 dp_4 \\
q\left(f\right) &= f_3 f_4 (1+f)(1+f_2) - f\, f_2 (1+f_3)(1+f_4) \\
f_i &= f(p_i),\ i = 2, 3, 4.
\end{aligned}$$

L. W. Nordheim 1928.

(In the Boltzmann equation for classical particles: $q\left(f\right) = f_3 f_4 - f\, f_2$)

# General Remarks

- One may use Born approximation:

$$W(p, p_2, p_3, p_4) = \delta(p + p_2 - p_3 - p_4)\ \delta\left(|p|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2\right)$$

- Given $f$ we define: $M(f) = \int_{\mathbb{R}^3} f(p)dp,$

$$P(f) = \int_{\mathbb{R}^3} f(p)pdp, \quad E(f) = \int_{\mathbb{R}^3} f(p)|p|^2 dp$$

If $f = f(t, p)$ solves the Nordheim equation then, formally:

$$\frac{d}{dt}M(f(t)) = \frac{d}{dt}P(f(t)) = \frac{d}{dt}E(f(t)) = 0.$$

# In radial variables

If $f(t,p) = f(t,|p|^2)$, the Nordheim equation for bosons reads:

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,\epsilon_1) = \dfrac{8\pi^2}{\sqrt{2}} \displaystyle\int_{D(\epsilon_1)} w\left(\epsilon_1,\epsilon_3,\epsilon_4\right) q(f) d\epsilon_3 d\epsilon_4 \\ \\ q\left(f\right) = f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \end{cases}$$

$$\epsilon = |p|^2, \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

$$D\left(\epsilon_1\right) \equiv \{(\epsilon_3,\epsilon_4) : \epsilon_3 > 0, \epsilon_4 > 0, \epsilon_3+\epsilon_4 \geq \epsilon_1 > 0\}$$

$$w\left(\epsilon_1,\epsilon_3,\epsilon_4\right) = \frac{\min\left(\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right)}{\sqrt{\epsilon_1}}.$$

# Mild solutions

**Definition** Suppose that $\gamma > 3$ and $0 \leq T_1 < T_2 < +\infty$. We will say that a function $f \in L^{\infty}_{loc}\left([T_1, T_2); L^{\infty}\left(\mathbb{R}^+; (1+\epsilon)^{\gamma}\right)\right)$ is a mild solution if it satisfies:

$$\boxed{f(t,\epsilon_1) = f_0(\epsilon_1)\Psi(t,\epsilon_1) + \frac{8\pi^2}{\sqrt{2}}\int_{T_1}^{t}\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^{\infty}\int_0^{\infty} f_3 f_4 (1+f_1+f_2)\, W d\epsilon_3 d\epsilon_4 ds}$$

a.e. $t \in [T_1, T_2)$, where:

$$a(t,\epsilon_1) = \frac{8\pi^2}{\sqrt{2}}\int_0^{\infty}\int_0^{\infty} f_2(1+f_3+f_4)\, W d\epsilon_3 d\epsilon_4$$

$$\Psi(t,\epsilon_1) = \exp\left(-\int_{T_1}^{t} a(s,\epsilon_1)\, ds\right).$$

# Existence of Mild Solutions

**Theorem.** Suppose that $f_0 \in L^\infty\left(\mathbb{R}^+; (1+\epsilon)^\gamma\right)$ with $\gamma > 3$. There exists $T > 0$, depending only on $\|f_0(\cdot)\|_{L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)}$, and there exists a unique mild solution, $f \in L^\infty_{loc}\left([0,T); L^\infty\left(\mathbb{R}^+; (1+\epsilon)^\gamma\right)\right)$ in the sense of the previous definition. The solution $f$ satisfies (mass & energy conservation):

$$\int_0^\infty f_0(\epsilon)\,\epsilon^r d\epsilon = \int_0^\infty f(t,\epsilon)\,\epsilon^r d\epsilon\,, \quad t \in (0,T)\ , \quad r = \frac{1}{2}, \frac{3}{2}.$$

The function $f$ is in $W^{1,\infty}\left((0,T); L^\infty\left(\mathbb{R}^+\right)\right)$ and it satisfies the equation for $a.e.\ \epsilon \in \mathbb{R}^+$ and for any $t \in (0, T_{\max})$. Moreover, $f$ can be extended as a mild solution to a maximal time interval $(0, T_{\max})$ with $T_{\max} \leq \infty$. If $T_{\max} < \infty$ we have:

$$\lim_{t \to T_{\max}^-} \sup \|f(t,\cdot)\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

In order to state our main result we need...

# Equilibria I

- Stationary solutions of the equation:

$$F(p) = \frac{1}{e^{\beta|p-p_0|^2-\mu} - 1}, \quad \beta > 0,\ p_0 \in \mathbb{R}^3,\ \ \mu \leq 0.$$

- $\beta = 1/T$, where $T$ is the temperature of the gas.
- They satisfy: $q(F) \equiv 0$ and then $Q(F) = 0$.
- It may be easily checked that for all such equilibria:

$$M(F) \leq \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}} \left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E(F)^{\frac{3}{5}}$$

- If $M \leq \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}} \left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E^{\frac{3}{5}}$, there is $F$ such that $M(F) = M$ and $E(F) = E$.

# Blow up Theorem

**Theorem.** Suppose that $f_0 \in L^\infty\left(\mathbb{R}^+; (1+\epsilon)^\gamma\right)$ with $\gamma > 3$. Define:

$$M = 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon}\,d\epsilon, \quad E = 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon^3}\,d\epsilon$$

Let $f \in L^\infty_{loc}\left([0, T_{\max}); L^\infty\left(\mathbb{R}^+; (1+\epsilon)^\gamma\right)\right)$ be the mild solution of the radial Nordheim equation with initial data $f_0$ where $T_{\max}$ is the maximal existence time. Suppose that:

$$M > \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}} \left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E^{\frac{3}{5}}.$$

Then:

$$T_{\max} < \infty,$$

and therefore

$$\lim_{t \to T^-_{max}} ||f(t)||_{L^\infty} = +\infty.$$

# Motivation: Relation with B-E Condensation?

B-E Condensation: predicted by S. N. Bose (1924) and A. Einstein (1925).

Observed first by E. Cornell, C. Wieman & al. in 1995:

Given a non interacting gas of bosons at equilibrium:

-The temperature is slowly lowered.

-Below a critical temperature a macroscopical fraction of particles appears at the minimum energy level of the system: a BE condensate forms.

-All the particles in the condensate are described by the same wave function.

# Equilibrium II

We have seen equilibria $F$ for "sub-critical" pairs $(M, E)$. Since $\beta = 1/T$:

$$M(F) \leq \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}}\left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E(F)^{\frac{3}{5}} \Longleftrightarrow T \geq T_{cr}$$

where $T_{cr} = T_{cr}(M, E)$ is a critical temperature. A new family of equilibria:

$$F(p) = \frac{1}{e^{\beta|p-p_0|^2} - 1} + \alpha\,\delta(p - p_0), \ \ \alpha \geq 0; \quad Q(F) = 0$$

are such that: $M(F) > \frac{\zeta\left(\frac{3}{2}\right)}{\zeta\left(\frac{5}{2}\right)^{\frac{3}{5}}}\left(\frac{4\pi}{3}\right)^{\frac{3}{5}} E(F)^{\frac{3}{5}} \quad (\Longleftrightarrow T < T_{cr}.)$

The presence of a Dirac measure is the precise formulation (in this setting) of having a large fraction of particles at the minimum energy level of the system: a B-E condensate. (Bose 1924, Einstein 1925).

# At non equilibrium ?

In several cases (like spatially homogeneous gases): possible to deduce a simplified model (Khalatnikov ('64), Kirkpatrick&al. ('85), Gardiner&al. ('98), Stoof ('99))

The density of particles in the gas is described by the Nordheim equation.

In presence of condensate: a system of a Boltzmann type equation coupled with the Gross Pitaevskii (cubic Schrödinger) equation.

The two unknown functions are: density $f$ of particles in the gas and the wave function $\Psi$ of the particles in the condensate.

For homogeneous gases the G-P equation reduces to an ODE for the number of particles in the condensate: $n_c(t) = ||\Psi(t)||_2^2$:

$$\begin{cases} \dfrac{\partial f}{\partial t} = Q(f) + n_c(t)\mathcal{Q}(n, f); \;\; \mathcal{Q}(n_c, f) : \text{describes the gas-condensate collisions} \\ n_c'(t) = -n_c(t) \displaystyle\int_0^\infty \mathcal{Q}(n_c, f)(t, x)dx \end{cases}$$

# Dirac measures and evolution

Nordheim evolution equation makes sense for radial distributions of the form $f(t) + n(t)\delta$ (D. V. Semikoz & I. I. Tkachev '95), and even for general radial measures (X. Lu '05, see below).

If we plug $f(t,p) + n(t)\delta$ in the equation, we obtain system above! (D. V. Semikoz & I. I. Tkachev '95)

"Then", since a BE condensation is observed in the experiments... one expects a Dirac mass to appear in finite time in the density function of the particles that solves the Nordheim equation.

Problem: describe dynamically how does the Dirac measure appears.

# Criticism to the simplified approximation.

A remark in all the physical references: kinetic processes can not lead to a macroscopic occupation of the one particle ground state. If no condensate initially it will not appear at any finite time.

In terms of the system: The equation for the density of particles in the condensate:

$$n_c'(t) = n_c(t) \int_0^\infty \mathcal{Q}(n_c, f)(t,x)dx.$$

That means: if $n_c(\tau) = 0$ and $\int_0^\infty \mathcal{Q}(n_c, f)(t,x)dx$ is a bounded function of time near $\tau$ then $n_c(t)$ remains zero for $t > \tau$......

We then need $\int_0^\infty \mathcal{Q}(n_c, f)(t,x)dx$ not bounded at some finite time $t_c$.

A first step is to prove that $f(t)$ itself does not remain bounded for all times.

# Previous references:

1.- E. Levich & V. Yakhot Study a more general equation:

$$\frac{\partial f}{\partial t}(t,p) = Q(f)(t,p) + \widetilde{Q}(f)(t,p)$$

$\widetilde{Q}$ describes collisions of bosons with a heat bath of fermions.

Collisions with the heat bath form a peak for $f$ at small values of $\varepsilon$.

1.-A (Phys. Rev. B.'77): $Q(f) + \widetilde{Q}(f)$ drive $f$ close to an equilibrium of $Q(f)$. Then $\widetilde{Q}(f)$ is dominant $\longrightarrow$ Delta formation in infinite time.

1.-B (J. Low Temp.'77): When $f >> 1$: neglect $\widetilde{Q}$ (quadratic) in front of $Q$ (cubic). Consider a simplified version of the Nordheim equation. Explicitely solvable. $\longrightarrow$ Delta formation in finite time: $\sqrt{\varepsilon} f(t,\varepsilon) \rightharpoonup C\delta(\varepsilon)$, as $t \to t_0$.

2.- D. V. Semikoz & I. I. Tkachev (PRL 1995) On the ground of numerics and previous work by B. S. Svistunov (J. Moscow Phys. Soc. '91) , propose that the Nordheim equation has solutions of the self similar form:

$$f(t,\epsilon) = C(t_c - t)^{-\alpha}\phi\left(\frac{\epsilon}{(t_c - t)^{\beta}}\right)$$

for some $t_c > 0$ and $\phi$ a bounded integrable function.

- If $\phi(x) \sim x^{-\nu}$ as $x \to +\infty$ and $\beta\nu = \alpha$ we would have $f(t,\epsilon) \sim \epsilon^{-\nu}$ as $t \to t_c$. Numerics suggests that $\nu \approx 1,24$.

3.- Similar in R. Lacaze, P. Lallemand, Y.Pomeau & S. Rica (Physica D 2001). $\nu \approx 1,234$

# Rigorous result

**Theorem** (X. Lu J. Stat. Phys. 2005). For any initial data $f_0$, finite radially symmetric measure such that $M(f_0)$ and $E(f_0)$ are well defined, there exists a global in time weak radially symmetric measure valued solution $f$ of the Nordheim equation such that $f(t)$ is still a finite measure with finite second moment. That solution has constant mass, momentum and energy. Finally, as $t \to +\infty$:

$$f(t) \rightharpoonup F \quad \text{in the weak sense of measures}$$

where $F$ is the unique equilibrium such that $M(F) = M(f_0)$, $E(F) = E(f_0)$.

Important result:

-Gives sense (weak) to the equation for radial measures (in particular $f + \delta$)

-If the initial data $f_0$ is supercritical the Dirac mass is here, at least in infinite time...

# Remark

X. Lu has also proved (JSP 2000) that if the transition rate $W(p, p_1, p_2, p_3)$ is such that:

$$W(p, p_2, p_3, p_4) = w(p, p_2, p_3, p_4)\delta(p + p_2 - p_3 - p_4)\delta\left(|p|^2 + |p_2|^2 - |p_3|^2 - |p_4|^2\right)$$

with $w(p, p_2, p_3, p_4)$ a function such that:

$$w(p, p_2, p_3, p_4) \leq C \min\left(|p - p_3||p - p_4|, 1\right)$$

(for some constant $C > 0$) then the modified Nordheim equation is globally well posed in the space $L^1_w(\mathbb{R}^3)$ of radially symmetric functions $f$ such that $f$ and $|p|^2 f(p)$ are in $L^1(\mathbb{R}^3)$. In that case, the collision integral may be estimated as:

$$\int_{\mathbb{R}^3} Q[f](p)\, dp \leq C||f||^3_{L^1_w(\mathbb{R}^3)}.$$

# Proof of the blow up result

Two different parts:

- In the first we prove a local criterium for blow up.

- In the second we prove that every super critical solution satisfies the local criterium at some finite time.

The first part uses mainly measure theory to describe the "local" properties of the solutions.

The second part uses more functional analysis arguments. It strongly uses the entropy and dissipation of entropy.

# The local criterium for blow up

**Theorem** Let $M > 0$, $E > 0$, $\nu > 0$, $\gamma > 3$. There exist $\rho = \rho(M, E, \nu) > 0$, $K^* = K^*(M, E, \nu) > 0$, $T_0 = T_0(M, E)$ and a numerical constant $\theta_* > 0$ independent on $M$, $E$, $\nu$ such that for any $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ satisfying $M(f_0) = M$, $E(f_0) = E$,

$$(i) \qquad \int_0^R f_0(\epsilon)\sqrt{\epsilon}d\epsilon \geq \nu R^{\frac{3}{2}} \text{ for } 0 < R \leq \rho,$$

$$(ii) \qquad \int_0^\rho f_0(\epsilon)\sqrt{\epsilon}d\epsilon \geq K^* \rho^{\theta_*},$$

the unique mild solution $f \in L^\infty_{loc}([0, T_{\max}); L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma))$ with initial data $f_0$ and its maximal existence time $T_{\max}$ satisfy:

$$T_{\max} < +\infty, \text{ and } \limsup_{t \to T_{\max}^-} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

# Proof of the local criterium

The proof has several steps:

① Monotonicity of the kernel. For any $f \in L^1(\mathbb{R}^+)$ :

$$\cdot \int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_3 d\epsilon_4 \, w(\epsilon_1, \epsilon_3, \epsilon_4) \;\; q_3(f)(\epsilon_1)\sqrt{\epsilon_1}\,\varphi(\epsilon_1) =$$

$$= \int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 \, f_1 \, f_2 \, f_3 \mathcal{G}_\varphi(\epsilon_1 \, \epsilon_2 \, \epsilon_3),$$

where: $q_3(f) = f_3 f_4 (f + f_2) - f \, f_2 (f_3 + f_4)$

$$\cdot \;\; \mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{1}{6} \sum_{\sigma \in \mathcal{S}^3} H_\varphi\left(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \epsilon_{\sigma(3)}\right) \Phi\left(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}; \epsilon_{\sigma(3)}\right),$$

$$H_\varphi(x, y, z) = \varphi(z) + \varphi(x + y - z) - \varphi(x) - \varphi(y),$$

$$\Phi(\epsilon_1, \epsilon_3, \epsilon_4)) = \min\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\} \quad , \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$$

- $\varphi$ convex $\Longrightarrow \mathcal{G}_\varphi\left(\epsilon_1, \epsilon_2, \epsilon_3\right) \geq 0;\quad \varphi$ concave $\Longrightarrow \mathcal{G}_\varphi\left(\epsilon_1, \epsilon_2, \epsilon_3\right) \leq 0$

(Proved independently by X. Lu (unpublished).)

We deduce: if $g = 4\pi\sqrt{2\epsilon}\, f$ :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^+} g(\epsilon_1)\varphi(\varepsilon_1)d\epsilon_1 &= \frac{1}{(4\pi)^2}\int_{(\mathbb{R}_+)^3}\frac{g_1g_2g_3}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}\mathcal{G}_\varphi(\epsilon_1\,\epsilon_2\,\epsilon_3)d\epsilon_1 d\epsilon_2 d\epsilon_3\\ &+ \frac{1}{2}\int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 \Phi\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\left(\varphi_3+\varphi_4-\varphi_1-\varphi_2\right)\end{aligned}$$

We seem to be in good shape, but:

$\mathcal{G}_\varphi\left(\epsilon_1, \epsilon_2, \epsilon_3\right)$ vanishes along the diagonal $\left\{(\epsilon_1, \epsilon_2, \epsilon_3) \in \left(\mathbb{R}^+\right)^3 : \epsilon_1 = \epsilon_2 = \epsilon_3\right\}$.

② Step 1 allows to estimate the number of collisions that happen between particles with small energy:

Suppose that $f$ is a solution of Nordheim equation. Let $g = \sqrt{\epsilon}\, f$ and $0 < T < T_{\max}$. Then, there exists a numerical constant $B > 0$, independent on $f_0$ and $T$, such that, for any $R \in (0, 1/2)$ any $0 < \rho < 1$ and all $T > 0$

$$B \int_0^T dt \int_{\mathcal{S}_{R,\rho}} \left[ \prod_{m=1}^{3} g_m d\epsilon_m \right] \leq \frac{2 b^{\frac{7}{2}} R}{\rho^2 \left( \sqrt{b} - 1 \right)^2} \left[ 2\pi \int_0^T dt \left( \int_{[0,1]} g\left(\epsilon\right) d\epsilon \right)^2 + M \right],$$

with $b = 1/(1-\rho)$, $\mathcal{S}_{R,\rho} = \left\{ (\epsilon_1, \epsilon_2, \epsilon_3) \in [0, R]^3 : |\epsilon_0 - \epsilon_-| > \rho\epsilon_0 \right\}$ and

$$\epsilon_+ \left(\epsilon_1, \epsilon_2, \epsilon_3\right) = \max\left\{\epsilon_1, \epsilon_2, \epsilon_3\right\}, \;\; \epsilon_- \left(\epsilon_1, \epsilon_2, \epsilon_3\right) = \min\left\{\epsilon_1, \epsilon_2, \epsilon_3\right\},$$

$$\epsilon_0 \left(\epsilon_1, \epsilon_2, \epsilon_3\right) = \epsilon_k \in \left\{\epsilon_1, \epsilon_2, \epsilon_3\right\} \text{ such that } \epsilon_- \left(\epsilon_1, \epsilon_2, \epsilon_3\right) \leq \epsilon_k \leq \epsilon_+ \left(\epsilon_1, \epsilon_2, \epsilon_3\right).$$

③ Alternative. Suppose that $b > 1$ and let us define for all $k = 1, 2, \ldots$:

$$\mathcal{I}_k(b) = b^{-k}\left(b^{-1}, 1\right], \quad \mathcal{I}_k^{(E)} = \mathcal{I}_{k-1}(b) \cup \mathcal{I}_k(b) \cup \mathcal{I}_{k+1}(b)$$

$$\mathcal{P}_b = \left\{ A \subset [0,1] : A = \bigcup_j \mathcal{I}_{k_j}(b) \text{ for some set of indexes } \{k_j\} \subset \{1, 2, ...\} \right\}.$$

The elements of $\mathcal{P}_b$ are unions of elements of $\{\mathcal{I}_k(b)\}$. The set $\{k_j\}$ can contain a finite or infinity number of elements.

Suppose that $A = \cup_{j=1}^{\infty} \mathcal{I}_{k_j}(b)$. We then define:

$$A^{(E)} = \bigcup_{j=1}^{\infty} \mathcal{I}_{k_j}^{(E)}(b).$$

Given $0 < \delta < \frac{2}{3}$, we define $\eta = \min\left\{\left(\frac{1}{3} - \frac{\delta}{2}\right), \frac{\delta}{6}\right\} > 0$. Then, for any $g \in \mathcal{M}^+[0,1]$ satisfying $\int_{\{0\}} g d\epsilon = 0$, at least one of the following statements is satisfied:

(i) There exists an interval $\mathcal{I}_k(b)$ such that: $\int_{\mathcal{I}_k^{(E)}(b)} gd\epsilon \geq (1-\delta)\int_{[0,1]} gd\epsilon$

(ii) There exists two sets $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{P}_b$ such that $\mathcal{U}_2 \cap \mathcal{U}_1^{(E)} = \varnothing$ and:

$$\min\left\{\int_{\mathcal{U}_1} gd\epsilon, \int_{\mathcal{U}_2} gd\epsilon\right\} \geq \eta \int_{[0,1]} gd\epsilon.$$

In that case the set $\mathcal{U}_1$ can be written in the form: $\mathcal{U}_1 = \bigcup_{j=1}^{L} \mathcal{I}_{k_j}(b)$, for some set of integers $\{k_j\} \subset \{1,2,3,...\}$ and some finite $L$. We have:

$$\mathcal{I}_{k_m}(b) \cap \left(\cup_{j=1}^{m-1} \mathcal{I}_{k_j}^{(E)}(b)\right) = \varnothing \quad , \quad m = 2,3,...L$$

$$\sum_{j=1}^{L}\left(\int_{\mathcal{I}_{k_j}(b)} gd\epsilon\right)^2 \leq \left(\int_{\mathcal{I}_{k_1}(b)} gd\epsilon\right)^2 + \sum_{j=2}^{L}\int_{\mathcal{I}_{k_1}(b)} gd\epsilon \int_{\mathcal{I}_{k_j}(b)} gd\epsilon,$$

$$\int_{\mathcal{I}_{k_1}(b)} gd\epsilon < (1-\delta)\int_{[0,1]} gd\epsilon.$$

# Last part of the proof

Using the monotonicity property:

$$\int_0^{\rho/2} g_0(\epsilon) d\epsilon \geq m_0 \Longrightarrow \exists T_0 > 0; \ \int_0^{\rho} g(t, (\epsilon) d\epsilon \geq \frac{m_0}{4} \ \forall t \in [0, T_0].$$

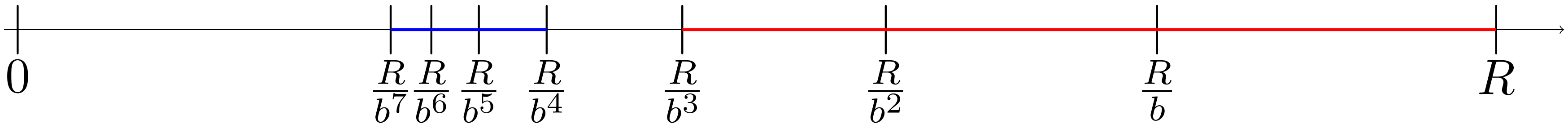
If we define:

$$B_\ell = \left\{ t \in [0, T_0] : \int_{[0, R_\ell]} g\left(\epsilon, t\right) d\epsilon \geq \left(R_\ell\right)^{\theta_1} \right\}, \ R_\ell = 2^{-\ell}, \ \ \ell = 0, 1, 2, ...$$

then, for $L$ and $\theta_1 > 0$ such that $2^{-\theta_1 L} \leq m_0/4$: $\boxed{B_L = [0, T_0]}$

This says that the mass is not spreading too much.

But $[0, 2^{-\theta_1 L}]$ is still too large:




$\mathcal{I}_1^{(E)}(b, R)$ in red; $\mathcal{I}_5^{(E)}(b, R)$ in blue

Consider $b_\ell = 1 + (R_\ell)^{\theta_2}$, $\ell = 0, 1, 2, \ldots$ and the sets:

$$A_{n,\ell} = \left\{ t \in [0, T_0] : \text{ such that } \int_{\mathcal{I}_n^{(E)}(b_\ell, R_\ell)} g(t, \epsilon)\, d\epsilon \geq (R_{\ell+1})^{\theta_1} \right\}, \quad n = 1, 2, 3, \ldots$$

$$\mathcal{A}_\ell = \bigcup_{n=1}^{\left[\frac{\log(2)}{\log(b_\ell)}\right]+1} A_{n,\ell} \quad \text{where: } \mathcal{I}_k^{(E)}(b, R) = \left( \frac{R}{b^{k+2}}, \frac{R}{b^{k-1}} \right].$$

At times $t \in A_{n,\ell}$ the solution is quite concentrated in $\mathcal{I}_n^{(E)}(b_\ell, R_\ell)$

We now use: $B_L = \bigcup_{\ell=L}^{\infty} B_\ell \setminus B_{\ell+1}$ and so:

$$T_0 = |B_L| \leq \sum_{\ell=L}^{\infty} |B_\ell \setminus B_{\ell+1}| \leq \sum_{\ell=L}^{\infty} \left( |(B_\ell \setminus B_{\ell+1}) \setminus A_\ell| + |A_\ell| \right).$$

The contradiction comes from the estimates of the right hand side.

In the set $(B_\ell \setminus B_{\ell+1}) \setminus A_\ell$ the solution is “not too much concentrated”

In $A_\ell$ the solution is “very concentrated”.

- $$(B_\ell \setminus B_{\ell+1}) \setminus A_\ell \subset \Omega_\ell; \qquad \Omega_\ell = B_\ell \setminus \bigcup_{n\geq 1} A_{n,\ell} \quad , \quad \ell = 0,1,2,...$$

The definition of the sets $B_\ell$ and $A_{n,\ell}$ show that $\Omega_\ell$ is a set of times $t$ in $[0, T_{\max}]$ for which the alternative (i) does not take place.

Therefore, the alternative (ii) takes place in $\Omega_\ell$.

Using the estimate ② of the number of collisions between particles with small energy, we deduce that there exists $\theta_0 > 0$ such that, if $\min\{\theta_1, \theta_2\} < \theta_0$, we have:

$$|(B_\ell \setminus B_{\ell+1}) \setminus A_\ell| \leq |\Omega_\ell| \leq K\,(1 + T_{\max})\, R_\ell^{1-3\theta_1-4\theta_2},$$

for some $K = K\,(M, \theta_1)$ and for any $\ell = 0, 1, 2, ..$

- Measure of $A_\ell$: there exists $\rho \in (0,1)$ such that, if $\ell > \frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$ then:

$$|\mathcal{A}_\ell| \leq K_2 \left(R_\ell\right)^{1-2\theta_1-\theta_2}.$$

This follows from the definition of $\mathcal{A}_\ell$ and the estimate:

$$\int_0^{T_0} \chi_{\mathcal{A}_\ell}(t) \left( \int_{\left\{\epsilon \in \mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}} g(t,\epsilon)\, d\epsilon \right)^2 dt \leq K_2 \left(R_\ell\right)^{1-\theta_2},$$

Such an estimate is obtained by contradiction, using an adjoint equation and suitable test functions.

$$\begin{aligned} \text{Then: } \quad T_0 &= |B_L| \leq \sum_{\ell=L}^{\infty} \left( |(B_\ell \setminus B_{\ell+1}) \setminus A_\ell| + |A_\ell| \right) \leq K_3 \sum_{\ell=L}^{\infty} R_\ell^\beta \\ &= \frac{K_3}{1-2^{-\beta}} \left(R_L\right)^\beta, \quad (\beta = \min\{1-2\theta_1-\theta_2, 1-3\theta_1-4\theta_2\}). \end{aligned}$$

Contradiction if $L$ large enough.

# Supercritical solutions satisfy local criterium

① Using the monotonicity property the first condition of the criteria holds:

**Proposition** There exist $T_0 = T_0(E, M)$, $\rho = \rho(E, M)$ and $K = K(E, M)$ such that if $T_{\max} \geq T_0(E, M)$, we have:

$$\int_0^R g(\epsilon, t)\, d\epsilon \geq K R^{\frac{3}{2}} \text{ for any } 0 < R \leq \rho$$

for any $t \geq T_0(E, M)$.

② Entropy, dissipation of entropy, etc...

$$
\begin{aligned}
S[f] &= \int_{\mathbb{R}^+} \left[(1+f)\log(1+f) - f\log(f)\right]\sqrt{\epsilon}\,d\epsilon \\
D[f] &= \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (1+f_1)(1+f_2)(1+f_3)(1+f_4)\left[Q_{1,2} - Q_{3,4}\right] \times \\
&\quad \times \left[\log(Q_{1,2}) - \log(Q_{3,4})\right]\Phi\, d\epsilon_1 d\epsilon_2 d\epsilon_3 \\
Q_{j,k} &= \frac{f_j}{(1+f_j)}\frac{f_k}{(1+f_k)} \quad , \quad j,k \in \{1,2,3,4\} \\
\Phi &= \min\left\{\sqrt{(\epsilon_1)_+}, \sqrt{(\epsilon_2)_+}, \sqrt{(\epsilon_3)_+}, \sqrt{(\epsilon_4)_+}\right\}
\end{aligned}
$$

$$
S[f](T_2) - S[f](T_1) = \int_{T_1}^{T_2} D[f(\cdot,t)]\,dt
$$

$$
|S[f(t)]| \leq C(E,M) \quad , \; 0 \leq t < T_{\max}.
$$

③ Using:

$$D[f] \geq \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(\epsilon_1) f(\epsilon_2)\, \Psi\left(\frac{Q(\epsilon_3)\, Q(\epsilon_4)}{Q(\epsilon_1)\, Q(\epsilon_2)} - 1\right) \Phi d\epsilon_1 d\epsilon_2 d\epsilon_3$$

$$\text{with } \ \Psi(s) = s \log(1+s)\,, \quad Q(t,\epsilon) = \frac{f(t,\epsilon)}{1 + f(t,\epsilon)}$$

We deduce the existence of a sequence $t_n \to +\infty$ such that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(t_n,\epsilon_1) f(t_n,\epsilon_2)\, \Psi\left(\frac{Q(t_n,\epsilon_3)\, Q(t_n,\epsilon_4)}{Q(t_n,\epsilon_1)\, Q(t_n,\epsilon_2)} - 1\right) \Phi d\epsilon_1 d\epsilon_2 d\epsilon_3 \to 0$$

$$Q(t_n,\cdot) \rightharpoonup Q_*(\cdot)$$

for some $Q_* \in L^\infty(\mathbb{R}^+)$ such that $0 \leq Q_*(\epsilon) \leq 1$.

This may be seen as a weak formulation of the equation:

$$Q_*(\epsilon_3)Q_*(\epsilon_4) = Q_*(\epsilon_1)Q_*(\epsilon_2)$$

“from where” $Q_*(\epsilon) = e^{-\beta_*(\epsilon+\alpha_*)}$ and so,

$$Q\left(t_n, \epsilon\right) \equiv \frac{f\left(t_n, \epsilon\right)}{1 + f\left(t_n, \epsilon\right)} \rightharpoonup e^{-\beta_*(\epsilon+\alpha_*)}.$$

We deduce the existence of $m_* > 0$ and $\rho > 0$ such that, for all $R \in (0, \rho)$ there exists a sequence $t_n \to +\infty$ satisfying:

$$\int_0^R g\left(t_n, \epsilon\right) d\epsilon = 4\pi \int_0^R \sqrt{2\epsilon} f\left(t_n, \epsilon\right) d\epsilon \geq m_*.$$

If $R$ is chosen small enough we may ensure that $m_* > K_* R^{\theta_*}$.

# Another blow up problem in kinetic equations.

Boltzmann equation withCoulomb interactions would be:

$$\frac{\partial f}{\partial t}(t,p) = \int_{\mathbb{R}^N} dp_2 \int_{S^{N-1}} \frac{d\sigma}{|p-p_2|^3 \sin^4(\theta/2)} (f_3\, f_4 - f\, f_2)$$

$\cos\theta = \langle \frac{p-p_2}{|p-p_2|}, \sigma \rangle$ But too singular: makes no sense.

Then, approximation arguments lead to Landau equation:

$$\frac{\partial f}{\partial t} = \sum_{i,j} \overline{a}_{i,j} \frac{\partial^2 f}{\partial p_i \partial p_j} + f^2, \quad t > 0,\ p \in \mathbb{R}^N$$

$$\overline{a}_{i,j} = \frac{1}{|p|} \left( \delta_{ij} - \frac{p_i\, p_j}{|p|^2} \right) * f$$

If $f$ is smooth, the matrix $(\overline{a}_{i,j})_{i,j}$ is locally positive definite and bounded. Blow up in finite time? (C. Villani, 2001).

# About the existence result

Method introduced by T. Carleman in his work about the well posedness of the spatially homogeneous Boltzmann equation.

Local well-posedness reduces then to find a class of functions whose behaviour for large $\epsilon$ is preserved in some suitable iterative scheme.
For large values of $\epsilon$ the dominant terms in the equation are the quadratic ones while the cubic terms can be treated as some kind of perturbation. T. Carleman found that a suitable class of functions are those bounded as $\frac{C}{(1+\epsilon)^\gamma}$ with $\gamma > 3$.
In the Boltzmann case, due to the conservation of the energy and the number of particles it is possible to prove global existence of solutions.

Although Carleman's method requires decay estimates on the solutions more restrictive than some of the more recent approaches for the Boltzmann equation (Povzner '65, Mischler and Wennberg '99), it uses simpler arguments.

In spite of this, such approach is enough to obtain a large class of initial data yielding blow-up in finite time.

# Previous existence results

1.- X. Lu '04, '05: Global measure valued weak solutions+conservation laws (solutions 1).

2.-E. & Velázquez: Local classical unbounded solutions. No particle conservation. (solutions 2)

X. Lu has shown that solutions 2 are not weak solutions as defined in 1. Nor are the solutions 2 the regular part of solutions 1.

But the initial data that we consider here are admissible for the solutions 1 by Lu.