Inside Structure of Pulled and Pushed Fronts

François HAMEL

Aix-Marseille University & Institut Universitaire de France

In collaboration with J. Garnier, T. Giletti, E. Klein and L. Roques

5th Euro-Japanese Workshop on Blow-up

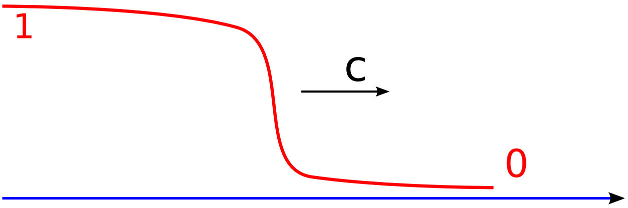
CIRM, September 2012

I. TRAVELING FRONTS

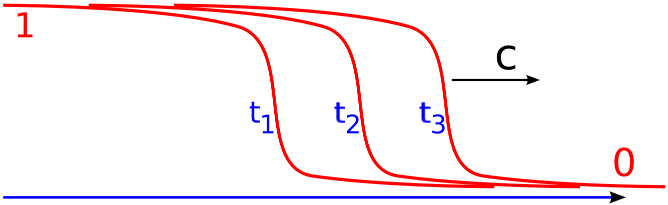
- Reaction-diffusion equation

$$u_t = u_{xx} + f(u)$$

- A traveling front (with $f(0) = f(1) = 0$)

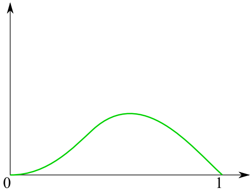


$$\boxed{u(t,x) = U(x - ct)}$$

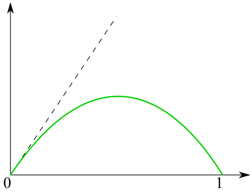


$$\begin{cases} U'' + cU' + f(U) = 0 & \text{in } \mathbb{R}, \\ U(-\infty) = 1, \ U(+\infty) = 0, \\ 0 < U < 1 & \text{in } \mathbb{R} \end{cases}$$

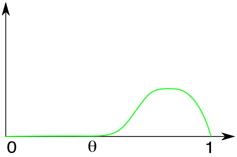
Nonlinearities f



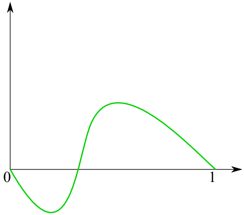
(a) monostable



(b) monostable KPP (no Allee effect)



(c) ignition



(d) bistable, Allee effect, with $\int_0^1 f > 0$

Existence results

- Monostable case: $\{c\} = [c^*, +\infty)$ with $c^* \geq 2\sqrt{f'(0)}$ and $c^* > 0$
- KPP (Kolmogorov, Petrovskii, Piskunov): $c^* = 2\sqrt{f'(0)}$
- Ignition and bistable: there is a unique speed c and $c > 0$

[Aronson and Weinberger, Fife and McLeod, Kanel']

Uniqueness of the profile U (up to shifts) for each speed c, and $U' < 0$

Stability for the Cauchy problem with

$$u_0 = U + \text{perturbation}$$

[Bramson, Eckmann and Wayne, Fife and McLeod, Kametaka, Kanel', Lau, McKean, Sattinger, Uchiyama...]

Pulled and pushed fronts in the monostable case [Stokes, 1976]

- Pulled front:
  - *Either* a critical front with $c = c^* = 2\sqrt{f'(0)}$

    Same speed as the solution of the linearized problem
  - *Or* any super-critical front, that is $c > c^*$

    Slow exponential decay

$$U_c(y) \sim A\, e^{-\lambda_- y} \text{ as } y \to +\infty$$

with $\lambda_- = \dfrac{c - \sqrt{c^2 - 4f'(0)}}{2}$ smallest root of $\lambda^2 - c\lambda + f'(0) = 0$

- Pushed front: a critical front with $c = c^* > 2\sqrt{f'(0)}$

Variational approach by Lucia, Muratov and Novaga (2004, 2008)

See also Rothe (1981) and van Saarloos (2003)

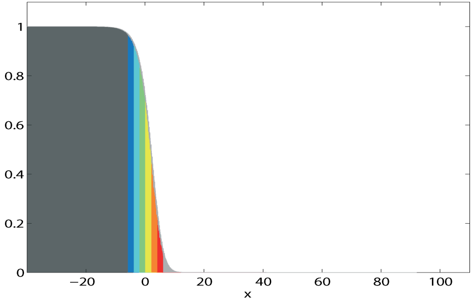
II. INSIDE SPATIAL STRUCTURE OF THE FRONTS

Decomposition of the front

$$u(t,x)=U(x-ct)$$

into a sum of components at initial time

$$u(0,x)=U(x)=\sum_{i\in I} v_0^i(x),\quad 0\ \leq\not\equiv\ v_0^i(x)\leq U(x)$$



All groups υ^i are neutral and share identical characteristics:

- Same diffusion rate, equal to 1
- Same per capita growth rate as the global front, equal to

$$g(u(t,x)) = \frac{f(u(t,x))}{u(t,x)}$$

[Hallatschek and Nelson, 2008, 2009], [Vlad, Cavalli-Sforza and Ross, 2004]

The components υ^i could be viewed as genetic groups.

Questions:

- Evolution of the spatial genetic structure as time runs?
- Loss of biodiversity along the front?
- Gene surfing?

Each component $\upsilon^i(t,x)$ satisfies the linear equation

$$\left\{\begin{array}{l} \upsilon^i_t = \upsilon^i_{xx} + g(u(t,x))\,\upsilon^i, \quad t>0, \quad x\in\mathbb{R}, \\ 0 \leq\not\equiv \upsilon^i_0(x) \leq U(x), \quad x\in\mathbb{R} \end{array}\right.$$

By uniqueness:

$$u(t,x) = \sum_{i\in I} \upsilon^i(t,x) \ \text{ and } \ g(u(t,x)) = g\Big(\sum_{i\in I} \upsilon^i(t,x)\Big)$$

Comparison principle:

$$0 < \upsilon^i(t,x) \leq u(t,x) = U(x-ct), \quad t>0, \quad x\in\mathbb{R}$$

Space-time heterogeneity $g(u(t,x)) = g(U(x-ct))$ with forced speed c, no periodicity or monotonicity

Can υ^i follow the global front?

Theorem (Pulled case)

Assume f is monostable and (c, U) is a pulled front, that is

$$\text{either } c = c^* = 2\sqrt{f'(0)} \quad \text{or} \quad c > c^*.$$

If

$$\int_0^{+\infty} e^{c\,x}\, v_0(x)^2 dx < +\infty,$$

then

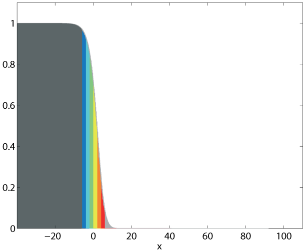
$$v(t, x + ct) \to 0 \text{ as } t \to +\infty \text{ locally uniformly in } x \in \mathbb{R}$$

and, more precisely,

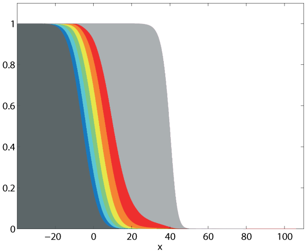
$$\limsup_{t\to+\infty} \Big(\max_{x\geq\alpha\sqrt{t}} v(t,x) \Big) \to 0 \text{ as } \alpha \to +\infty.$$

Furthermore, if $v_0(-\infty)=0$ or if $v_0 \in L^p(\mathbb{R})$ with $1 \leq p < +\infty$, then

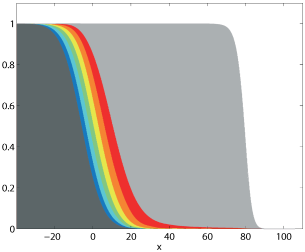
$$v(t,\cdot)\to 0 \text{ as } t\to+\infty \text{ uniformly in } \mathbb{R}.$$



(e) $t = 0$



(f) $t = 20$ (speed=2)



(g) $t = 40$

Consequence

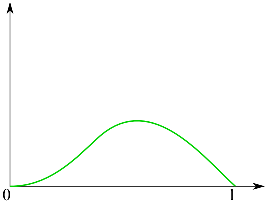
$$\forall\,\varepsilon>0,\quad \max_{x\geq\varepsilon t} v(t,x)\to 0 \text{ as } t\to+\infty$$

Right spreading speed of v is equal to 0

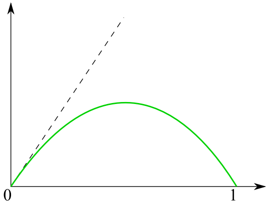
Examples: v_0 is compactly supported, or supported in $(-\infty,a)$.

- The propagation is due to the leading edge of the front (the "rightmost" component).
- The fronts are pulled when $c=c^*=2\sqrt{f'(0)}$ *or* when $c>c^*$.

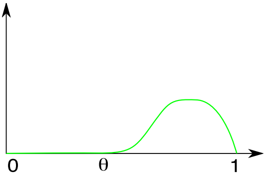
Nonlinearities f



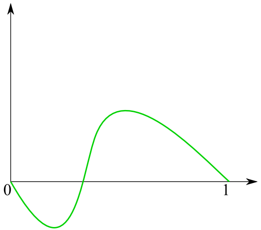
(h) monostable



(i) monostable KPP (no Allee effect)



(j) ignition



(k) bistable, with $\int_0^1 f > 0$

Theorem (Pushed case)

Assume f is monostable with $c = c^* > 2\sqrt{f'(0)}$, or ignition, or bistable.

Then there is $p = p(\upsilon_0) \in (0,1]$ such that

$$\upsilon(t, x + ct) \to p\,U(x) \text{ as } t \to +\infty \text{ locally uniformly in } x \in \mathbb{R}.$$

More precisely,

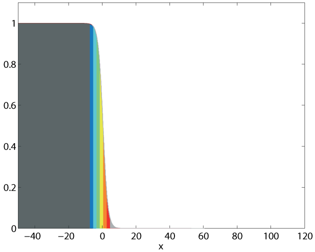
$$\limsup_{t\to+\infty} \Big(\max_{x \geq \alpha\sqrt{t}} \big|\upsilon(t,x) - p\,U(x - ct)\big| \Big) \to 0 \text{ as } \alpha \to +\infty.$$

Furthermore,

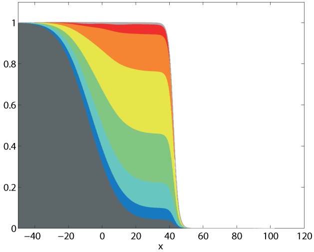
$$\forall\,\alpha \in \mathbb{R},\ \forall\,x_0 \in \mathbb{R}, \quad \liminf_{t\to+\infty} \Big(\min_{\alpha\sqrt{t} \leq x \leq x_0 + ct} \upsilon(t,x) \Big) > 0.$$

If $\upsilon_0(-\infty) = 0$ or if $\upsilon_0 \in L^p(\mathbb{R})$ with $1 \leq p < +\infty$, then

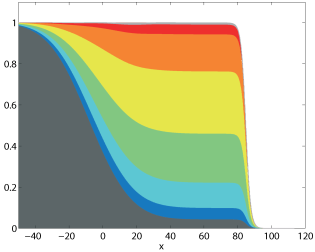
$$\limsup_{t\to+\infty} \Big(\max_{x \leq \alpha\sqrt{t}} \upsilon(t,x) \Big) \to 0 \text{ as } \alpha \to -\infty$$



(l) $t = 0$



(m) $t = 20$ (speed=2)



(n) $t = 40$

Consequence: right spreading speed of v is equal to c.

- In the pushed case, every component v^i contributes to a positive proportion of the global front (even if it is initially compactly supported).
- In other words, all components push the front (not only the rightmost one).

- The bistable and ignition fronts are pushed in the above sense.
- Extension and new light on the definition given by Stokes in the monostable case.

Strong contrast with the pulled case.

Definition of the proportion $p(\upsilon_0)$ in the pushed case

$$p(\upsilon_0)=\frac{\displaystyle\int_{\mathbb{R}}\upsilon_0(x)\,U(x)\,e^{cx}\,dx}{\displaystyle\int_{\mathbb{R}}U(x)^2\,e^{cx}\,dx}$$

Exponential behavior of the fronts

- Bistable case:

  $$U(y)\sim A\,e^{-\lambda_+ y}\ \text{as } y\to+\infty,\ \text{where } \lambda_+=\frac{c+\sqrt{c^2-4f'(0)}}{2}>\frac{c}{2}.$$

- Ignition case: $U(y)\sim A\,e^{-cy}$ as $y\to+\infty$.
- Monostable critical fronts $c=c^*$:
  - Pushed case $c=c^*>2\sqrt{f'(0)}$: $U(y)\sim A\,e^{-\lambda_+ y}$, with $\lambda_+>c^*/2$.
  - Pulled case $c=c^*=2\sqrt{f'(0)}$: $U(y)\sim(Ay+B)\,e^{-c^*y/2}$.
- Monostable super-critical fronts $c>c^*$:

  $$U(y)\sim A\,e^{-\lambda_- y}\ \text{as } y\to+\infty,\ \text{where } \lambda_-=\frac{c-\sqrt{c^2-4f'(0)}}{2}<\frac{c}{2}.$$

Transition from pushed to pulled critical fronts for monostable f

$$f_a(u) = u(1-u)(1+au), \quad \text{parameter } a \geq 0$$

Minimal speed c_a^* [Hadeler and Rothe]

$$c_a^* = \begin{cases} 2 & \text{if } 0 \leq a \leq 2, \quad \text{pulled case: } c_a^* = 2 = 2\sqrt{f_a'(0)} \\ \sqrt{\dfrac{2}{a}} + \sqrt{\dfrac{a}{2}} & \text{if } a > 2, \quad \text{pushed case: } c_a^* > 2 = 2\sqrt{f_a'(0)} \end{cases}$$

Minimal front for $a \geq 2$:

$$U_a(x) = \frac{1}{1+e^{\sqrt{a/2}\,x}} \text{ with } \int_{\mathbb{R}} U_a(x)^2 e^{c_a^* x} dx \geq \frac{1}{4(\sqrt{a/2}-\sqrt{2/a})} \underset{a\to 2^+}{\longrightarrow} +\infty$$

Fixed initial condition $0 \leq\not\equiv v_0 \leq U_a$ with $\mathrm{supp}(v_0) \subset [-B, B]$

$$0 < p(v_0, a) = \frac{\displaystyle\int_{\mathbb{R}} v_0(x)\, U_a(x)\, e^{c_a^* x}\, dx}{\displaystyle\int_{\mathbb{R}} U_a^2(x)\, e^{c_a^* x}\, dx} \longrightarrow 0^+ \text{ as } a \to 2^+$$

Set $p(v_0, a) = 0$ for $0 \leq a \leq 2$. Then $p(v_0, \cdot)$ is continuous on $[0, +\infty)$ and

$$v(t, x + c_a^* t) \to p(v_0, a)\, U_a(x) \text{ as } t \to +\infty, \quad \text{for all } a \geq 0.$$

Notions of pulled and pushed generalized transition fronts

$$u_t = \mathcal{D}(u) + f(t,x,u)$$

Assume $f(t,x,0) = 0$ and there is a solution $p^+(t,x) > 0$.

Generalized transition fronts connecting 0 and $p^+(t,x)$ [H. Berestycki, F.H.]:

$$\begin{cases} u(t,x) - p^+(t,x) & \to & 0 \text{ as } x - x_t \to -\infty, \\ u(t,x) & \to & 0 \text{ as } x - x_t \to +\infty. \end{cases}$$

Pulled front: for all $0 \le\not\equiv v_0 \le u(0,\cdot)$ with v_0 compactly supported, the solution v of

$$\begin{cases} v_t & = & \mathcal{D}(v) + g(t,x,u(t,x))\, v, \\ v(0,\cdot) & = & v_0 \end{cases}$$

where $g(t,x,u) = f(t,x,u)/u$, satisfies

$$\forall\, M \ge 0, \quad \sup_{x \in [x_t - M, x_t + M]} v(t,x) \to 0 \text{ as } t \to +\infty.$$

Pushed front: there is $M \ge 0$ such that

$$\limsup_{t \to +\infty} \Big(\sup_{x \in [x_t - M, x_t + M]} v(t,x) \Big) > 0.$$

III. PROOFS

Local extinction in the (pulled) KPP case

$$\begin{cases} v_t = v_{xx} + g(u(t,x))\, v \leq v_{xx} + f'(0)\, v, & t > 0, \quad x \in \mathbb{R}, \\ 0 \leq\not\equiv v_0(x) \leq U(x), & x \in \mathbb{R} \end{cases}$$

Assume for simplicity that v_0 has a compact support.

$$v(t,x') \leq \frac{e^{f'(0)t}}{\sqrt{4\pi t}} \int_{-B}^{B} e^{-\frac{(x'-y)^2}{4t}}\, v_0(y)\, dy$$

Then, for $x' = c^* t + x = 2\sqrt{f'(0)}t + x$,

$v(t, c^* t + x) \to 0$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}$.

Local extinction in the pulled (monostable) case

The function

$$r(t,x)=\frac{\upsilon(t,ct+x)}{U(x)}$$

satisfies

$$r_t+\mathcal{L}r=0$$

with $\mathcal{L}=-\partial_x^2-\psi'(x)\partial_x$ and $\psi(x)=cx+2\ln(U(x))$

Weight function $\sigma(x)=U^2(x)\,e^{cx}$.

Weighted spaces $L^2_\sigma(\mathbb{R})$ and $L^\infty_\sigma(\mathbb{R})$. Initially: $r(0,\cdot)\in L^2_\sigma(\mathbb{R})$.

Estimates

$$\frac{d}{dt}\left(\|r(t,\cdot)\|^2_{\sigma,2}\right) = -2\|r_x(t,\cdot)\|^2_{\sigma,2}$$

and

$$\frac{d}{dt}\Big(K\|r(t,\cdot)\|^2_{\sigma,2} + \|r_x(t,\cdot)\|^2_{\sigma,2}\Big) \leq -2\Big(\|r_x(t,\cdot)\|^2_{\sigma,2} + \|r_{xx}(t,\cdot)\|^2_{\sigma,2}\Big)$$

for some $K > 0$

Then

$$\|r_x(t,\cdot)\|_{\sigma,2} \to 0 \text{ as } t \to +\infty$$

and

$$\|r^2(t,\cdot)\|_{\sigma,\infty} \leq C\|r(t,\cdot)\|_{\sigma,2}\|r_x(t,\cdot)\|_{\sigma,2} \to 0 \text{ as } t \to +\infty$$

Conclusion: $\|r^2(t,\cdot)\|_{\sigma,\infty} = \|v^2(t, ct+\cdot)\, e^{cx}\|_{L^\infty(\mathbb{R})}$ and

$$v(t, ct+x) \to 0 \text{ as } t \to +\infty \text{ unif. in } x \in [A, +\infty)$$

Uniform extinction in the pulled case

- Locally extinction $v(t, ct+x) \to 0$ as $t \to +\infty$
- Remember that

$$v_t = v_{xx} + g(u(t,x))\, v$$

and

$$g(u(t,x)) = \frac{f(U(x-ct))}{U(x-ct)} \quad \text{and} \quad U(-\infty) = 1$$

Thus

$$v_t \simeq v_{xx} \quad \text{when } x - ct \ll -1$$

and

$$\limsup_{t\to+\infty} \Big(\max_{\alpha\sqrt{t} \le x \le x_0 + ct} v(t,x) \Big) \simeq 0 \ \text{ for } \alpha \gg 1$$

- If $v_0(-\infty) = 0$ or if $v_0 \in L^p(\mathbb{R})$ with $1 \le p < +\infty$, then

$$v(t, -\infty) = 0 \text{ for all } t > 0$$

Conclusion (comparison with the heat equation):

$$v(t,x) \to 0 \ \text{ as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}$$

Local persistence in the pushed case

The function

$$\widetilde{v}(t,x) = v(t, ct + x)$$

obeys

$$\widetilde{v}_t = \widetilde{v}_{xx} + c\widetilde{v}_x + g(U(x))\,\widetilde{v}$$

The function

$$v^*(t,x) = e^{cx/2}\,\widetilde{v}(t,x) = e^{cx/2}\,v(t, ct+x)$$

satisfies

$$\begin{cases} v_t^*(t,x) + \mathcal{L}^* v^*(t,x) = 0, & t > 0, \quad x \in \mathbb{R}, \\ v^*(0,x) = e^{cx/2}\,v_0(x), & x \in \mathbb{R}. \end{cases}$$

with

$$\mathcal{L}^* = -\partial_x^2 + \left(\frac{c^2}{4} - g(U(x))\right)$$

The function

$$\phi(x) = e^{cx/2}\, U(x)$$

belongs to $H^2(\mathbb{R})$ and $\mathrm{Ker}(\mathcal{L}^*) = \mathbb{R}\phi$. Moreover [Henry], [Pazy]

$$\left\| e^{-t\mathcal{L}^*} w \right\|_{L^\infty(\mathbb{R})} \le C\, e^{-\eta\, t} \left\| w \right\|_{L^\infty(\mathbb{R})} \ \text{ for } w \in (\mathbb{R}\phi)^\perp$$

Then

$$\upsilon^*(0,\cdot) = \big(\upsilon^*(0,\cdot), \varphi\big)_{L^2(\mathbb{R})}\, \varphi + w \ \text{ with } \varphi = \frac{\phi}{\|\phi\|_{L^2(\mathbb{R})}} \text{ and } w \in (\mathbb{R}\phi)^\perp$$

and

$$\upsilon^*(t,\cdot) = \big(\upsilon^*(0,\cdot), \varphi\big)_{L^2(\mathbb{R})}\, \varphi + e^{-t\mathcal{L}^*} w$$

Conclusion:

$$\upsilon(t, ct + x) = e^{-cx/2} \upsilon^*(t, x) = p(\upsilon_0)\, U(x) + o(1)$$

as $t \to +\infty$ unif. in $x \in [A, +\infty)$, with

$$p(\upsilon_0) = \frac{\displaystyle\int_{\mathbb{R}} \upsilon_0(x)\, U(x)\, e^{cx}\, dx}{\displaystyle\int_{\mathbb{R}} U(x)^2\, e^{cx}\, dx} \in (0, 1]$$

"Global" persistence in the pushed case

- Locally persistence $v(t, ct + x) \to p(v_0)\, U(x)$ as $t \to +\infty$
- Remember that

$$v_t = v_{xx} + g(u(t,x))\, v$$

and

$$g(u(t,x)) = \frac{f(U(x-ct))}{U(x-ct)} \quad \text{and} \quad U(-\infty) = 1$$

Thus $v_t \simeq v_{xx}$ when $x - ct \ll -1$ and

$$\limsup_{t\to+\infty} \Big(\max_{\alpha\sqrt{t} \le x \le x_0 + ct} \big| v(t,x) - p(v_0) \big| \Big) \simeq 0 \ \text{ for } \alpha \gg 1 \text{ and } x_0 \ll -1$$

- If $v_0(-\infty) = 0$ or if $v_0 \in L^p(\mathbb{R})$ with $1 \le p < +\infty$, then

$$\limsup_{t\to+\infty} \Big(\max_{x \le \alpha\sqrt{t}} v(t,x) \Big) \simeq 0 \ \text{ for } \alpha \ll -1$$

and

$$v(t,x) \to \frac{p(v_0)}{2} \ \text{ as } t \to +\infty \text{ loc. unif. } x \in \mathbb{R}$$