Blow-up set for type I blowing up solutions for a semilinear heat equation

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Let u be a solution of

$$(P) \quad \begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x) \geq 0, & x \in \Omega, \end{cases}$$

where Ω is a domain in $\mathbf{R}^N$, $N \geq 1$, and $p > 1$.

Aim of this talk

- Give a sufficient condition for type I blowing up solutions not to blow up on the boundary of the domain Ω.
- If Ω is an annulus, then radially symmetric solutions don't blow up on the boundary.
- O.D.E. blow-up and No Boundary blow-up

1. Introduction

Consider the problem

$$(P) \quad \begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x) \geq 0, & x \in \Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbf{R}^N$, $N \geq 1$, $p > 1$, and $\varphi \in C(\Omega) \cap L^\infty(\Omega)$.

Definitions:

- T : the maximal existence time of the unique bounded classical solution u of (P)

$$(P)\quad \begin{cases} \partial_t u = \Delta u + u^p \ \text{ in } \ \Omega \times (0,T), \\ u = 0 \ \text{ on } \ \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \ \text{ in } \ \Omega. \end{cases}$$

Definitions:

- T : the maximal existence time of the unique bounded classical solution u of (P)

$T < \infty$ ➡ $\limsup_{t \to T} \|u(t)\|_{L^\infty(\Omega)} = \infty$

(Blow-up Time)

➡ $\liminf_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} \geq \kappa,$

where $\kappa := (p-1)^{-1/(p-1)}.$

(a) $\limsup_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} < \infty$

➡ type I blow-up

(b) Blow-up of u is not type I ➡ type II blow-up

$$(P)\begin{cases} \partial_t u = \Delta u + u^p & \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), & u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega. \end{cases}$$

$T < \infty$: Blow-up time

- Blow-up Set

$$B := \Big\{ x \in \overline{\Omega} \;:\; {}^{\exists}\{(x_n, t_n)\} \subset \overline{\Omega} \times (0,T) \;\; \text{s.t.}$$
$$\lim_{n\to\infty} (x_n, t_n) = (x, T), \;\; \lim_{n\to\infty} u(x_n, t_n) = \infty \Big\}$$

⟹ B is a closed set in $\overline{\Omega}$

Problem: Is the blow-up set compact in Ω?

- Boundedness of B
- $B \cap \partial\Omega = \emptyset$?

$$(P)\begin{cases}\partial_t u = \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega.\end{cases}$$

Known Results

- Friedman & McLeod '85 :

  Ω is convex ⇒ $B \cap \partial\Omega = \emptyset$

  (moving plane method, comparison principle)

- Giga & Kohn '89 : They established a blow-up criterion in the case $(N-2)p < N+2$

  (a) Ω is convex , $\varphi \in H^1(\Omega)$ ⇒ B is bounded

  (b) Ω is strictly starshaped about $a \in \partial\Omega$

  ⇒ $a \notin B$

$$(P)\begin{cases} \partial_t u = \Delta u + u^p \text{ in } \Omega \times (0,T), \\ u = 0 \text{ on } \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \text{ in } \Omega. \end{cases}$$

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(b) Ω is strictly starshaped about $a \in \partial\Omega$

⇒ $a \notin B$

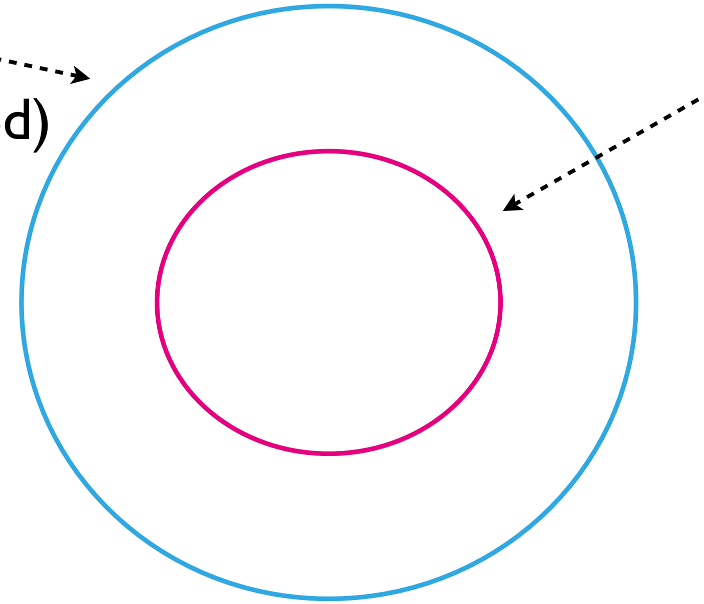
- I & Mizoguchi '03:

Ω : bounded smooth domain, $(N-2)p \leq N+2$

⇒ Type I blowing up solutions don't blow up on the boundary $\partial\Omega$

Example: $\Omega = \{a < |x| < b\} \quad (0 < a < b < \infty)$

No Blow-up
(Friedman & McLeod)



No Blow-up
if $(N-2)p \leq N+2$
and type I
(I & Mizoguchi)

If the solution is radially symmetric,
then does it blow up on the boundary?

Fila & Winkler '08: A boundary blowing up solution exists for

$$\partial_t u = u_{xx} + k(u^m)_x + u^{2m-1} \quad (m > 1,\ k \gg 2/\sqrt{m})$$

Fujishima & I. ('10)

Let u be a type I blowing up solution of

$$\begin{cases} \partial_t u = \epsilon \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x) \geq 0, & x \in \Omega, \end{cases}$$

where $N \geq 1,\ p > 1,$ and $\varphi \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\varphi = 0$ on $\partial\Omega$. Then, for any $\delta > 0,$

$$B_\epsilon \subset \{x \in \Omega : \varphi(x) > \|\varphi\|_{L^\infty(\Omega)} - \delta\}$$

for all sufficiently small $\epsilon > 0.$

Fujishima & I. (to appear in IUMJ) Assume $\varphi \in C^2(\Omega) \cap C(\overline{\Omega})$.

Then $^\exists C > 0$ s.t.

$$B_\epsilon \subset \{x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - C\epsilon\}$$

for all sufficiently small $\epsilon > 0.$

Let u be a type I blowing up solution of

$$(P)\quad \begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = \varphi(x) \geq 0, & x \in \Omega, \end{cases}$$

where $N \geq 1$ and $p > 1.$

Aim of this talk

Study the relationship between
the blow-up set and the level sets of the solution
just before the blow-up time.

⇒ Boundedness of the blow-up set

No boundary blow-up

O.D.E. blow-up and No boundary blow-up

2. Main Results

Consider the problem

$$(P) \quad \begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega, \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases}$$

where $N \geq 1$ and $p > 1$.

Notation:

- For any $f \in C(\overline{\Omega})$ and $\eta > 0$, put

$$M(f,\eta) := \left\{ x \in \overline{\Omega} : \ f(x) \geq \|f\|_{L^\infty(\Omega)} - \eta \right\}.$$

- $\kappa := \left(\dfrac{1}{p-1} \right)^{1/(p-1)}$

$$(P)\begin{cases} \partial_t u = \Delta u + u^p & \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), & u(x,0) = \varphi(x) \ge 0 \quad \text{in} \quad \Omega. \end{cases}$$

Theorem A

Let u be a solution type I blowing up at $t = T < \infty$.

Assume

$$\varphi \in L^\infty(\Omega) \cap L^q(\Omega)$$

for some $1 \le q < \infty$. Then the blow-up set is bounded.

Remark:

- Galaktionov & Posashkov '85:

$$N = 1 \quad \& \quad 0 \le \varphi(x) \le C|x|^{-2/(p-1)}$$

- Giga & Kohn '89: Geometric Assumptions &

$$(N-2)p < N+2 \quad \& \quad \varphi \in H^1(\Omega)$$

$$(P)\begin{cases}\partial_t u = \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega.\end{cases}$$

Theorem B

Let u be a solution type I blowing up at $t = T < \infty$.

Assume

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$$

Then

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} = \kappa.$$

(O.D.E. type blow-up)

Furthermore, for any $\eta > 0$, $\exists T' \in (0,T)$ s.t.

$$B \subset \bigcap_{T'<t<T} M\left((T-t)^{\frac{1}{p-1}} u(t), \eta\right).$$

⇒ No Boundary Blow-up

$$(P)\begin{cases} \partial_t u = \Delta u + u^p & \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), & u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega. \end{cases}$$

Corollary A

Let $\Omega = \{a < |x| < b\} \quad (0 < a < b < \infty).$

Then the radially symmetric solutions don't blow up on the boundary.

Remark: In this case, we have

$$\limsup_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} < \infty,$$

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$$

Thus Theorem B implies Corollary A.

$$(P)\quad \begin{cases} \partial_t u = \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega. \end{cases}$$

Theorem C

Let Ω be a bounded domain in $\mathbf{R}^N$ satisfying the exterior sphere condition. If

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} = \kappa,$$

(O.D.E. type blow-up)

then the solution u does not blow up on the boundary.

Type I blow-up

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$$

→ (Thm. B) O.D.E. type blow-up

→ (Thm. B) No Boundary Blow-up

O.D.E. type blow-up → (Thm. C) No Boundary Blow-up

3. Proof of Theorem B

Let u be a type I blowing up solution of

$$(P)\quad \begin{cases} \partial_t u = \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \geq 0 \quad \text{in} \quad \Omega. \end{cases}$$

For any $\epsilon \in (0,T)$, put

$$v(x,t) := \epsilon^{\frac{1}{p-1}} u(x, T - \epsilon + \epsilon t).$$

Then

$$\begin{cases} \partial_t v = \epsilon \Delta v + v^p & \text{in} \quad \Omega \times (0,1), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ v(x,0) = \varphi_\epsilon(x) & \text{in} \quad \Omega, \end{cases}$$

(A semilinear heat equation with small diffusion)

where $\varphi_\epsilon(x) := \epsilon^{\frac{1}{p-1}} u(x, T - \epsilon).$

Fujishima & I (JDE '10) Let v_ϵ be a solution of

$$\begin{cases} \partial_t v = \epsilon \Delta v + v^p & \text{in} \quad \Omega \times (0,1), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ v(x,0) = \varphi_\epsilon(x) \geq 0 & \text{in} \quad \Omega. \end{cases}$$

Let $t = 1$ be the blow-up time of v_ϵ $(0 < \epsilon \ll 1)$.

Assume that (Type I)

$$\limsup_{\epsilon \to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty, \quad \lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1} + \frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0$$

Then, for any $\eta > 0$,

$$B_\epsilon \subset M(\varphi_\epsilon, \eta)$$

$$\equiv \{x \in \overline{\Omega} : \varphi_\epsilon(x) \geq \|\varphi_\epsilon\|_{L^\infty(\Omega)} - \eta\}$$

for all sufficiently small $\epsilon > 0$.

Fujishima & I (JDE '10) Let v_ϵ be a solution of

$$\begin{cases} \partial_t v = \epsilon \Delta v + v^p & \text{in} \quad \Omega \times (0,1), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,1), \\ v(x,0) = \varphi_\epsilon(x) \geq 0 & \text{in} \quad \Omega. \end{cases}$$

Let $t = 1$ be the blow-up time of v_ϵ $(0 < \epsilon \ll 1)$.

Assume that

Type I

$$\limsup_{\epsilon \to 0} \|\varphi_\epsilon\|_{L^\infty(\Omega)} < \infty, \quad \lim_{\epsilon \to 0} \epsilon^{1/2} \|\nabla \varphi_\epsilon\|_{L^\infty(\Omega)} = 0.$$

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1} + \frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0$$

Then, for any $\eta > 0$,

$$B(u) = B_\epsilon \subset M\left(\epsilon^{\frac{1}{p-1}} u(\cdot, T-\epsilon), \eta\right)$$

for all sufficiently small $\epsilon > 0$. ⇒ Theorem B

4. Proof of Corollary A

Assume that $\Omega = \{a < |x| < b\}$ $(0 < a < b < \infty)$.

Let u be a blowing up solution of

$$\begin{cases} \partial_t u = \partial_r^2 u + \dfrac{N-1}{r}\partial_r u + u^p & \text{in} \quad (a,b)\times(0,T), \\ u = 0 & \text{on} \quad \{a,b\}\times(0,T), \\ u(r,0) = \varphi(r) & \text{in} \quad (a,b). \end{cases}$$

By Theorem B it suffices to prove

$$\limsup_{t\to T}\,(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty(\Omega)} < \infty,$$

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}+\frac{1}{2}}\|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$$

These are proved by the arguments in [Fila & Souplet '01] and [Mizoguchi '03] with the aid of the Liouville theorem (Merle & Zaag, Polàčik, Quittner, & Souplet).

Proof of $\lim_{t \to T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla u(t)\|_{L^\infty(\Omega)} = 0.$

Assume $^\exists\{(r_n, t_n)\} \subset [a,b] \times (0,T)$ s.t. $t_n \to T$ and

$$M_n := (T-t_n)^{\frac{p+1}{2(p-1)}} |\partial_r u(r_n, t_n)| \geq {}^\exists m > 0$$

for $n = 1, 2, \ldots$. Put

$$\mu_n := (T-t_n)^{\frac{1}{2}} M_n^{-\frac{p-1}{p+1}}, \quad \longrightarrow \quad \lim_{n\to\infty} \mu_n = 0$$

$$w_n(\tau, s) := \mu_n^{\frac{2}{p-1}} u(r_n + \mu_n \tau, t_n + \mu_n^2 s),$$

for $(\tau, s) \in I_n \times (-\alpha_n, 0]$, where $\quad \longrightarrow \quad |\partial_r w_n(0,0)| = 1$

$$I_n := \{\tau \in \mathbf{R} : \mu_n \tau + r_n \in (a,b)\},$$

$$\alpha_n := \mu_n^{-2} t_n. \quad \longrightarrow \quad \lim_{n\to\infty} \alpha_n = \infty$$

Then we have

$$\partial_s w_n = \partial_\tau^2 w_n + \mu_n \frac{N-1}{r_n + \mu_n \tau} \partial_\tau w_n + w_n^p \quad \text{in} \quad I_n \times (-\alpha_n, 0].$$

⇨ $^\exists$ unbounded open interval I, $^\exists w \in C^{2;1}(\bar{I} \times (-\infty, 0])$

s.t.

$$\partial_s w = \partial_\tau^2 w + w^p \quad \text{in} \quad I \times (-\infty, 0],$$

$$w(\tau, s) = 0 \quad \text{in } (-\infty, 0] \quad \text{if } \tau \in \partial I,$$

$$|\partial_\tau w(0,0)| = 1.$$

Furthermore, since the blow-up of u is type I, we have

$$w(\tau, s) \leq {}^\exists C(-s)^{-\frac{1}{p-1}} \quad \text{in} \quad I \times (-\infty, 0].$$

⇨ (Liouville type theorems)

$$w(\tau, s) \equiv 0 \quad \text{or} \quad w(\tau, s) = \kappa (T_0 - s)^{-1/(p-1)} \ (T_0 \geq 0).$$

⇨ This contradicts to $|\partial_\tau w(0,0)| = 1.$

5. Proof of Theorems A and C

Theorem A

Let u be a solution type I blowing up at $t = T < \infty$.

Assume

$$\varphi \in L^\infty(\Omega) \cap L^q(\Omega)$$

for some $1 \leq q < \infty$. Then the blow-up set is bounded.

Theorem C

Let Ω be a bounded domain in $\mathbf{R}^N$ satisfying the exterior sphere condition. If

$$\lim_{t \to T} (T-t)^{\frac{1}{p-1}} \|u(t)\|_{L^\infty(\Omega)} = \kappa,$$

(O.D.E. type blow-up)

then the solution u does not blow up on the boundary.

$$(P)\begin{cases}\partial_t u = \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \quad u(x,0) = \varphi(x) \ge 0 \quad \text{in} \quad \Omega.\end{cases}$$

Improvement of Theorem B

Let u be a solution of (P) blowing up at $t = T$. Assume

$^\exists \Omega'$: domain with $\Omega \subset \Omega'$, $^\exists \eta \in (0, \kappa)$,

$^\exists \tilde{u} = \tilde{u}(x,t) \in L^\infty_{loc}(0, T : W^{1,\infty}(\Omega'))$ satisfying

- $0 \le u(x,t) \le \tilde{u}(x,t)$ in $\Omega \times (0,T)$,
- $\sup_{0<t<T} (T-t)^{\frac{1}{p-1}} \|\tilde{u}(t)\|_{L^\infty(\Omega')} < \infty$,
- $\sup_{x \in \partial\Omega'} (T-t)^{\frac{1}{p-1}} \tilde{u}(x,t) < \kappa - \eta.$

- $0 \le u(x,t) \le \tilde{u}(x,t)$ in $\Omega \times (0,T)$,
- $\sup_{0<t<T} (T-t)^{\frac{1}{p-1}} \|\tilde{u}(t)\|_{L^\infty(\Omega')} < \infty$,
- $\sup_{x\in\partial\Omega'} (T-t)^{\frac{1}{p-1}} \tilde{u}(x,t) < \kappa - \eta.$

Let $\delta > 0$. Put

$$\Sigma(t) := \{x \in \overline{\Omega'} \, : \, \kappa - \eta \le (T-t)^{\frac{1}{p-1}} \tilde{u}(x,t) \le \kappa\}.$$

If

$$\sup_{0<t<T} (T-t)^{\frac{1}{p-1}+\frac{1}{2}} \|\nabla \tilde{u}(t)\|_{L^\infty(\Sigma(t))}$$

is sufficiently small, then ${}^{\exists}T' \in (0,T)$ s.t.

$$B \subset \bigcap_{T'<t<T} \left\{x \in \overline{\Omega} \, : \, (T-t)^{\frac{1}{p-1}} \tilde{u}(x,t) \ge \kappa - \delta\right\}.$$

Construction of $\tilde{u}$

In the case of Thm. A Since $\varphi \in L^\infty(\Omega) \cap L^q(\mathbf{R}^N)$ $(1 \le q < \infty)$,

for any $t \in (0, T)$,

$$\lim_{L\to\infty} \|u(t)\|_{L^\infty(\Omega\setminus B(0,L))} = 0.$$

Then, for any $t \in (0,T)$, taking a sufficiently large $L_t > 0$,

we put

$$\tilde{u}(x,t) := \begin{cases} \|u(t)\|_{L^\infty(\Omega)} & \text{if} \quad |x| \le L_t, \\ -(|x| - L_t) + \|u(t)\|_{L^\infty(\Omega)} & \\ \qquad \text{if} \quad L_t < |x| \le L_t + \|u(t)\|_{L^\infty(\Omega)} - \kappa/2, & \\ \kappa/2 & \text{otherwise.} \end{cases}$$

⇒ Theorem A

In the case of Thm. C

Let $x_0 \in \partial\Omega$. Then ${}^{\exists}y_0 \in \mathbf{R}^N \setminus \overline{\Omega}$, ${}^{\exists}R > 0$ s.t.

$$\Omega \subset \Omega' := \{x : |x_0 - y_0| < |x - y_0| < R\}.$$

For any $t \in (0, T)$, we take a suitable $t' \in (0, T)$, and consider the solution $v = v(x, t : t')$ of

$$\begin{cases} \partial_t v = \Delta v + v^p, & x \in \Omega',\ t > t', \\ v = 0 & x \in \partial\Omega',\ t > t', \\ v(x, t') = \|u(t')\|_{L^\infty(\Omega)} & x \in \Omega'. \end{cases}$$

Put

$$\tilde{u}(x, t) := v(x, t : t').$$

⇒ Theorem C

6. Open Problems

- How about the other nonlinear parabolic equations?
- Type I blow up ⟹ No boundary blow-up?
- In an annulus, are there any solutions blowing up on the boundary?

(This is still open!)

Thank you for your attention !!