Small stochastic perturbations of Hamiltonian flows in $2D$: a PDE approach

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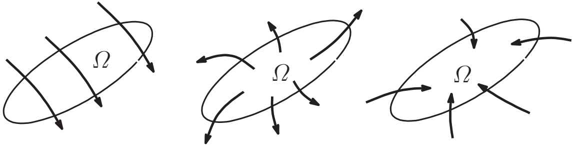
Introduction

Let $\Omega \subset \mathbb{R}^2$ **open set,** $\varepsilon > 0$.

$$\begin{cases} -\operatorname{div}(a(x)Du^\varepsilon) - b_0 \cdot Du^\varepsilon - \dfrac{b \cdot Du^\varepsilon}{\varepsilon} = g & \text{in } \Omega, \\ \text{Dirichlet BC} & \text{on } \partial\Omega. \end{cases}$$

Question: $\lim_{\varepsilon \to 0} u^\varepsilon(x)$

$$-b \cdot Du = 0 \quad \text{in } \Omega.$$



Freidlin-Wentzell, book '79, Kamin '78, Perthame '90

H **Hamiltonian,**
Hamiltonian flow:

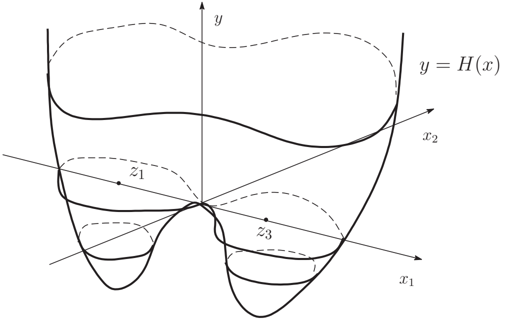
$$\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t)) = (H_{x_2}(x(t)),\ -H_{x_1}(x(t))).$$

$b(x) = (H_{x_2}(x),\ -H_{x_1}(x))$.
$b(x)$ **is rotation of** $DH(x) = (H_{x_1}, H_{x_2})$ **by** $\pi/2$.

$$\dot{x}(t) = b(x(t))$$

$$H \in C^4, \qquad \lim_{|x|\to\infty} H(x) = \infty,$$

$$DH(x) = 0 \quad \Longleftrightarrow \quad x = z_i \ (i = 1, 3) \ \text{ or } x = 0$$

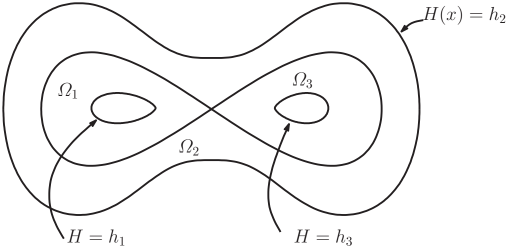


Nondegeneracy at the critical points:

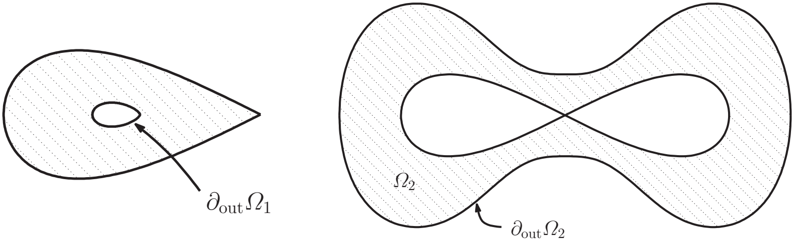
$$\begin{cases} D^2H(z_i) > 0 \quad (i = 1, 3), \\ \det D^2H(0) < 0. \end{cases}$$

Let $h_1, h_3 < 0 < h_2$ such that $H(z_i) < h_i$ $(i = 1, 3)$.

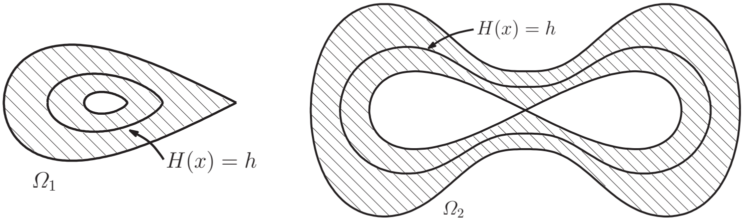
$$J_1 := (h_1,\, 0), \quad J_3 := (h_3,\, 0), \quad J_2 := (0,\, h_2).$$



$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \{H = 0\}.$$



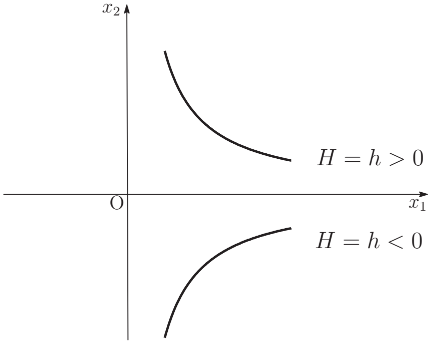
$$\boldsymbol{\partial\Omega = \partial_{\mathrm{out}}\Omega_1 \cup \partial_{\mathrm{out}}\Omega_2 \cup \partial_{\mathrm{out}}\Omega_3.}$$



$$c_i(h) = \{H = h\} \cap \bar{\Omega}_i \qquad \textbf{loop.}$$

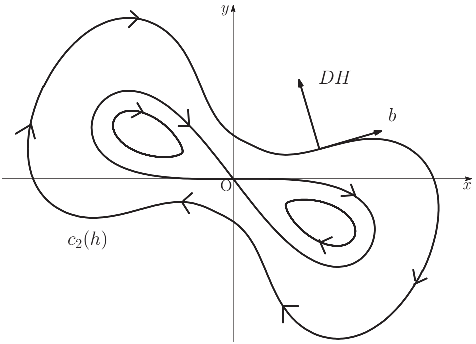
Morse Lemma: By C^2 change of variables, we may assume:

$$H(x_1, x_2) = x_1 x_2 \quad \text{near } x = 0.$$



Then

$$\begin{cases} DH = (x_2, x_1), \quad b = (x_1, -x_2), \\ D^2H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{cases}$$



Let $t \mapsto \Phi_i(t,h)$, $i = 1,2,3$, **be the Hamiltonian flow such that** $\Phi_i(t,h) \in c_i(h)$.

We may assume that $\Phi_i \in C^3(\mathbb{R} \times \bar{J}_i)$, $i = 1,2,3.$

For $h \neq 0$, $T_i(h) =$ **the minimal period of** $t \mapsto \Phi_i(t,h)$.

Note: $\det D\Phi_i(t,h) = 1.$

Rewrite

$$-\operatorname{tr}\frac{\sigma^2}{2}D^2u^\varepsilon - \tilde{b}_0\cdot Du^\varepsilon - \frac{b\cdot Du^\varepsilon}{\varepsilon} = g.$$

The LHS is the generator of the SDE

$$\mathrm{d}X_t = \left(\tilde{b}_0 + \frac{b}{\varepsilon}\right)\mathrm{d}t + \sigma\,\mathrm{d}B_t,$$

where B_t is a 2D Brownian motion.
Assume that $\tilde{b}_0 = 0$ and make the change of variables: $\widetilde{X}_{t/\varepsilon} = X_t$.

$$\mathrm{d}\widetilde{X}_t = b(\widetilde{X}_t)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma(\widetilde{X}_t)\,\mathrm{d}B_t.$$

Stochastic perturbation of the Hamiltonian flow.

2.1. Assumptions

Symmetry: $$a(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \qquad a_{12} = a_{21}.$$

Smoothness: $$a_{ij},\ b_0 \in C^2, \qquad g \in C^0 \quad \text{on } \bar{\Omega}.$$

Degenerate Ellipticity: $$a(x) \geq 0 \ \ \text{(nonnegative definite)}$$

Nondegeneracy 1:
$$\begin{cases} \forall i,\ h \in J_i \ \ \exists x^* = x^*(h,i) \in c_i(h) \quad \text{such that} \\ \qquad a(x^*)DH(x^*) \cdot DH(x^*) > 0. \end{cases}$$

Nondegeneracy 2: $\begin{cases} \text{In the coordinates where } D^2H(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ a_{ii}(0) > 0 \quad i = 1, 2. \end{cases}$

If $H = x_1 x_2$, then $D^2H(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$a(x)DH(x) \cdot DH(x) = a_{11}x_2^2 + a_{22}x_1^2 + 2a_{12}x_1x_2.$$

Hence

$$x \sim 0,\ x \neq 0,\ H = 0 \implies a(x)DH(x) \cdot DH(x) > 0$$

(same as in nondegeneracy 1)

Notation:
$$\begin{cases} |D_0 f|^2 = a(x)Df(x)\cdot Df(x), \\ \Delta_0 f(x) = \mathrm{div}(a(x)Df(x)). \end{cases}$$

Let $d_i \in \mathbb{R} \quad i = 1, 2, 3.$

(DP)$^\varepsilon$
$$\begin{cases} -\Delta_0 u^\varepsilon - b_0\cdot Du^\varepsilon - \dfrac{b\cdot Du^\varepsilon}{\varepsilon} = g \quad \text{in } \Omega, \\ u^\varepsilon(x) = d_i \quad \text{on } \partial_{\mathrm{out}}\Omega_i, \quad i = 1, 2, 3. \end{cases}$$

2.3. Coefficients

$$
\begin{aligned}
A_i(h) &= \frac{1}{T_i(h)} \int_0^{T_i(h)} |D_0 H(\Phi_i(t,h))|^2 \, \mathrm{d}t, \\
B_i(h) &= \frac{1}{T_i(h)} \int_0^{T_i(h)} |\Delta_0 H(\Phi_i(t,h))|^2 \, \mathrm{d}t, \\
B_{0i}(h) &= \frac{1}{T_i(h)} \int_0^{T_i(h)} (b_0 \cdot DH)(\Phi_i(t,h)) \, \mathrm{d}t, \\
\hat{g}_i(h) &= \frac{1}{T_i(h)} \int_0^{T_i(h)} g(\Phi_i(t,h)) \, \mathrm{d}t, \\
\beta_i &= \lim_{h \to 0+} (A_i T_i)((-1)^i h).
\end{aligned}
$$

2.4. ODE system

$$
\text{(ODE)} \qquad \begin{cases} A_i u_i'' + (B_i + B_{0i}) u_i' + \hat{g}_i = 0 \ \text{ in } J_i, \\ \left.\begin{aligned} & u_1(0) = u_2(0) = u_3(0), \\ & \beta_2 u_2'(0) = \beta_1 u_1'(0) + \beta_3 u_3'(0), \end{aligned}\right\} \text{Gluing Condition} \\ u_i(h_i) = d_i, \quad i = 1, 2, 3. \end{cases}
$$

- $\exists\,!$ a solution $u = (u_1, u_2, u_3)$ of (ODE).

MAIN THEOREM

Let u^ε be a solution of (DP)$^\varepsilon$ and u_i, $i = 1, 2, 3$, be the solution of (ODE). Then, as $\varepsilon \to 0$,

$$u^\varepsilon(x) \;\to\; u_i(H(x)) \quad \text{uniformly on } \bar{\Omega}_i, \quad i = 1, 2, 3.$$

- **Freidlin-Wentzell, '94** $(\Delta_0 = \Delta,\ b_0 = 0)$ **probabilistic method**
- **Freidlin-Weber, '98** $(\Delta_0 = \partial_{x_2}^2,\ b_0 = 0,$ $H(x_1, x_2) = f(x_1) + \frac{1}{2}x_2^2$, f **a double-well potential) probabilistic method & hypoelliptic estimates**
- **Sowers, '07** $(\Delta_0$ **uniformly elliptic) probabilistic method & perturbed test function method.**

Let $\phi \in C_0^\infty(J_i)$, $\Phi = \Phi_i(t,h)$.

$$0 = \iint_{\Omega_i} \phi(H)(\Delta_0 u^\varepsilon + b_0 \cdot Du^\varepsilon + \frac{1}{\varepsilon} b \cdot Du^\varepsilon + g)\,\mathrm{d}x$$

(use: $DH \cdot b = (H_{x_1}, H_{x_2}) \cdot (H_{x_2}, -H_{x_1}) = 0$ and
$\operatorname{div} b = H_{x_2 x_1} - H_{x_1 x_2} = 0$,)

$$= \iint_{\Omega_i} \{u^\varepsilon (\phi' \Delta_0 H + \phi'' |D_0 H|^2 - \phi' b_0 \cdot DH - \phi \operatorname{div} b_0) + \phi\, g\}\,\mathrm{d}x$$

Send $\varepsilon \to 0$ and use the change of variables: $x = \Phi(t,h)$,

$$0 = \int_{J_i} \mathrm{d}h \int_0^{T_i(h)} \{u_i(h)(\phi' \Delta_0 H + \phi'' |D_0 H|^2 - \phi'\, b_0 \cdot DH - \phi \operatorname{div} b_0) + \phi(h) g\}\,\mathrm{d}t$$

(writing

$$C_{0i}(h) = \frac{1}{T_i(h)} \int_0^{T_i(h)} \mathrm{div}\, b_0(\Phi_i(t,h))\, \mathrm{d}t,)$$

$$= \int_{J_i} \left\{ u_i \big(T_i A_i \phi'' + T_i B_i \phi' - T_i B_{0i} \phi' - T_i C_{0i} \phi \big) + T_i \hat{g}_i \phi \right\} \mathrm{d}t$$
$$= \int_{J_i} \phi \big((T_i A_i u_i)'' - (T_i B_i u_i)' + (T_i B_{0i} u_i)' - T_i C_{0i} u_i + T_i \hat{g}_i \big)\, \mathrm{d}t$$

Use the identities:

$$(T_i A_i)' = T_i B_i \quad \text{and} \quad (T_i B_{0i})' = T_i C_{0i},$$

to conclude:

$$A_i u_i'' + B_i u_i' + B_{0i} u_i' + \hat{g}_i = 0 \quad \text{in } J_i.$$

- **Gluing condition:**

By continuity, $u_1(0) = u_2(0) = u_3(0)$.

Let $\phi \in C_0^\infty(-r, r)$, with $0 < r \ll 1$.

$$0 = \lim_{\delta \to 0+} \iint_{\Omega \setminus \{|H| < \delta\}} \left\{ u^\varepsilon \Big(\phi' \Delta_0 H + \phi'' |D_0 H|^2 \right. \\ \left. - \phi' \, b_0 \cdot DH - \phi \operatorname{div} b_0 \Big) + g\phi \right\} \mathrm{d}x$$

after letting $\varepsilon \to 0$,

$$= \lim_{\delta \to 0+} \Big\{ \sum_{i=1,3} \int_{-r}^{-\delta} \mathrm{d}h \int_0^{T_i(h)} \big\{ u_i(\cdots) + g(\Phi_i(t,h))\phi(h) \big\} \, \mathrm{d}t \\ + \int_\delta^r \mathrm{d}h \int_0^{T_2(h)} \big\{ u_2(\cdots) + g(\Phi_2(t,h))\phi(h) \big\} \, \mathrm{d}t.$$

Integration by parts & taking limit as $\delta \to 0+ \implies$ the second gluing condition.

Outline of proof

1. L^∞ **estimate: a blow-up argument. Suppose that** $\exists \varepsilon_j \to 0+$ **for which**

$$M_j := \|u^{\varepsilon_j}\|_{\infty,\Omega} \to \infty.$$

Introduce the new unknown functions

$$w_j := u^{\varepsilon_j}/M_j.$$

Then we get a contradiction,

$$\begin{cases} \|w_j\|_{\infty,\Omega} = 1, \\ w_j(x) \to 0 \quad \text{as } j \to \infty. \end{cases}$$

The last claim is a consequence of the argument which follows. Here note that $(0,0,0)$ **is the solution of the ODE system**

(ODE)

$$\begin{cases} A_i u_i'' + (B_i + B_{0i}) u_i' = 0 \ \ \text{in } J_i, \\ u_1(0) = u_2(0) = u_3(0), \\ \beta_2 u_2'(0) = \beta_1 u_1'(0) + \beta_3 u_3'(0), \\ u_i(h_i) = 0, \quad i = 1, 2, 3. \end{cases}$$

2. L^2 **gradient estimate:** $\forall K \subset \Omega$ **compact**

$$\iint_K |D_0 u^\varepsilon|^2 \, dx \le {}^{\exists}C_K \left(\|u^\varepsilon\|_\infty^2 + 1 \right).$$

Use cut-off functions $\phi(H(x))$. Multiply (DP)$^\varepsilon$ by $\phi(H(x))u^\varepsilon(x)$, integrate by parts, and use Hölder's inequality.

We assume henceforth that there exists a sequence $\varepsilon_j \to 0+$ such that

$$\sup_{j \in \mathbb{N}} \|u^{\varepsilon_j}\|_{\infty,\Omega} < \infty.$$

When needed, we will pass to a subsequence of $\{u^{\varepsilon_j}\}$.

3. Set

$$N = \{x \in \Omega \,:\, |D_0 H(x)|^2 > 0\}.$$

For any $z \in N$, after a change of variables, we are typically in the situation that

$$\begin{cases} H(x) = x_2 \quad \text{in } U, \\ C := \sup_j \iint_U (u_{x_2}^{\varepsilon_j})^2 \,\mathrm{d}x < \infty, \end{cases}$$

where U is a neighborhood of z.

Then, $b = (1, 0)$ and the PDE for $u = u^{\varepsilon_j}$ is

$$-u_{x_1} = \varepsilon_j(\Delta_0 u + b_0 \cdot Du + g) \approx 0.$$

By translation, assume that $z = 0$. Let

$$U = [-\alpha,\, \alpha] \times [-\beta,\, \beta], \quad \alpha > 0,\ \beta > 0.$$

We have

$$\min_{\alpha/2 \le x_1 \le \alpha} \int_{-\beta}^{\beta} \left(u^{\varepsilon_j}_{x_2}(x_1, x_2)\right)^2 \,\mathrm{d}x_2 \le 2C/\alpha.$$

$\exists x_1^j \in [\alpha/2,\, \alpha]$, $\exists \phi \in C([-\beta,\, \beta])$ such that

$$u^{\varepsilon_j}(x_1^j, \cdot) \to \phi \quad \text{uniformly on } [-\beta,\, \beta].$$

For $\delta > 0$, if $\varepsilon_j \ll \delta$, then

$x \mapsto \pm\delta(x_1 - x_1^j)$ are a $\begin{matrix}\text{subsolution} \\ \text{supersolution}\end{matrix}$ of

$$-u_{x_1} = \varepsilon_j(\Delta_0 u + b_0 \cdot Du + g).$$

Hence, if $j \gg 1$, then

$$|u^{\varepsilon_j}(x) - \phi(0)| < \delta \quad \text{for } x \text{ near } [-\alpha/2,\, \alpha/2] \times \{0\}.$$

Thus,

$$u^+(0) = u^-(0),$$

where

where $u^{\pm}$ denote the half-relaxed limits:

$$\begin{cases} u^{+}(x) = \limsup_{j\to\infty}{}^{*}\, u^{\varepsilon_j}(x), \\ u^{-}(x) = \liminf_{j\to\infty}{}_{*}\, u^{\varepsilon_j}(x). \end{cases}$$

We have

$$u^{+}(x) = u^{-}(x) \quad \text{for } x \in N.$$

4. We have in the viscosity sense,

$$\begin{cases} -b \cdot Du^{+} \leq 0 & \text{in } \Omega, \\ -b \cdot Du^{-} \geq 0 & \text{in } \Omega. \end{cases}$$

For $h \neq 0$,
$t \mapsto u^{\pm}(\Phi_i(t, h))$ monotone and periodic;
hence constant in t.
$\exists v_i^{\pm}(h)$ such that

$$u^{\pm}(x) = v_i^{\pm}(h) \quad \text{for} \quad x \in c_i(h), \; h \neq 0.$$

5. Since $u^+(x) = u^-(x)$ for $x \in N$ and $N \cap c_i(h) \neq \emptyset$,

$$v_i^+(h) = v_i^-(h) \quad \text{for } h \in J_i,\ i = 1, 2, 3.$$

Since $N \cup \{0\}$ is a neighborhood of the origin and the geometry of $c_i(0)$,

$$u^+(x) = u^-(x) = \lim_{h \to 0+} v_i^{\pm}((-1)^i h) \quad \text{for } x \in \{H = 0\} \setminus \{0\}.$$

A conclusion here is:

$$u^{\varepsilon_j}(x) \ \text{ converge locally uniformly on } \Omega \setminus \{0\},$$

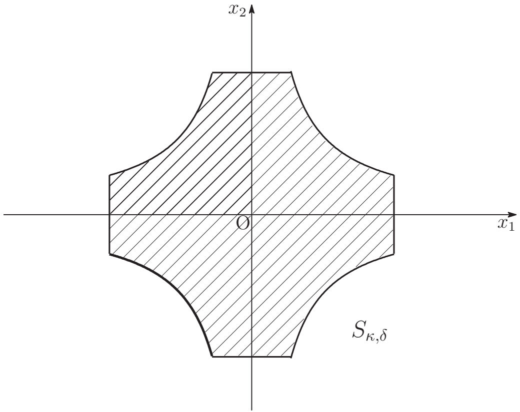
and the limit on Ω_i is: $v_i(H(x))$ for $v_i := v_i^{\pm}$.

6. Some barrier arguments show that the uniform convergence of u^{ε_j} on $\bar{\Omega} \setminus \{0\}$.

7. Assume that $H = x_1 x_2$ near the origin. Consider the set

$$S_{\kappa,\delta} = \{x \in \mathbb{R}^2 \,:\, |x|_\infty \le \kappa,\ |H| \le \delta\},$$

where $\kappa > 0$ and $\delta > 0$ are small.



Set

$$v_i(0) = \lim_{h \to 0+} v_i((-1)^i h) \quad \text{for } i = 1, 2, 3.$$

Note that $v_1(0) = v_2(0) = v_3(0)$. If $\delta \ll 1$, then

$$v_i(t) \approx v_i(0) \quad \text{for } |t| \leq \delta,$$

So, if $\delta \ll 1$ and $j \gg 1$, then

$$u^{\varepsilon_j}(x) \approx v_1(0) = v_2(0) = v_3(0) \qquad \text{for } x \in \partial S_{\kappa,\delta}.$$

Start with the function

$$E(r)=\int_0^r \mathrm{e}^{-t^2}\,\mathrm{d}t\int_0^t \mathrm{e}^{s^2}\,\mathrm{d}s,$$

and scale it appropriately, to find functions E_ε on $\mathbb{R}$ and v_ε defined on $S_{\kappa,\delta}$ as

$$v_\varepsilon(x)=E_\varepsilon(x_1),$$

which has the properties

$$\begin{aligned}
&-\big(\Delta_0+(b_0+\varepsilon^{-1}b)\cdot D\big)v_\varepsilon < -|g(x)| \quad \text{on } S_{\kappa,\delta},\\
&\lim_{\varepsilon\to 0}\|v_\varepsilon\|_{\infty,S_{\kappa,\delta}}=0.
\end{aligned}$$

By comparison, if $\delta \ll 1$ and $j \gg 1$, then

$$|u^{\varepsilon_j}(x) - v_1(0)| \lesssim -v_{\varepsilon_j}(x) + \|v_{\varepsilon_j}\|_{\infty, S_{\kappa,\delta}} \quad \text{on } S_{\kappa,\delta}.$$

Hence,

$$u^+(0) = u^-(0) \ (= v_1(0) = v_2(0) = v_3(0)).$$

8. **Finally, identify (v_1, v_2, v_3) with the unique solution (u_1, u_2, u_3) of (ODE).**

With this identification, we complete the argument for the uniform boundedness of u^{ε_j}, and then complete the proof of the uniform convergence of u^{ε} on $\bar{\Omega}$ as $\varepsilon \to 0$.

Reference

I & P. E. Souganidis, A pde approach to small stochastic perturbations of Hamiltonian flows. J. Differential Equations 252 (2012), no. 2, 1748–1775.