Asymptotic behavior of Palais-Smale sequences in the critical problem and its application to semilinear heat equation

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Introduction.

- **Notation.**
  $N \geq 3, \quad \Omega \subset \mathbb{R}^N, \quad K(\cdot) := e^{\frac{|\cdot|^2}{4}}, \quad \|u\|_{r,K}^r := \int |u(\cdot)|^r K(\cdot),$
  $\dot{H}^1(K) := \overline{C_0^\infty}^{\|\nabla \cdot\|_{2,K}}, \ L^r(K) := \overline{C_0^\infty}^{\|\cdot\|_{r,K}}.$

- **Problem.**
  $u_0 \in \dot{H}^1(K) \cap L^\infty$: initial data, $\quad p > 2, \quad u = u(x,t),$

(P)

$$\begin{array}{rcll} \partial_t u - \Delta u &=& u|u|^{p-2} & \text{in} \quad \Omega \times (0,T), \\ u &=& 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u|_{t=0} &=& u_0 & \text{in} \quad \Omega, \end{array}$$

where T: maximal existence time in the classical sense.

Asymptotic behavior of solutions of (P) with
$p = 2^* := \frac{2N}{N-2}, \quad \Omega = \mathbb{R}^N$ and u_0: "threshold"
from **the variational point of view**. $(\dot{H}^1(\Omega) \hookrightarrow L^p(\Omega))$

Plan of this talk.

- Known results. (*Results for* $p < 2^*$ *and* $p = 2^*$)
- Main results.
- Background facts for main results.
  (*Scaling structure, (PS)-sequence, energy structure*)
- Sketch of the proof.
- Application.

Known Results for $p < \frac{2N}{N-2}$, $\Omega = \mathbb{R}^N$.

Kavian ('87), Kawanago ('96): $\varphi \in \dot{H}^1(K) \cap L^\infty$, $\varphi \geq 0$.

- For any φ above, $\exists \overline{\lambda} > 0$ s.t. sol. u_λ with $u(0) = \lambda\varphi$ $(\lambda > 0)$,

$$\begin{array}{lll} \lambda < \overline{\lambda}: & T = \infty, & t^{N/2}\|u_\lambda(t)\|_\infty = O(1) \\ \lambda = \overline{\lambda}: & T = \infty, & t^{1/(p-2)}\|u_\lambda(t)\|_\infty = O(1) \\ \lambda > \overline{\lambda}: & T < \infty, & \|u_\lambda(t)\|_\infty \to \infty \end{array}$$

- $u_{\overline{\lambda}}(x,t) \simeq \frac{1}{t^{\frac{1}{p-2}}} f\left(\frac{x}{\sqrt{t}}\right)$ where $f = f(z)$ satisfies

$$0 = \Delta f + \frac{1}{2} z \cdot \nabla f + \frac{1}{p-2} f + f|f|^{p-2}$$

$\Rightarrow$ Threshold behavior: Decay with the self-similar profile

Known Results for $p = \frac{2N}{N-2}$, $\Omega = \mathbb{R}^N$, radial case.

$\varphi \geq 0$, radial, $\lim_{r\to\infty} \varphi(r) r^{\frac{2}{p-2}} = 0$.

- Poláčik-Quittner ('08), Quittner ('09)
  For any φ above, $\exists \overline{\lambda} > 0$ s.t. sol. u_λ with $u(0) = \lambda\varphi$ $(\lambda > 0)$,

$$\begin{array}{lll} \lambda < \overline{\lambda}: & T = \infty, & t^{N/2}\|u_\lambda(t)\|_\infty = O(1) \\ \lambda = \overline{\lambda}: & T = \infty, & t^{1/(p-2)}\|u_\lambda(t)\|_\infty \to \infty \ (\limsup_{t\to\infty}) \\ \lambda > \overline{\lambda}: & T < \infty, & \|u_\lambda(t)\|_\infty \to \infty \end{array}$$

- (I.('10)) In addition φ: decreasing in r, fast decay at $\infty \Rightarrow$

$$u_{\overline{\lambda}}(x,t) = \mu(t) w(\mu(t)^{\frac{2}{N-2}} x) + o(1) \text{ in } \dot{H}^1(\mathbb{R}^N)$$

where $w(x)$ is a stationary solution of (P) with $\|w\|_\infty = 1$

$\Rightarrow$ **Threshold behavior: rescale of the stationary solution**

if $\varphi \geq 0$ and radial. Nonradial case?

Known Results for $p = \frac{2N}{N-2}$, $\Omega = \mathbb{R}^N$, general case.

T. Suzuki-I. (submitted): $\varphi \geq 0$, $\varphi \in \dot{H}^1(K)$.

- For any φ above, $\exists \overline{\lambda} > 0$ s.t. sol. u_λ with $u(0) = \lambda\varphi$ $(\lambda > 0)$,

$$\begin{array}{lll} \lambda < \overline{\lambda}: & T = \infty, & t^{N/2}\|u_\lambda(t)\|_\infty = O(1) \\ \lambda = \overline{\lambda}: & T = \infty, & t^{1/(p-2)}\|u_\lambda(t)\|_\infty \to \infty \text{ or} \\ & T < \infty, & \|u_\lambda(t)\|_\infty \to \infty \\ \lambda > \overline{\lambda}: & T < \infty, & \|u_\lambda(t)\|_\infty \to \infty \end{array}$$

radial case:	intersection comparison + ODE argument,
nonradial case:	energy method + blow-up argument.

- Additionally φ: symmetric in $x_i = 0$, decreasing in $|x_i|$, $\forall i$
  $\Rightarrow$ $T = \infty$ for $\lambda = \overline{\lambda}$

Threshold behavior for general case?

Hereafter we assume $\Omega = \mathbb{R}^N$ and $p = \frac{2N}{N-2}$ unless stated.

Main Result. Asymp. behavior of threshold solution.

Let u: time-global solution of (P) with $\|\nabla u(t)\|_2 \not\to 0$ as $t \to \infty$
$\Rightarrow$
There exist $M \in \mathbb{N}$, (t_k) with $t_k \to \infty$ as $k \to \infty$ and

- $\mu_k^{(m)} \in \mathbb{R}_+$,
- $a_k^{(m)} \in \mathbb{R}^N$, $\qquad (m = 1, \cdots, M)$
- $w^{(m)} (\neq 0)$: sol. of $-\Delta w = w|w|^{p-2}$

s.t. for $\quad \langle w^{(m)} \rangle_k (x) := \frac{1}{(\mu_k^{(m)})^{\frac{N-2}{2}}} w^{(m)} \left(\frac{\cdot - a_k^{(m)}}{\mu_k^{(m)}} \right)$

: w^m rescaled by $\mu_k^{(m)}$ and translated to $a_k^{(m)}$, the following holds.

- $u(\cdot, t_k) = \sum_{m=1}^{M} \langle w^{(m)} \rangle_k + o(1) \quad \text{in } \dot{H}^1(\mathbb{R}^N)$,
- $J(u(t_k)) = \sum_{m=1}^{M} J(w^{(m)}) + o(1)$,

where $J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p$.

($J(\langle w^{(m)} \rangle_k) = J(w^{(m)})$ by the scale-invariance)

- Theorem says $(u(t_k))$ splits into a finite pieces of rescaled stationary solutions.
- u: (Nonradial) changing-sign solution is allowed.
- $(u(t_k))$ is a (PS)-sequence of J in $\dot{H}^1$.
- u: nonnegative or radially symmetric $\Rightarrow$

$w^{(m)}(x) = \left[\frac{C_N}{1+|x|^2}\right]^{\frac{N-2}{2}} =: w(x)$: m-indep., hence

$J(u(t_k)) = MJ(w) + o(1)$.

Energy structure associated with (P).

- (P) has a "L^2-negative gradient flow" structure:

$$\|\partial_t u\|_2^2 = -\partial_t J(u(t)),\ J(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p}\int |u|^p.$$

- Scale invariance of the structure associated with (P):

Let $u_\lambda(y,s) = \lambda^{\frac{2}{p-2}} u(x,t),\ y = \lambda x,\ s = \lambda^2 t$ $\quad \lambda \in \mathbb{R}_+$

- "u satisfies (P) $\Leftrightarrow$ u_λ satisfies (P)" , $\forall p$.
- "$J(u) = J(u_\lambda)$" only holds for $p = \frac{2N}{N-2}$.

(P) with $p = \frac{2N}{N-2}$

(P) and the energy structure is invariant under the action of $\mathbb{R}_+$

Difficulty. Absence of a priori (PS)-sequence.

Let u be a threshold solution of (P) (i.e., time-global, $\|\nabla u(t)\|_2 \not\to 0$).

- If $\exists (t_k)$ s.t. $u(t_k) =: u_k$ is a (PS)-sequence of J,

  (i.e., $J(u_k) \to c$, $(dJ)_{u_k}(\varphi) \to 0$ unif. in $\|\nabla \varphi\|_2 = 1$)

  then

  Struwe ('84), Benci-Cerami ('92)

  $$u_k \simeq \sum_{m=1}^{M} \frac{1}{(\mu_k^{(m)})^{\frac{N-2}{2}}} w^{(m)} \left(\frac{\cdot - a_k^{(m)}}{\mu_k^{(m)}} \right) \text{ in } \dot{H}^1 \text{ as } k \to \infty$$

  which implies a main result. $\exists (t_k)$ s.t. $u(t_k)$: (PS) of J?

- We have $\exists (t_k)$ s.t. $\|u_t(t_k)\|_2 \to 0$ as $n \to \infty$, indeed,
  - $T = \infty \Rightarrow \lim_{t\to\infty} J(u(t)) > -\infty \Rightarrow \exists (t_k)$ s.t. $\frac{d}{dt} J(u(t_k)) \to 0$.
  - By the energy equality $\|u_t(t)\|_2^2 = -\frac{d}{dt} J(u(t))$, conclusion.

- We have t_k s.t. $\|u_t(t_k)\|_2 = o(1)$. We want to show

$$(dJ)_{u(t_k)} \to 0 \text{ in } (\dot{H}^1)^*$$

  - Take any $\varphi \in \dot{H}^1$ with $\|\nabla\varphi\|_2 = 1$. We compute $(dJ)_{u(t_k)}(\varphi)$.

$$\begin{aligned} |(dJ)_u(t_k)(\varphi)| &= \left|\int_\Omega u_t(t_k)\varphi\right| \\ &\le \|u_t(t_k)\|_2\|\varphi\|_2 \text{"} \le \text{"} C\|u_t(t_k)\|_2\|\nabla\varphi\|_2 \\ &\le C\|u_t(t_k)\|_2 \end{aligned}$$

  : OK?

  - A priori NO when the Poincaré inequality fails in Ω

Dynamical structure: L^2 vs. Energy structure: $\dot{H}^1$

Sketch of the Proof for Main Theorem.

Strategy

Take a forward self-similar transformation to (P) and analyze the lack of the compactness of the associated variational problem.

Let u: $T = \infty$, $\|\nabla u(t)\|_2 \not\to 0$ ("threshold solution").

- **Forward self-similar transformation:**
  - $v(y,s) = (1+t)^{1/(p-1)} u(x,t), \quad s = \ln(1+t), \quad y = \frac{x}{\sqrt{1+t}}$
  - (P) $\Leftrightarrow$

    (P') $\quad v_s = \frac{1}{K}\nabla(K\nabla v) + \frac{1}{p-2} v + v|v|^{p-2}$, $K(y) := e^{\frac{|y|^2}{4}}$
  - u: threshold $\Rightarrow$ v: threshold ($\|\nabla v(s)\|_2 \not\to 0$).
  - v: stationary $\Rightarrow$ $v \equiv 0$ (Escobedo-Kavian ('87))

- - $\|\partial_s v\|_2^2 = -\frac{d}{ds} J_K(v(s)) \quad \left(J_K(v) = \frac{\|\nabla v\|_{2,K}^2}{2} - \frac{\|v\|_{2,K}^2}{2(p-2)} - \frac{\|v\|_{p,K}^p}{p} \right)$
  - $\|\varphi\|_{2,K} \leq C\|\nabla\varphi\|_{2,K}$ due to $\lim_{|y|\to\infty} K(y) = \infty$

  $\Rightarrow \exists (s_n)$ s.t. $v(s_k) =: v_k$: (PS)-sequence of J_K in $\dot{H}^1(K)$.

  $\Rightarrow$ Behavior of noncompact (PS)-sequence of J_K?

  - J: scale-invariant. Struwe '84, Benci-Cerami '92
  - J_K: not scale-inv. due to K. But $J_K(v) \simeq K(a)J(v)$ if $v \simeq \delta_a$.
- **Detection of the singular set:**
  (v_k): noncompact (PS) of $J_K \Rightarrow v_k \rightharpoonup 0$ in $\dot{H}^1(k)$ and

  $$\begin{cases} |v_k|^p & \rightharpoonup \quad \nu = \sum_{z\in\mathcal{S}} \nu_z\delta_z \\ |\nabla v_k|^2 & \rightharpoonup \quad \mu \geq \sum_{z\in\mathcal{S}} \mu_z\delta_z \end{cases} \quad \text{in } \mathcal{M}(\mathbb{R}^N).$$

Lemma 1.

$$\nu_z = \mu_z \geq S^{\frac{N}{2}}, \quad \forall z \in \mathcal{S}.$$

Indeed, let

$$\varphi_{z,\varepsilon} \geq 0,\ \varphi_{z,\varepsilon} = 1 \text{ in } B_\varepsilon(z),\ \varphi_{z,\varepsilon} = 0 \text{ in } B_{2\varepsilon}(z)$$

: cut-off around z with $\operatorname{supp} \varphi_{z,\varepsilon} = B_\varepsilon(z)$. Then

- $\mu_z = \lim_{\varepsilon \to 0} \lim_{k \to \infty} \|\nabla(v_k \varphi_{z,\varepsilon})\|_{2,K}^2$,
  $\nu_z = \lim_{\varepsilon \to 0} \lim_{k \to \infty} \|v_k \varphi_{z,\varepsilon}\|_{p,K}^p$.
- $(dJ_K)_{v_k}(v_k \varphi_{z,\varepsilon}) = o(1)$
  $\Rightarrow \|\nabla(v_k \varphi_{z,\varepsilon})\|_{2,K}^2 = \|v_k \varphi_{z,\varepsilon}\|_{p,K}^p + o(1)$ as $k \to \infty$, $\forall \varepsilon > 0$
  $\Rightarrow \boxed{\nu_z = \mu_z}$ (taking $\lim\limits_{\varepsilon \to 0} \lim\limits_{k \to \infty}$)
- $S \|v_k \varphi_{z,\varepsilon}\|_{p,K}^2 \leq \|\nabla(v_k \varphi_{z,\varepsilon})\|_{2,K}^2$: Sobolev inequality
  $\Rightarrow \boxed{S {\nu_z}^{\frac{2}{p}} \leq \mu_z}$ (taking $\lim\limits_{\varepsilon \to 0} \lim\limits_{k \to \infty}$).

Lemma 2.

$|\mathcal{S}| < \infty$ and there exists $R > 0$ s.t. $\lim_{k\to\infty} \int_{R\le|y|} |\nabla v_k|^2 = 0.$

- First: By Lemma 1. Especially $\boxed{\exists R_0 > 0 \text{ s.t. } \mathcal{S} \subset B_{R_0}}$
- Second: $\int_{2R_0\le|y|} |\nabla v_k|^2 = \int_{2R_0\le|y|\le\rho} |\nabla v_k|^2 + \int_{\rho\le|y|} |\nabla v_k|^2.$
  - For $\int_{\rho\le|y|} |\nabla v_k|^2$:
    - $\|\nabla v_k\|_{2,K}$: b'dd. since (v_k): (PS) of J_K. Then
    - $\int_{\rho\le|y|} |\nabla v_k|^2 e^{\frac{\rho^2}{4}} \le \int |\nabla v_k|^2 e^{\frac{|y|^2}{4}} \le \exists C$

      $$\Rightarrow \int_{\rho\le|y|} |\nabla v_k|^2 \le e^{-\frac{\rho^2}{4}} C \Rightarrow \lim_{\rho\to\infty} \lim_{k\to\infty} \int_{\rho\le|y|} |\nabla v_k|^2 = 0.$$
    - Hence $\int_{\rho\le|y|} |\nabla v_k|^2 = o(1)$ for $\rho \gg R_0$ and $k \gg 1$.
  - For $\int_{2R_0\le|y|\le\rho} |\nabla v_k|^2$:

    $|\nabla v_k|^2 \rightharpoonup \sum_{z\in\mathcal{S}} \mu_z \delta_z$, $\mathcal{S} \subset B_{R_0} \Rightarrow \int_{2R_0\le|y|\le\rho} |\nabla v_k|^2 = o(1)$ as $k \to \infty$.

Lemma 3.

There exists $\varepsilon_k > 0$ with $\varepsilon_k \to 0$, $y_k \in \mathbb{R}^N$,
w: nontrivial solution of $-\Delta w = w|w|^{p-2}$ in $\mathbb{R}^N$ s.t.
$v_k^{(1)}(z) := (\varepsilon_k)^{\frac{N-2}{2}} v(y_k + \varepsilon_k z) \rightharpoonup w(z)$ in $\dot{H}^1(\mathbb{R}^N)$.

- **Find appropriate scaling:**

  $\boxed{Q_k(r) := \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |\nabla v_k|^2}$: concentration function.

  Then

  - $Q_k(0) =$ integration over ball with radius $0 = 0$,
  - $Q_k(\infty) = \int_{\mathbb{R}^N} |\nabla v_k|^2 \geq \exists\delta$
    (since otherwise $\|\nabla u_k\|_2 = \|\nabla v_k\|_2 \to 0$: contradictiong the assumption "u: threshold.")

  $\Rightarrow$ (by mean-value theorem) $\exists\varepsilon_k > 0$, $y_k \in \mathbb{R}^N$ s.t.

  $$\frac{\delta}{2} = Q_k(\varepsilon_k) = \int_{B_{\varepsilon_k}(y_k)} |\nabla v_k|^2.$$

- **Rescaling:**
  For $v_k^{(1)}(z) := (\varepsilon_k)^{\frac{N-2}{2}} v_k(\varepsilon_k z + y_k)$, there exists w s.t.
  $v_k \rightharpoonup w$ in $\dot{H}^1(\mathbb{R}^N)$. What is w?
  - (v_k): (PS) with weight $K(y) = e^{\frac{|y|^2}{4}}$ $\Rightarrow$ How about $(v_k^{(1)})$?
  - ex. $\int \nabla_y v_k(y) \nabla_y \varphi(y) K(y) dy = \int \nabla_z v_k(z) \nabla_z \varphi(z) K_k(z) dz$,

    where $K_k(z) = e^{\frac{|\varepsilon_k z|^2}{4}}$
  - Tails of rescaled weight K_k:

$$\lim_{k\to\infty} K_k(z) = \lim_{k\to\infty} e^{\frac{|\varepsilon_k z|^2}{4}} = 1,$$

$$\lim_{|z|\to\infty} K_k(z) = \lim_{|z|\to\infty} e^{\frac{|\varepsilon_k z|^2}{4}} = \infty, \quad \forall k > 1.$$

  $\exists$tail of v_k at $z \simeq \infty$ causes a several technical difficulties

- Test $(dJ_{K(\varepsilon_k \cdot)})_{v_k^{(1)}}$ with φ: compact support. Then we can show that w is a nontrivial stationary solution of (P)

By iterating if $\|\nabla v_k^{(1)}\|_2 \not\to 0$, we obtain:
$\beta > 0$, $J_{\beta,K}(v) := \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\beta}{2}\|v\|_{2,K}^2 - \frac{1}{p}\|v\|_{p,K}^p$,

Proposition.

(v_k): (PS)-sequence of $J_{\beta,K}$ in $\dot{H}^1(K)$ with $\|\nabla v_k\|_2 \not\to 0 \Rightarrow$
$\exists M \in \mathbb{N}$, $\varepsilon_k^{(m)} \in \mathbb{R}_+$, $y_k^{(m)} \in \mathbb{R}^N$, $w^{(m)}$: sol. $-\Delta w = w|w|^{p-2}$
s.t., as $k \to \infty$, $(m = 1, \cdots, M)$

$$v_k = \sum_{m=1}^M \frac{1}{(\varepsilon_k^{(m)})^{\frac{N-2}{2}}} w^{(m)}\left(\frac{\cdot - y_k^{(m)}}{\varepsilon_k^{(m)}}\right) + o(1) \quad \text{in } \dot{H}^1(\mathbb{R}^N),$$

$$J(v_k) = \sum_{m=1}^M J(w^{(m)}) + o(1), \quad J(v) = \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{p}\|v\|_p^p.$$

- Invariance of $\|\nabla \cdot\|_2$ under the forward self-similar transformation with $\beta = \frac{1}{p-2} (= \frac{N-2}{4})$
  $\Rightarrow$ Main Result (on $u(t_k)$).
- $\mu_k^{(m)} := \sqrt{1+t_k}\varepsilon_k^{(m)}$, $a_k^m := \sqrt{1+t_k}y_k^{(m)}$:
  no infomation for $\lim_{k\to\infty}$ is available.

- **Proposition** gives the energy level for which J_K does not satisfy the (PS)-condition: $c = c_0 + \sum_{m=1}^{M} c_j$ (c_0: critical value of J_K, c_j: critical value of J, $M \in \mathbb{N}$).

Several applications in variational analysis

- **Example:** We can show that, if $N \geq 7$, then
  - $\exists$2-critical points v_j $(j = 1, 2)$ of J_K for $\beta \in (\frac{N}{4}, \frac{N}{2})$, thus
  - $\partial_t u = \Delta u + \frac{\omega}{1+t} u + u|u|^{p-2}$ has at least 2 time-global solution of the form $u_j(x,t) = \frac{1}{t^{\frac{1}{p-2}}} v_j \left(\frac{x}{\sqrt{t}} \right)$ for $\omega \in (\frac{1}{2}, \frac{1}{2} + \frac{N}{4})$.

$\Leftarrow$ construction of the critical point by min-max over

1. 1-dimensioinal family (moutain pass) which gives a lower energy solution , and
2. 2-dimensioinal family which gives a higher energy solution.

Thank you for your kind attention!