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Global attractivity properties of stationary solutions for semilinear heat equations

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Introduction

In this talk we will consider the problem

$$(1.1) \qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $N > 2$, $p > 1$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, $u_0 \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$.

It is well known that there exists $T \in (0,\infty]$ such that (1.1) has a unique classical solution on $t \in (0,T)$, and if $T < \infty$ then $\lim_{t\to T} \|u(t)\|_\infty = \infty$.

We are interested in the case $T = \infty$, and consider the attractivity property of positive stationary solutions.

Plan of this talk

Part I. Remarks on attractivity properties for $u_t = \Delta u + u^p$.

1.1. Known results

(i) Local stability property by Gui, Ni, and Wang (1992, 2001)

(ii) Global attracitivty property by Poláčik and Yanagida (2003)

1.2. Main remarks

1.3. Sketch of Proof

Part II. Application to the problem $u_t = \Delta u + e^u$

Part III. Application to self-similar solutions to $u_t = \Delta u + u^p$

Part I.

Remarks on attractivity properties for

$$u_t = \Delta u + u^p.$$

Stationary solutions: Definitions

Let us recall the properties of solutions to the problem

$$\Delta\phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N.$$

Define

$$p_c = \begin{cases} \infty, & 3 \le N \le 10, \\ 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \ge 11. \end{cases}$$

Put

$$L = \left(\frac{2}{p-1}\left(N - 2 - \frac{2}{N-2}\right)\right)^{1/(p-1)}.$$

Note that there is a singular solution $\phi_\infty(|x|) = L|x|^{-2/(p-1)}$ if $p > N/(N-2)$.

Stationary solutions: Positive solutions

For $\alpha > 0$, we denote by $\phi_\alpha(r)$ a solution of

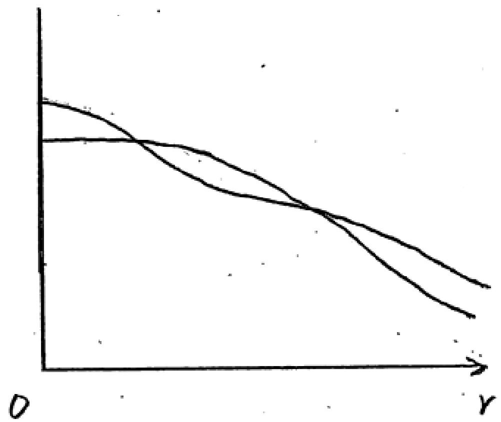
$$\begin{cases} \phi'' + \dfrac{N-1}{r}\phi' + \phi^p = 0, \quad r > 0, \\ \phi(0) = \alpha \quad \text{and} \quad \phi'(0) = 0. \end{cases}$$

- When $p > (N+2)/(N-2)$, ϕ_α satisfies $\phi_\alpha(r) > 0$ for $r > 0$ and

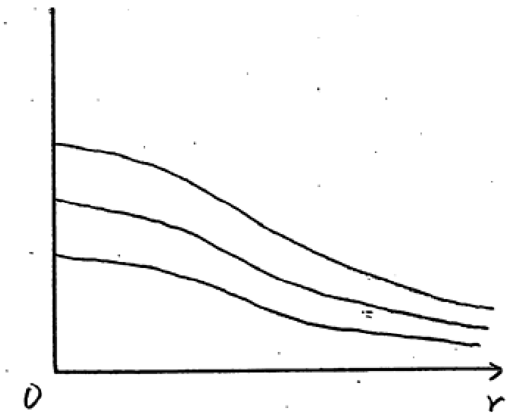
$$r^{2/(p-1)}\phi_\alpha(r) \to L \quad \text{as } r \to \infty.$$

- When $(N+2)/(N-2) < p < p_c$,
  $\phi_\alpha(r)$ and $\phi_\beta(r)$ intersect infinity many times on $(0, \infty)$.

- When $p \geq p_c$, $\phi_\alpha(r) > \phi_\beta(r)$ for $r \geq 0$ if $\alpha > \beta$.



$P < P_c$



$P \geq P_c$

Known results by Gui-Ni-Wang

Let $\psi \in C(\mathbf{R}^N)$. For $\lambda > 0$, define $\|\psi\|_\lambda = \sup_{x \in \mathbf{R}^N} (1+|x|)^\lambda |\psi(x)|$.

Let λ_1, λ_2 be the roots of

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0 \quad \text{with } m = \frac{2}{p-1}.$$

When $p > p_c$, we have $0 < \lambda_1 < \lambda_2$. (If $p = p_c$, then $0 < \lambda_1 = \lambda_2$.)

Theorem A (Gui-Ni-Wang 1992, 2001). Let $p > p_c$ and $\alpha > 0$.

(i) For any $\varepsilon > 0$ there is $\delta > 0$ such that, if $\|u_0 - \phi_\alpha\|_{m+\lambda_1} < \delta$, then

$$\|u(\cdot, t, u_0) - \phi_\alpha\|_{m+\lambda_1} < \varepsilon \quad \text{for all } t > 0.$$

(ii) For any $\lambda \in (\lambda_1, \lambda_2]$, there is $\delta > 0$ such that, if $\|u_0 - \phi_\alpha\|_{m+\lambda} < \delta$, then

$$\lim_{t \to \infty} \|u(\cdot, t, u_0) - \phi_\alpha\|_{m+\lambda'} = 0 \quad \text{for any } \lambda' \in [0, \lambda).$$

i.e., ϕ_α is stable w.r.t. $\|\cdot\|_{m+\lambda_1}$, and
ϕ_α is weakly asymptotically stable w.r.t. $\|\cdot\|_{m+\lambda}$.

Known results by Gui-Ni-Wang: Remark

The condition $p \geq p_c$ is crucial.

Theorem (Gui-Ni-Wang 1992)

Assume that $(N+2)/(N-2) < p < p_c$.

(i) If $u_0 \leq \phi_\alpha$ and $u_0 \not\equiv \phi_\alpha$ for some $\alpha > 0$,

then $u(x,t,u_0)$ is global and $\lim_{t\to\infty} \|u(\cdot,t,u_0)\|_{L^\infty} = 0$.

(ii) If $u_0 \geq \phi_\alpha$ and $u_0 \not\equiv \phi_\alpha$ for some $\alpha > 0$,

then $u(x,t,u_0)$ blows up in finite time.

Known results by Gui-Ni-Wang: Remark

Let us recall a sketch of proof of Theorem A.

Let $p > p_c$. For each $\alpha > 0$, we have

$$\phi_\alpha(x) = \frac{L}{|x|^m} + \frac{a_1(\alpha)}{|x|^{m+\lambda_1}} + o\left(\frac{1}{|x|^{m+\lambda_1+\varepsilon}}\right) \quad \text{as } |x| \to \infty,$$

where $a_1(\alpha) < 0$ and $a_1(\alpha)$ is continuous and increasing in $\alpha > 0$.

Then we have

$$\|\phi_\alpha - \phi_\beta\|_{m+\lambda_1} \to 0 \quad \text{as } \beta \to \alpha.$$

Thus, for any $\varepsilon > 0$, there is $\eta > 0$ such that

$$\|\phi_{\alpha\pm\eta} - \phi_\alpha\|_{m+\lambda_1} < \varepsilon.$$

Known results by Gui-Ni-Wang: Remark

Let $\eta > 0$ satisfy

$$\|\phi_{\alpha\pm\eta} - \phi_\alpha\|_{m+\lambda_1} < \varepsilon.$$

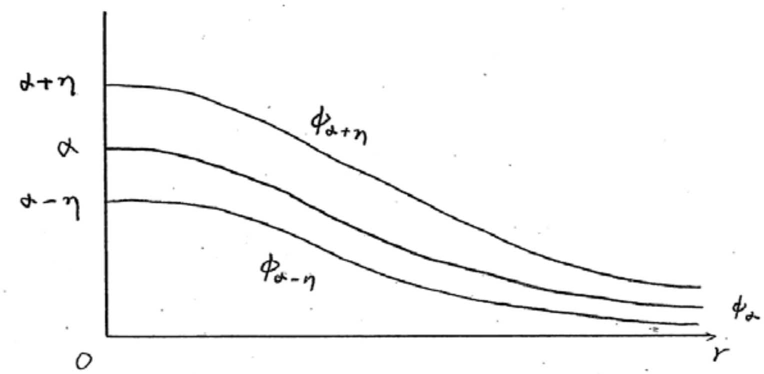
There exists $\delta > 0$ such that, if $\|\phi_\alpha - u_0\|_{m+\lambda_1} < \delta$, then

$$\phi_{\alpha-\eta}(x) \le u_0(x) \le \phi_{\alpha+\eta}(x) \quad \text{in } \mathbf{R}^N.$$

By the separation property of $\phi_{\alpha+\eta}$ and $\phi_{\alpha-\eta}$, we have

$$\phi_{\alpha-\eta}(x) \le u(x,t,u_0) \le \phi_{\alpha+\eta}(x) \quad \text{for } t > 0,\ x \in \mathbf{R}^N.$$

Thus we obtain Theorem A (i).



Known results by Gui-Ni-Wang: Remark

Let us consider the sketch of proof of Theorem A (ii). Recall that

$$\phi_\alpha(x) = \frac{L}{|x|^m} + \frac{a_1(\alpha)}{|x|^{m+\lambda_1}} + o\left(\frac{1}{|x|^{m+\lambda_1+\varepsilon}}\right) \quad \text{as } |x| \to \infty.$$

Let $\lambda > \lambda_1$. Then we have

$$\{u_0 : \|u_0 - \phi_\alpha\|_{m+\lambda} < \delta\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$

If $\|u_0 - \phi_\alpha\|_{m+\lambda} < \delta$, we can show that

$$\|u(\cdot, t; u_0) - \phi_\alpha\|_{\lambda'} \to 0 \quad \text{as } t \to \infty$$

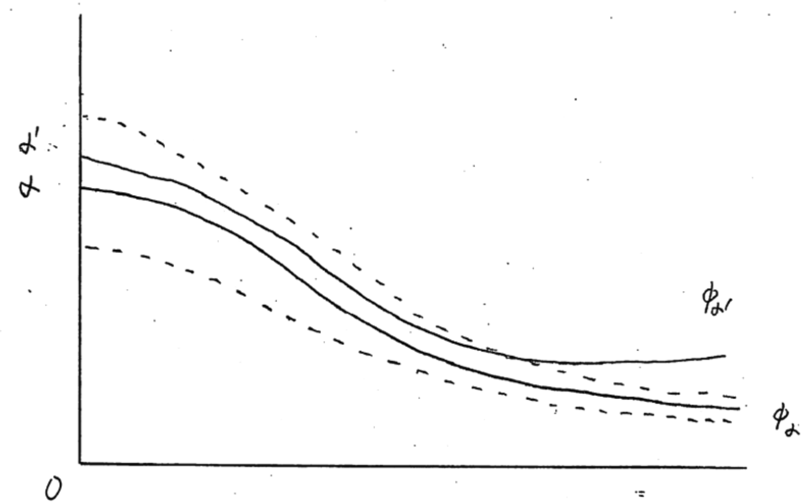
for any $\lambda' \in (0, m + \lambda)$. Thus we obtain Theorem A (ii).

Recall that

$$\phi_\alpha(x) = \frac{L}{|x|^m} + \frac{a_1(\alpha)}{|x|^{m+\lambda_1}} + o\left(\frac{1}{|x|^{m+\lambda_1+\varepsilon}}\right) \quad \text{as } |x| \to \infty.$$

Let $\lambda > \lambda_1$. Then we have

$$\{u_0 : \|u_0 - \phi_\alpha\|_{m+\lambda} < \delta\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$



Known results by Poláčik-Yanagida

The global attractivity property of steady states was shown by Poláčik-Yanagida(2003).

Theorem B (Poláčik-Yanagida 2003)

Let $p \geq p_c$. Assume that

$$-\phi_\infty(|x|) \leq u_0(x) \leq \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

If u_0 satisfies

$$\lim_{|x|\to\infty} |x|^{m+\lambda_1} |u_0(x) - \phi_\alpha(|x|)| = 0$$

with some $\alpha > 0$, then

$$\|u(\cdot, t, u_0) - \phi_\alpha\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } t \to \infty.$$

Known results by Poláčik-Yanagida: Applications

As applications of Theorem B, they showed the following.

Define the ω-set of the solution u by

$$\omega(u) = \{\psi : u(\cdot, t_n) \to \psi \text{ for some sequence } t_n \to \infty\}.$$

Theorem (Existence of nonstabilizing bounded solution)

Let $p \geq p_c$. For given $-\infty < \alpha < \beta < \infty$,
there is u_0 such that $u = u(x, t; u_0)$ satisfies

$$\omega(u) = \{\phi_\gamma : \alpha \leq \gamma \leq \beta\}.$$

Theorem (Existence of global unbounded solution)

Let $p \geq p_c$. Then there exists u_0 such that $u(x, t, u_0)$ is global and

$$\|u(\cdot, t, u_0)\|_{L^\infty(\mathbf{R}^N)} \to \infty \quad \text{as } t \to \infty.$$

See also Poláčik-Yanagida (2004).
(The existence of solution which undergoes a sort of birth-and-death process.)

Known results (summary)

In Part I, we will give some remarks on these results.

Theorem A. Let $p > p_c$ and $\alpha > 0$.

(i) For any $\varepsilon > 0$ there is $\delta > 0$ such that,
if $\|u_0 - \phi_\alpha\|_{m+\lambda_1} < \delta$, then

$$\|u(\cdot, t, u_0) - \phi_\alpha\|_{m+\lambda_1} < \varepsilon \quad \text{for all } t > 0.$$

(ii) For any $\lambda \in (\lambda_1, \lambda_2]$, there is $\delta > 0$ such that,
if $\|u_0 - \phi_\alpha\|_{m+\lambda} < \delta$, then

$$\lim_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha\|_{m+\lambda'} = 0 \quad \text{for any } \lambda' \in [0, \lambda).$$

Theorem B. Let $p \geq p_c$. Assume that

$$-\phi_\infty(|x|) \leq u_0(x) \leq \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

If u_0 satisfies $\lim_{|x|\to\infty} |x|^{m+\lambda_1} |u_0(x) - \phi_\alpha(|x|)| = 0$ with some $\alpha > 0$, then

$$\|u(\cdot, t, u_0) - \phi_\alpha\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } t \to \infty.$$

Main remarks: The case $p > p_c$

We assume that $u_0 \in C(\mathbf{R}^N)$ satisfies

$$(*) \qquad -\phi_\infty(|x|) \le u_0(x) \le \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

By a small change, we obtain the following.

Theorem 1.1. Let $p > p_c$ and $\alpha > 0$. Assume $(*)$.
For any $\varepsilon > 0$, there is $\delta > 0$ such that, if

$$\limsup_{|x|\to\infty} |x|^{m+\lambda_1} |u_0(x) - \phi_\alpha(|x|)| < \delta,$$

then

$$\limsup_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{m+\lambda_1} < \varepsilon.$$

Corollary 1.1. Let $p > p_c$. Assume $(*)$. If

$$\lim_{|x|\to\infty} |x|^{m+\lambda_1} |u_0(x) - \phi_\alpha(|x|)| = 0$$

with some $\alpha > 0$, then

$$\lim_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{m+\lambda_1} = 0.$$

Main remarks: The case $p > p_c$

The asymptotically stability result is extended as follows.

Theorem 1.2. Let $p > p_c$ and $\alpha > 0$. Assume $(*)$.
If there exist $\lambda \in (\lambda_1, \lambda_2]$ such that

$$\limsup_{|x|\to\infty} |x|^{m+\lambda} |u_0(x) - \phi_\alpha(|x|)| < \infty$$

then, for any $\lambda' \in [0, \lambda)$,

$$(**) \qquad \lim_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{m+\lambda'} = 0.$$

Remark.
(i) It is open whether $(**)$ holds with $\lambda' = \lambda$.
(ii) In Theorem 1.2, we can not replace $\lambda \in (\lambda_1, \lambda_2]$ by $\lambda = \lambda_1$.
(iii) In the proof, we will employ weak super- and subsolutions to stationary problem.

Main remarks: The case $p > p_c$

Theorem (Fila-Winkler-Yanagida 2005, see also Hoshino-Yanagida 2008)
Let $p > p_c$ and $\lambda \in (\lambda_1, \lambda_2)$. Assume $(*)$.
If u_0 satisfies

$$|u_0(x) - \phi_\alpha(|x|)| \le c|x|^{-m-\lambda}, \quad x \in \mathbf{R}^N,$$

for some $\alpha > 0$ and $c > 0$, then there exists $C > 0$ such that

$$\|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{L^\infty} \le C(1+t)^{-\frac{\lambda-\lambda_1}{2}}$$

for all $t > 0$. (This estimate is optimal.)

Remark. The decay rate may depend on the weight of the norm.
We have

$$\|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{\lambda'} = \begin{cases} O\left(t^{-\frac{\lambda-\lambda_1}{2}}\right) & \text{if } \lambda' \in (0, m+\lambda_1] \\ O\left(t^{-\frac{\lambda-\lambda_1}{2\lambda}(\lambda-\lambda')}\right) & \text{if } \lambda' \in (m+\lambda_1, m+\lambda) \end{cases}$$

as $t \to \infty$.

Main remarks: The case $p = p_c$

Next let us consider the case where $p = p_c$. For each $\alpha > 0$, we have

$$\phi_\alpha(x) = \frac{L}{|x|^m} + \frac{a_1(\alpha)\log|x|}{|x|^{m+\lambda_1}} + O\left(\frac{1}{|x|^{m+\lambda_1}}\right) \quad \text{as } |x| \to \infty,$$

where $a_1(\alpha) < 0$ and $a_1(\alpha)$ is increasing in $\alpha > 0$.
For $\lambda > 0, \mu > 0$, define

$$|||\psi|||_{\lambda,\mu} = \sup_{x\in\mathbf{R}^N} \frac{(1+|x|)^\lambda}{(\log(2+|x|))^\mu}|\psi(x)|.$$

Theorem (Gui-Ni-Wang 1992, 2001) Let $p = p_c$ and $\alpha > 0$.
(i) For any $\varepsilon > 0$ there is $\delta > 0$ such that, if $|||u_0 - \phi_\alpha|||_{m+\lambda_1,1} < \delta$, then

$$|||u(\cdot,t,u_0) - \phi_\alpha|||_{m+\lambda_1,1} < \varepsilon \quad \text{for all } t > 0.$$

(ii) For any $\mu \in [0,1)$, there is $\delta > 0$ such that, if $|||u_0 - \phi_\alpha\|_{m+\lambda_1,\mu} < \delta$, then

$$\lim_{t\to\infty} |||u(\cdot,t,u_0) - \phi_\alpha|||_{m+\lambda_1,\mu'} = 0 \quad \text{for any } \mu' \in (\mu, 1].$$

Main remarks: The case $p = p_c$

Theorem 2.1. Let $p = p_c$ and $\alpha > 0$. Assume $(*)$.
For any $\varepsilon > 0$, there is $\delta > 0$ such that, if

$$\limsup_{|x|\to\infty} \frac{|x|^{m+\lambda_1}}{\log|x|} |u_0(x) - \phi_\alpha(|x|)| < \delta,$$

then

$$\limsup_{t\to\infty} |||u(\cdot,t,u_0) - \phi_\alpha(|\cdot|)|||_{m+\lambda_1,1} < \varepsilon.$$

Corollary 2.1. Let $p = p_c$. Assume $(*)$.
If

$$\lim_{|x|\to\infty} \frac{|x|^{m+\lambda_1}}{(\log|x|)} |u_0(x) - \phi_\alpha(|x|)| = 0$$

with some $\alpha > 0$, then

$$\lim_{t\to\infty} |||u(\cdot,t,u_0) - \phi_\alpha(|\cdot|)|||_{m+\lambda_1,1} = 0.$$

Main remarks: The case $p = p_c$

Theorem 2.2. Let $p = p_c$. Assume $(*)$.
If there exist $\alpha > 0$ and $\mu \in [0,1)$ such that, if

$$\limsup_{|x|\to\infty} \frac{|x|^{m+\lambda_1}}{(\log|x|)^{\mu}} |u_0(x) - \phi_\alpha(|x|)| < \infty,$$

then, for any $\mu' \in (\mu, 1]$,

$$\lim_{t\to\infty} |||u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)|||_{m+\lambda_1,\mu'} = 0.$$

Sketch of Proof:Weak super- and subsolutions

Let us consider the stationary problem

$$\Delta\phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N.$$

We say that ϕ is a (continuous) weak subsolution, if $\phi \in C^\infty(\mathbf{R}^N)$ satisfies, for any $\eta \in C^2(\mathbf{R}^N)$ with $\eta \geq 0$, compact support in $\mathbf{R}^N$,

$$\int_{\mathbf{R}^N} \left(\phi(x)\Delta\eta(x) + \phi(x)^p\eta(x)\right) dx \geq 0.$$

Note here that, if $\phi \in C^2(\mathbf{R}^N)$, then

$$\Delta\phi + \phi^p \geq 0 \quad \text{in } \mathbf{R}^N.$$

Sketch of Proof:Weak super- and subsolutions

Let us recall fundamental results for the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

Lemma 1. Let $\overline{u}_0$ and $\underline{u}_0$ are weak super- and subsolutions satisfying $\underline{u}_0 \le u_0 \le \overline{u}_0$. Then

$$\underline{u}_0(x) \le u(x,t,u_0) \le \overline{u}_0(x) \quad \text{for all } t > 0.$$

Lemma 2. If u_0 is a bounded weak subsolution, then $u(x,t,u_0)$ is non-decreasing in $t > 0$ for each fixed $x \in \mathbf{R}^N$.

Lemma 3. Assume that $u = u(|x|,t)$ is decreasing or increasing in $t > 0$, and that

$$\lim_{t\to\infty} u(|x|,t) = v(|x|) \in L^\infty.$$

Then v is a stationary solution, i.e, $v \equiv \phi_\alpha$ with some $\alpha \ge 0$.

Sketch of Proof:Weak super- and subsolutions

By using previous lemmas, we obtain the following.

Lemma 4. (i) Assume that $\overline{u}_0$ is a weak supersolution satisfying
$\overline{u}_0 > \phi_\alpha$ and

$$\{\psi : \phi_\alpha \leq \psi \leq \overline{u}_0\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$

Then we have $u(x, t, \overline{u}_0) \to \phi_\alpha(x)$ as $t \to \infty$.

(Because we have $\phi_\alpha(x) \leq u(x, t, \overline{u}_0) \leq \overline{u}_0(x)$ for $t \geq 0$.)

(ii) Assume that $\underline{u}_0$ is a weak subsolution satisfying
$\underline{u}_0 < \phi_\alpha$ and

$$\{\psi : \underline{u}_0 \leq \psi \leq \phi_\alpha\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$

Then we have $u(x, t, \underline{u}_0) \to \phi_\alpha(x)$ as $t \to \infty$.

Sketch of Proof: Proof of Theorem 1.2

We will give the proof of Theorem 1.2 (global attractivity part).

Theorem 1.2. Let $p > p_c$ and $\alpha > 0$. Assume $(*)$.
If there exist $\lambda \in (\lambda_1, \lambda_2]$ such that

$$\limsup_{|x|\to\infty} |x|^{m+\lambda} |u_0(x) - \phi_\alpha(|x|)| < \infty$$

then, for any $\lambda' \in [0, \lambda)$,

$$\lim_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{m+\lambda'} = 0.$$

Recall that $u_0 \in C(\mathbf{R}^N)$ satisfies $(*)$, i.e.,

$$(*) \qquad -\phi_\infty(|x|) \le u_0(x) \le \phi_\infty(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

We may assume that there exists $\beta > \alpha$ such that

$$-\phi_\beta(|x|) \le u_0(x) \le \phi_\beta(|x|) \quad \text{for } x \in \mathbf{R}^N.$$

Sketch of Proof: Proof of Theorem 1.2

The following result is crucial for the proof of Theorem 1.2.

Proposition. Assume that $u_0 \in C(\mathbf{R}^N)$ satisfies

$$\limsup_{|x|\to\infty} |x|^{m+\lambda}|u_0(x) - \phi_\alpha(|x|)| < \infty$$

for some $\lambda \in (\lambda_1, \lambda_2]$, and

$$-\phi_\beta(|x|) \le u_0(x) \le \phi_\beta(|x|) \quad \text{for } x \in \mathbf{R}^N \quad \text{with } \beta > \alpha.$$

(i) There exists a weak supersolution $\overline{u}_0$ satisfying

$u_0 < \overline{u}_0$, $\phi_\alpha < \overline{u}_0$ and

$$\{\psi : \phi_\alpha \le \psi \le \overline{u}_0\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$

(ii) There exists a weak subsolution $\underline{u}_0$ satisfying

$\underline{u}_0 < u_0$, $\underline{u}_0 < \phi_\alpha$ and

$$\{\psi : \underline{u}_0 \le \psi \le \phi_\alpha\} \cap \{\phi_\beta : \beta > 0\} = \{\phi_\alpha\}.$$

Sketch of Proof: Proof of Theorem 1.2

Once we obtain Proposition, it is easy to give the proof of Theorem 1.2.

Let $\overline{u}_0$ and $\underline{u}_0$ be super- and subsolutions obtained in Proposition.

Since $\underline{u}_0 \le u_0 \le \overline{u}_0$, we have

$$u(x,t,\underline{u}_0) \le u(x,t,u_0) \le u(x,t,\overline{u}_0) \quad \text{for } t \ge 0.$$

By Lemma 4, we have

$$u(x,t,\underline{u}_0) \to \phi_\alpha(x) \quad \text{and} \quad u(x,t,\overline{u}_0) \to \phi_\alpha(x) \quad \text{as } t \to \infty.$$

Thus we obtain $u(x,t,u_0) \to \phi_\alpha(x)$ as $t \to \infty$. $\square$

Part II.

Application to the problem $u_t = \Delta u + e^u$

Exponential nonlinearity: Problem

We consider the problem

$$\begin{cases} u_t = \Delta u + e^u & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $N > 2$ and $u_0 \in C(\mathbf{R}^N)$.

First, let us recall the properties of solutions to the problem

$$\Delta \phi + e^{\phi} = 0 \quad \text{in } \mathbf{R}^N.$$

Exponential nonlinearity: Stationary solutions

For $\alpha > 0$, we denote by $\phi_\alpha(r) = \phi(r;\alpha)$ a solution of

$$\begin{cases} \phi'' + \dfrac{N-1}{r}\phi' + e^{\phi} = 0, \quad r > 0, \\ \phi(0) = \alpha \quad \text{and} \quad \phi'(0) = 0. \end{cases}$$

- When $N \geq 3$, $\phi_\alpha(r)$ is decreasing and satisfies

$$\phi_\alpha(r) = -2\log r + O(1) \quad \text{as } r \to \infty.$$

- When $3 \leq N \leq 9$,
  $\phi_\alpha(r)$ and $\phi_\beta(r)$ intersect infinity many times on $(0,\infty)$.

- When $N \geq 10$, $\phi_\alpha(r) > \phi_\beta(r)$ for $r \geq 0$ if $\alpha > \beta$.

Exponential nonlinearity: Stationary solutions

Let us recall the precise behavior of $\phi_\alpha(r)$ as $r \to \infty$. (Tello 2006)
Let λ_1, λ_2 be the roots of

$$\lambda^2 - (N-2)\lambda + 2(N-2) = 0.$$

When $N \geq 10$, we have $0 < \lambda_1 \leq \lambda_2$, i.e.,

$$\lambda_1 = \frac{N-2-\sqrt{(N-2)(N-10)}}{2}, \quad \lambda_2 = \frac{N-2+\sqrt{(N-2)(N-10)}}{2}.$$

For each $\alpha > 0$, we have

$$\phi_\alpha(x) = \begin{cases} -2\log r + \log(2N-4) + a_1(\alpha) r^{-\lambda_1} + b_1(\alpha) r^{-\lambda_2} + o(r^{-\lambda_2}) \\ \hfill \text{if } N \geq 11, \\ -2\log r + \log(2N-4) + \dfrac{a_1(\alpha)\log r + b_1(\alpha)}{r^{\lambda_1}} + o(r^{-\lambda_1}) \\ \hfill \text{if } N = 10 \end{cases}$$

as $|x| \to \infty$, where $a_1(\alpha)$ is a negative constant.
Note here that $a_1(\alpha) < 0$ is continuous and increasing in α.

Exponential nonlinearity: Local stability

Theorem A (Tello 2006)

Let $N > 10$ and $\alpha > 0$.

(i) For any $\varepsilon > 0$ there is $\delta > 0$ such that, if $\|u_0 - \phi_\alpha\|_{\lambda_1} < \delta$, then

$$\|u(\cdot, t, u_0) - \phi_\alpha\|_{\lambda_1} < \varepsilon \quad \text{for all } t > 0.$$

(ii) There is $\delta > 0$ such that, if $\|u_0 - \phi_\alpha\|_{\lambda_2} < \delta$, then

$$\lim_{t \to \infty} \|u(\cdot, t, u_0) - \phi_\alpha\|_\lambda = 0 \quad \text{for any } \lambda \in (0, \lambda_2).$$

Exponential nonlinearity: Global attractivity

We assume that $u_0 \in C(\mathbf{R}^N)$ satisfies

$$(*) \qquad \phi_{\beta'}(|x|) \le u_0(x) \le \phi_\beta(|x|) \quad \text{for } x \in \mathbf{R}^N$$

with some $\beta', \beta \in \mathbf{R}$.

Theorem 3.1. Let $N > 10$, and assume $(*)$. Let $\alpha \in (\beta', \beta)$.
For any $\varepsilon > 0$, there is $\delta > 0$ such that, if

$$\limsup_{|x|\to\infty} |x|^{\lambda_1} |u_0(x) - \phi_\alpha(|x|)| < \delta,$$

then

$$\limsup_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{\lambda_1} < \varepsilon.$$

As a consequence of Theorem 1, we obtain the following.

Corollary 3.1.(Tello 2006) Let $N > 10$. Assume $(*)$. If

$$\lim_{|x|\to\infty} |x|^{\lambda_1} |u_0(x) - \phi_\alpha(|x|)| = 0$$

with some $\alpha \in (\beta', \beta)$, then

$$\lim_{t\to\infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{\lambda_1} = 0.$$

Exponential nonlinearity: Global attractivity

Theorem 3.2. Let $N > 10$, and assume $(*)$. Let $\alpha \in (\beta', \beta)$. If there exist $\lambda \in (\lambda_1, \lambda_2]$ such that

$$\limsup_{|x| \to \infty} |x|^{\lambda} |u_0(x) - \phi_\alpha(|x|)| < \infty$$

then, for any $\lambda' \in [0, \lambda)$,

$$(**) \qquad \lim_{t \to \infty} \|u(\cdot, t, u_0) - \phi_\alpha(|\cdot|)\|_{\lambda'} = 0.$$

Part III.

Application to self-similar solutions to

$$u_t = \Delta u + u^p$$

Self-similar solutions

We will consider the problem

$$(1.1) \qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $N > 2$, $p > 1$, and $u_0 \geq 0$, $u_0 \not\equiv 0$, $u_0 \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$.

We recall that u is a self-similar solution if and only if u has the form

$$u(x,t) = t^{-1/(p-1)} \phi(x/\sqrt{t}),$$

where ϕ satisfies the elliptic equation

$$\Delta\phi + \frac{1}{2} x \cdot \nabla\phi + \frac{1}{p-1}\phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N.$$

Similarity variables

For a solution u of (1.1), define

$$w(y,s) = (t+1)^{1/(p-1)} u(x,t) \quad \text{with } y = x/\sqrt{t+1},\ s = \log(t+1).$$

Then (1.1) is reduced to the problem

$$\text{(1.2)} \qquad \begin{cases} w_s = \Delta w + \dfrac{1}{2} y \cdot \nabla w + \dfrac{1}{p-1} w + w^p, & s \geq 0,\ y \in \mathbf{R}^N, \\ w(y,0) = u_0(y), & y \in \mathbf{R}^N. \end{cases}$$

A stationary problem for (1.2) is as follows.

$$\Delta \phi + \frac{1}{2} x \cdot \nabla \phi + \frac{1}{p-1} \phi + \phi^p = 0 \quad \text{in } \mathbf{R}^N.$$

Initial value problem

For $\alpha > 0$, we denote by ϕ_α a solution to

$$\begin{cases} \phi'' + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right)\phi' + \dfrac{1}{p-1}\phi + \phi^p = 0, \quad r > 0 \\ \phi(0) = \alpha \quad \text{and} \quad \phi'(0) = 0. \end{cases}$$

This ODE problem was studied extensively, by

Haraux-Weissler (1982), Weissler (1985),

Peletier-Terman-Weissler (1986), Yanagida (1996)

Dohmen-Hirose (1998), Galaktionov-Vazquez (1997)

Souplet-Weissler (2003), Bae (2004), N (2006)

We recall here the results on the asymptotic behavior of $\phi_\alpha(r)$ as $r \to \infty$.

Asymptotic behavior

Haraux-Weissler (1982)

Let $p > p_F := (N+2)/N$. For $\alpha > 0$, there exists $\ell = \ell(\alpha) \in \mathbf{R}$ such that

$$\lim_{r\to\infty} r^{2/(p-1)}\phi_\alpha(r) = \ell(\alpha),$$

and that $\ell(\alpha)$ is continuous in $\alpha > 0$.

N (2006) Let $p_F < p < p_c$.
Then there exists $\alpha^* > 0$ such that, if $0 < \alpha < \beta < \alpha^*$, then

$$0 < \phi_\alpha(r) < \phi_\beta(r) \quad \text{for } r \geq 0 \quad \text{and} \quad 0 < \ell(\alpha) < \ell(\beta).$$

Note here that

$$\phi_\alpha(r) = \ell(\alpha) r^{-2/(p-1)} + o(r^{-2/(p-1)}) \quad \text{as } r \to \infty.$$

We can expect similar results as Part I and II for self-similar solutions.

First let us consider the problem (1.2).

$$(1.2)\qquad \begin{cases} w_s = \Delta w + \dfrac{1}{2} y \cdot \nabla w + \dfrac{1}{p-1} w + w^p, & s \geq 0,\ y \in \mathbf{R}^N, \\ w(y,0) = u_0(y), & y \in \mathbf{R}^N. \end{cases}$$

We assume that

$$(*)\qquad 0 \leq u_0(y) \leq \phi_{\alpha^*}(|y|) \quad \text{for } y \in \mathbf{R}^N.$$

Theorem 3.1. Let $p_F < p < p_c$. Assume $(*)$. Let $\alpha \in (0, \alpha^*)$. For any $\varepsilon > 0$, there is $\delta > 0$ such that, if

$$\limsup_{|y| \to \infty} |y|^{2/(p-1)} |u_0(y) - \phi_\alpha(|y|)| < \delta,$$

then

$$\limsup_{t \to \infty} \|w(\cdot, s) - \phi_\alpha(|\cdot|)\|_{2/(p-1)} < \varepsilon.$$

Results: Remark and Corollary

Remark. Since $|y|^{2/(p-1)}\phi_\alpha(|y|) \to \ell(\alpha)$ as $|y| \to \infty$, the condition

$$\limsup_{|y|\to\infty} |y|^{2/(p-1)}|u_0(y) - \phi_\alpha(|y|)| < \delta,$$

in Theorem 3.1, can be written as

$$\limsup_{|y|\to\infty} \Big| |y|^{2/(p-1)}u_0(y) - \ell(\alpha) \Big| < \delta.$$

We note that, for any $\ell \in (0, \ell(\alpha^*))$, there exists a unique $\alpha \in (0, \alpha^*)$ satisfying $\ell = \ell(\alpha)$.

Corollary 3.1. Let $p_F < p < p_c$. Assume $(*)$. If

$$\lim_{|y|\to\infty} |y|^{2/(p-1)}u_0(y) = \ell$$

for some $\ell \in (0, \ell(\alpha^*))$, then

$$\lim_{t\to\infty} \|w(\cdot, s) - \phi_\alpha(|\cdot|)\|_{2/(p-1)} = 0,$$

where α satisfies $\ell = \ell(\alpha)$.

Results

Finally, we apply our result to the problem (1.1):

$$(1.1) \qquad \begin{cases} u_t = \Delta u + u^p & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

For $\alpha > 0$, define

$$u_\alpha(x,t) = t^{-1/(p-2)} \phi_\alpha(|x|/\sqrt{t}).$$

Corollary 3.2. Let $p_F < p < p_c$.
Assume that there exists $t_0 > 0$ such that

$$0 \le u_0(x) \le u_{\alpha^*}(x,t_0) \quad \text{for } x \in \mathbf{R}^N.$$

If $\lim_{|x| \to \infty} |x|^{2/(p-1)} u_0(x) = \ell$ for some $\ell \in (0, \ell(\alpha^*))$, then

$$\lim_{t \to \infty} t^{1/(p-1)} \| u(\cdot,t) - u_\alpha(|\cdot|,t) \|_{L^\infty} = 0,$$

where α satisfies $\ell = \ell(\alpha)$.

Thank you very much for your kind attention !