Exponential separation between positive and sign-changing solutions and its applications

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To be discussed:

(LE) $$v_t = L(x,t)v, \quad x \in \Omega, t \in J, \quad \text{(e.g.} \quad v_t = \Delta v + a(x,t)v)$$
$$v = 0, \quad x \in \partial\Omega, t \in J, \quad (\text{if } \partial\Omega \neq \emptyset \ )$$

Principal Floquet bundles and exponential separation:

relative exponential decay of sign-changing solutions compared to positive solutions

naturally extend the concepts of principal eigenvalues and eigenfunctions of $u \mapsto L(x)u$

A useful tool in studies o nonlinear *nonautonomous* equations

$$u_t = \Delta u + f(t, u, \nabla u), \quad x \in \Omega, t > 0,$$

(LE) is obtained by linearizing along a solution or by taking the difference of two solutions

-- will show some applications in equations with blowup on R^N

Exponential separation: bounded domains

$$v_t = \Delta v + a(x,t)v \quad x \in \Omega, t \in J,$$
$$v = 0, \quad x \in \partial\Omega, t \in J,$$

$\Omega \subset \mathbb{R}^N$, bounded, Lipschitz,
$J = (s, \infty)$ or $J = (-\infty, s)$ or $J = \mathbb{R}$,
$a \in L^\infty(\Omega \times \mathbb{R})$ (extend a by zero, if defined on a halfline only)

$$v(\cdot, t, s, v_0) := \text{ the solution with } v(\cdot, s) = v_0$$

$$v_0 \in X,\ X := L^2(\Omega) \ \ (\text{or } L^p(\Omega),\ p \in [1, \infty] \text{ or } X = C_0(\Omega))$$

$$v_t = \Delta v + a(x,t)v \quad x \in \Omega, t \in J,$$
$$v = 0, \quad x \in \partial\Omega, t \in J,$$

Theorem.

- $\exists$ positive solution $\varphi(x,t)$, $t \in \mathbb{R}$; unique with $\|\varphi(\cdot,0)\|_{L^2} = 1$.
- For any $s \in \mathbb{R}$, the set

$$X_2(s) := \{v_0 \in L^2(\Omega) : \text{the solution } v(\cdot,t,s,v_0) \text{ changes sign for all } t > s\}$$

  is a subspace of $L^2(\Omega)$ of codimension 1. So

$$L^2(\Omega) = X_1(s) \oplus X_2(s), \;\; X_1(s) := \text{span}\{\varphi(\cdot,s)\}.$$

  Clearly, $v_0 \in X_i(s) \implies v(\cdot,t,s,v_0) \in X_i(t)$, $i = 1,2$.

- (Exp. Separation) $\exists \gamma, C > 0$, determined by Ω, $\|a\|_{L^\infty}$,

$$\forall v_0 \in X_2(s) : \frac{\|v(\cdot,t,s,v_0)\|_{L^2}}{\|\varphi(\cdot,t)\|_{L^2}} \le Ce^{-\gamma(t-s)} \frac{\|v_0\|_{L^2}}{\|\varphi(\cdot,s)\|_{L^2}}$$

Remarks

- if $v_0 \in X_2(s)$, $u > 0$ then $\dfrac{\|v(\cdot,t,s,v_0)\|_{L^2}}{\|u(\cdot,t)\|_{L^2}} \le ce^{-\gamma(t-s)}$

$$u(\cdot,t) = \overbrace{\beta\varphi(\cdot,t) + w(\cdot,t)}, \quad \text{with } w(\cdot,t) \in X_2(t),\ \beta \neq 0$$

- $X_1(t) := \text{span}\,\{\varphi(\cdot,t)\}$, $t \in \mathbb{R}$, - principal Floquet bundle,
  $X_2(t)$, $t \in \mathbb{R}$, - its complementary Floquet bundle

- connection to principal eigenvalues, eigenfunctions:

$$a(x,t) \equiv a(x) \Rightarrow$$

$$X_i(t) \equiv X_i \quad (i = 1, 2)$$

$$\varphi(x,t) = \varphi_1(x)e^{\lambda_1 t},$$

λ_1, φ_1 - principal eigenvalue, eigenfuntion of $\Delta + a(x)$

exponetial separation $\Leftrightarrow \exists \gamma > 0 : \|e^{(\Delta+a)t}|_{X_2}\| \le Ce^{(\lambda_1-\gamma)t}$

- First results: [Mierczynski], [P. – Terescak '93], …
  (… = several results based on limiting arguments (with t→ ±∞) involving regularity of the coefficients)

General equations on Lip. domains [Huska – P. –Safonov '07]:

$$u_t = L(x,t)u, \quad x \in \Omega, t \in J,$$
$$u = 0, \quad x \in \partial\Omega, t \in J,$$

$$\textbf{(ND)} \quad L(x,t)u = a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u$$

or

$$\textbf{(D)} \quad L(x,t)u = \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i} + a_j(x,t)u\right) + b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u$$

Ω - bdd., Lipschitz, coefficients - measurable, bounded on $\Omega \times \mathbb{R}$ (for (ND), the a_{ij} are continuous), a_{ij} - uniformly elliptic

constants in the exponential separation determined only by Ω, the bound on the coefficients, and the ellipticity constant (gives interesting results even for $L(x,t) \equiv L(x)$)

Also have continuity and robustness results: the Floquet bundles X_i, i=1,2 depend continuously on the coefficients and Ω

Exp. separation with a constant γ $\Longrightarrow$ exp. separation with a constant $\tilde{\gamma} < \gamma$, $\tilde{\gamma} \approx \gamma$ after a small perturbation of the coefficients and Ω

The method relies on a new elliptic-type Harnack inequality for quotients of positive solutions:

if u, v are positive solutions on $\Omega \times (s, \infty)$

$$\Longrightarrow \quad \sup_{x\in\Omega} \frac{u(x,t)}{v(x,t)} \leq K \inf_{x\in\Omega} \frac{u(x,t)}{v(x,t)} \qquad (t > s + \delta)$$

K determined only by Ω, the bound on the coefficients, the ellipticity constant, and δ (δ>0 is arbitrary)

[Huska ‘06-’09]:

- a version of the exponential separation theorem for general bounded (not necessarily Lipschitz) domains extending [Berestycki, Nirenberg, Varadhan], [Birindelli] on principal eigenvalues, eigenfunctions of elliptic operators

- also oblique derivative problem on Lipschitz domains

[Mierczynski – Shen ’08] - monograph, includes random parabolic equations

Exponential separation: R^N

$$v_t = \Delta v + a(x,t)v \quad x \in \mathbb{R}^N,\ t \in J,$$

$J = (s,\infty)$ or $J = \mathbb{R}$,

$a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, measurable (extend suitably, if needed)

$|a(x,t)| \le d \ ((x,t) \in \mathbb{R}^N \times \mathbb{R}), \quad a(x,t) \le -\alpha \ (|x| \ge \rho)$

Example: $L = \Delta + a(x), \quad a(x) \le -\alpha \quad (|x| \ge \rho)$

- no exponential separation if

  $a(x) \to -\alpha$ as $|x| \to \infty$ and $\sigma(L) \subset (-\infty, -\alpha] \quad (= \sigma_{ess}(L))$

- $\exists$ exponential separation if $\sigma(L) \cap (-\alpha,\infty) \neq \emptyset$

Instability Condition (IC): for some $\epsilon>0$,

$\forall s \in \mathbb{R}\ \exists$ solution $\psi(\cdot,t) \in L^\infty(\mathbb{R}^N)\ (t > s)$ such that

$$\frac{\|\psi(\cdot,t)\|_{L^\infty}}{\|\psi(\cdot,s)\|_{L^\infty}} \ge c\, e^{(-\alpha+\epsilon)(t-s)} \ (t > s)$$

$v_t = \Delta v + a(x,t)v \qquad |a(x,t)| \le d, \quad a(x,t) \le -\alpha \ (|x| \ge \rho),$
$v(\cdot,t,s,v_0) :=$ the solution with $v(\cdot,s) = v_0 \in X := L^\infty(\mathbb{R}^N)$

Theorem [Huska – P. 08] Assume (IC).

- $\exists$ positive solution $\varphi(\cdot,t)$, $t \in \mathbb{R}$; such that

$$\exists c_1 : \frac{\varphi(x,t)}{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le c_1^{-\sqrt{\epsilon}|x|} \qquad (x \in \mathbb{R}^N, \ t \in \mathbb{R}).$$

- For any $s \in \mathbb{R}$, the set

$$X_2(s) := \{v_0 \in L^\infty(\mathbb{R}^N) : \exists \text{ a ball } B \subset \mathbb{R}^N \text{ and } t_k \to \infty, \ v(\cdot,t_k,s,v_0) \text{ changes sign in } B\}$$

is a subspace of $L^\infty(\mathbb{R}^N)$ of codimension 1. So

$$L^\infty(\mathbb{R}^N) = \text{span}\,\{\varphi(\cdot,s)\} \oplus X_2(s).$$

- (exp. separation) $\exists \gamma, C > 0$, determined by d, ρ, α, ϵ

$$\forall v_0 \in X_2(s) : \frac{\|v(\cdot,t,s,v_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le Ce^{-\gamma(t-s)} \frac{\|v_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}}$$

Remarks

The results extend to more general equations, e.g.

$$v_t = \Delta v + b(x,t) \cdot \nabla v + a(x,t)v \quad x \in \mathbb{R}^N, \; t \in J,$$

with $b \in \mathbb{C}^1, \quad \nabla_x \cdot b(x,t) \to 0$ as $x \to \infty$

- Continuity and robustness results under perturbations of the coefficients and also under “perturbation” of the domain: replacing $\mathbb{R}^N$ by B(0,R) with R large (and taking the Dirichlet boundary condition)

Remarks (cont'd)

Useful consequences of

$$\frac{\|v(\cdot,t,s,v_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le Ce^{-\gamma(t-s)}\frac{\|v_0\|_{L^\infty(\mathbb{R}^N)}}{\|\varphi(\cdot,s)\|_{L^\infty(\mathbb{R}^N)}}$$

- (Exp. growth of positive sol's) If there is $v_0 \in X_2(0)$ with

$$\inf_{t>0} \|v(\cdot,t,0,v_0)\|_{L^\infty} > 0$$

then

$$\|\varphi(\cdot,t)\|_{L^\infty} \ge c_2\, e^{\gamma(t-s)}\|\varphi(\cdot,s)\|_{L^\infty} \qquad (t > s > 0)$$

- (Positivity by growth rate) If v is a solution on (-∞,t) with

$$\limsup_{s\to-\infty} \frac{\log\|v(\cdot,s)\|_{L^\infty}}{|s|} > \limsup_{s\to-\infty} \frac{\log\|\varphi(\cdot,s)\|_{L^\infty}}{|s|} - \gamma$$

then v is of one sign (>0 everywhere, or <0 everywhere, or $\equiv 0$).
Similarly with lim inf.

a principal Lyapunov exponent

Exponential separation with backward self-similar variables

$$v_s = \Delta v - \frac{1}{2} y \cdot \nabla v + a(y,s) v \quad y \in \mathbb{R}^N, \; s \in \mathbb{R}$$

a – bounded

consider the solutions $v(\cdot, s, \sigma, v_0)$

with $v_0 \in X := L^2(\mathbb{R}^N, \rho), \; \rho(y) = e^{-|y|^2/4}$

Rm: Δ –(y/2)· ∇ has compact resolvent, no issues with $\sigma_{\mathbf{ess}}$

Extra assumption: $a(y,s) \to b(y) \; (s \to -\infty)$, locally uniformly.

Have the exponential separation on (-∞,0) with

$$\gamma = \lambda_1(\Delta - (y/2) \cdot \nabla + b(y)) - \lambda_2(\Delta - (y/2) \cdot \nabla + b(y)) > 0.$$

Moreover, the principal Lyapunov exponent at s=-∞ is equal to

$$\lambda_1(\Delta - (y/2) \cdot \nabla + b(y)).$$

- based on semigroup estimates, using [Escobedo-Kavian]

-a similar result on a halfspace under Dirichlet bounday condition (considering the equation on X_{odd}, positivity for $y_1>0$ only)

An application: instability of localized solutions of

$$u_t = \Delta u + f(t, u, \nabla u), \quad x \in \mathbb{R}^N, t > 0,$$

$$u(\cdot, 0) = u_0 \in C_0(\mathbb{R}^N),\ u_0 \geq 0.$$

where

$$f \in C^2,\ f(t, 0, 0) \equiv 0, \quad f_u(t, 0, 0) \leq -2\alpha < 0,$$

Assume u_0 yields a global *localized solution*:

$$0 < c_1 < \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} < c_2 < \infty$$

$$u(x, t) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } t$$

Linearization around u

$$v_t = \Delta v + b(x, t) \cdot \nabla v + a(x, t)v$$

$$b(x, t) = f_p(t, u(x, t), \nabla u(x, t)),\ a(x, t) = f_u(t, u(x, t), \nabla u(x, t))$$

The exp. separation theorem applies because

$$a(x, t) < -\alpha < 0 \quad (|x| \gg 1)$$

$$\nabla \cdot b(x, t) = f_{pu} \cdot \nabla u(x, t) + \ldots \to 0\ (|x| \to \infty)$$

and u_{x_1} is a sign-changing solution, which is bounded and stays away from zero (gives (IC))

Get a linear instability of $\boldsymbol{u}$: There is a positive solution of

$$v_t = \Delta v + b(x,t)\cdot\nabla v + a(x,t)v \quad x\in\mathbb{R}^N, t>0,$$

which grows exponentially

$$\|v(\cdot,t)\|_{L^\infty} \ge c_3\, e^{\tilde\gamma(t-s)}\|v(\cdot,s)\|_{L^\infty} \quad (t>s>0)$$

Also true for the Dirichlet problem on each sufficiently large ball

- good for comparison arguments in the nonlinear equation, leads to a nonlinear instability of the localized solution

- a stronger result by exponential separation and continuation arguments (moving hyperplanes, sliding method):

$$\text{if } \tilde u_0 \ge, \not\equiv u_0 \;\Rightarrow\; \liminf_{t\to\infty} u(x,t;\tilde u_0) > \liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty} =: q,$$

uniformly on compacts, or $u(x,t;\tilde u_0)$ blows up

-- $u(x,t;\tilde u_0)$ is not localized: uniqueness of localized solutions in any ordered family of sol's

-- the behavior of $u(x,t;\tilde u_0)$ depends on *f*, for a class of superlinear equations, it has to blow up

Example

$$u_t = \Delta u + r(t)(u^p - \lambda u)$$

$$r \text{ -cont., } 0 < r_1 < r(t) < r_2;\ \lambda > 0,\ p > 1$$

- each localized solution is on the (sharp) threshold between decay and blowup (decay to 0 below, blowup above), regardless of *p*

(No need to know the behavior of the threshold solution for this.)

- under some restrictions on *p* (in particular, $p<p_S$) and the initial conditions, localized solutions do occur as thresholds between blowup and decay [P. 2011]

An application: a proof of a Liouville theorem for radial ancient sol's

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^N,\ t \in (-\infty, T)$$
$$1 < p < p_S = (N+2)/(N-2)_+$$

Theorem. Assume u is a radial solution with finite (and bounded) zero number: $u = u(r,t),\ r = |x|,\ z(v(\cdot,t)) \le k,\ (t < T)$. Then it is spatially homogeneous: $u = u(t)$

(a) Under the assumption that $\|u(\cdot,t)\|_{L^\infty} \le c(T-t)^{-1/(p-1)}$, this follows from [Merle-Zaag '98, 2000] (no symmetry needed)

(b) A Liouville theorem for entire radial solutions with finite zero number [Bartsch, P., Quittner '11] and the doubling lemma [P., Quittner, Souplet '07] reduce the general case to (a)

(c) Exponential separation theorem can be used to give a simple proof of (a) (radial case only)

Sketch of the proof for positive radial solutions u

$$w(y,s) := (T-t)^{1/(p-1)} u(x,t), \quad y = x/\sqrt{T-t}, s = -\log(T-t)$$

- bounded solution of

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + w^p, \qquad y \in \mathbb{R}^N, s \in \mathbb{R}$$

and by [Giga-Kohn], unless $w \equiv 0$,

$$w(\cdot, s) \to \kappa := (1/(p-1))^{1/(p-1)} \quad (s \to -\infty),$$

uniformly on compacts. Want to prove that $w_{y_1} \equiv 0$.

The function $e^{s/2} w_{y_1}$ is a solution of

$$v_s = \Delta v - (y/2) \cdot \nabla v + a(y,s) v \quad y \in \mathbb{R}^N, \; s \in \mathbb{R}$$

$$a(y,s) = -1/(p-1) + p(w(y,s))^{p-1} \to -1/(p-1) + p\kappa^{p-1} = 1$$

(uniformly on compacts)

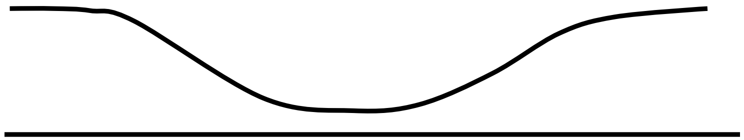
$$\lambda_1(\Delta - (y/2) \cdot \nabla + 1) = 1 \text{ and } \lambda_1(\Delta - (y/2) \cdot \nabla + 1 \Big|_{X_{odd}}) = 1/2$$

Since $e^{s/2} w_{y_1}$ is odd in y_1, $\approx e^{s/2}$ ($s \to -\infty$), it is of one sign.

So u_{x_1}, u_r are of one sign.

$u_r(r,t) < 0$ $(u(0,t) = \|u(\cdot,t)\|_\infty)$ is easy to rule out [Giga-Kohn]

Assume $u_r(r,t) > 0$



Take $\eta(t) := \lim_{r\to\infty} u(r,t) = \lim_{x_1\to\infty} u(x_1, x', t)$

It solves $\eta_t = \eta^p$ so $\eta(t) = \kappa(\tau - t)^{-1/(p-1)}$ for some $\tau \geq T$.

Replacing T with τ, returning to self-similar variables,

$$w_s = \Delta w - (y/2)\cdot\nabla w - \frac{1}{p-1}w + w^p,$$

we get $w \leq \kappa$ (since $u \leq \eta$)

Now $v = \kappa - w$ is a positive solution of

$$v_s = \Delta v - (y/2)\cdot\nabla v + a(y,s)v \quad y \in \mathbb{R}^N,\ s \in \mathbb{R}$$

$$a(y,s) = -1/(p-1) + p|\zeta(y,s)|^{p-1} \to 1$$

$\lambda_1(\Delta - (y/2)\cdot\nabla + 1) = 1$ gives $\|\kappa - w\|_X \approx e^s$ and

then also $\|w_{y_1}\|_X \approx e^s$ contradiction: too fast decay for $e^{s/2}w_{y_1}$