Oscillating solutions to a simplified chemotaxis system in high dimensional spaces

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Our system (PE)

(PE)
$$\begin{cases} U_t = \nabla \cdot (\nabla U - U\nabla V) \quad \text{in } \mathbf{R}^n \times (0,\infty), \\ 0 = \Delta V + U \quad \text{in } \mathbf{R}^n \times (0,\infty), \qquad V(0,\cdot) = 0 \qquad \text{in } (0,\infty), \\ U(\cdot,0) = U^{\mathcal{I}} \geq 0 \quad \text{in } \mathbf{R}^n. \end{cases}$$

- $n = 1, 2, 3, \cdots$.
- (PE) is a simplified version of Keller-Segel system, if $n = 2$.
- Keller-Segel system is introduced to describe the aggregation of cellular slime molds.
- $U(x,t)$ represents the density of the cells.
- We consider the behavior of the function U.

Plan of our talk

1. Fundamental properties of radial solutions.
2. Known results and behavior of solutions.
3. Our results (Stability of stationary solutions)
4. Application of our results (existence of oscillating solutions)
5. Idea of poof of our reslts.

Time local existence and uniqueness

If $U^{\mathcal{I}}$ is radial, positive and

$$U^{\mathcal{I}}(x) = \left\{ \begin{array}{l} O(1)/|x|^2 \ (n \geq 3) \\ O(1)/|x|^4 \ (n = 2) \end{array} \right\} \text{ as } |x| \to \infty,$$

there exists a unique solution (U, V) as follows.

$$\begin{aligned} U(x,t) &= \int_{\mathbf{R}^n} \mathcal{G}(x-\tilde{x}, t) U^{\mathcal{I}}(\tilde{x}) d\tilde{x} \\ &\quad - \int_0^t \int_{\mathbf{R}^n} \left\{ \nabla_{\tilde{x}} \mathcal{G}(x-\tilde{x}, t-\tilde{t}) \cdot \frac{\tilde{x}}{\omega_n |\tilde{x}|^n} \int_{|\hat{x}|<|\tilde{x}|} U(\hat{x}, \tilde{t}) d\hat{x} \right\} U(\tilde{x}, \tilde{t}) d\tilde{x} d\tilde{t} \end{aligned}$$

in $\mathbf{R}^n \times [0, T)$ with a constant $T \in (0, \infty]$.
$\mathcal{G}$ is the Gauss kernel of $\partial_t - \Delta$ in $\mathbf{R}^n$ and $\omega_n = |S^{n-1}|$.

$$V(x,t) = -\int_0^{|x|} \frac{1}{\omega_n r^{n-1}} \int_{|\tilde{x}|<r} U(\tilde{x}, t) d\tilde{x} dr \qquad \text{in } \mathbf{R}^n \times [0, T).$$

Fundamental properties of solutions

• U is non-negative in $\mathbf{R}^n \times (0, T)$, since $U^{\mathcal{I}}$ is non-negative.
• (PE) has radial stationary solutions (U_α, V_α) for any $\alpha > 0$ satisfying $U_\alpha(0) = \alpha$,

$$\begin{cases} U_\alpha(x) = \begin{cases} O(1)/|x|^2 & \text{as } |x| \to \infty \ (n \geq 3), \\ \alpha/(1 + (\alpha/8)|x|^2)^2 & (n = 2), \end{cases} \\ V_\alpha(x) = \log(U_\alpha(x)/\alpha). \end{cases}$$

† If $n = 2$, $\displaystyle\int_{\mathbf{R}^2} U_\alpha(x)dx = 8\pi$ for $\alpha > 0$.
† U_α is continuous with respect to α.
† Stationary solutions (U_α, V_α) satisfies $\begin{cases} 0 = \Delta V_\alpha + \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n, \\ V_\alpha(0) = 0, \quad U_\alpha = \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n \end{cases}$
† Singular stationary solution $U_\infty(x) = \begin{cases} \dfrac{2(n-2)}{|x|^2} & \text{if } n \geq 3, \\ 8\pi\delta_0 & \text{if } n = 2. \end{cases}$

Known results $\sim$ radial case $\sim$

- There exist solutions blowing up at finite time T.

$$\limsup_{t \to T} \|U(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} = \infty. \quad \text{[Nagai '95 etc]}$$

- Solutions exist globally in time, if

$$\begin{cases} 0 \le U^{\mathcal{I}} \le U_\infty, \quad U^{\mathcal{I}} \not\equiv U_\infty \quad (n \ge 3), \\ U^{\mathcal{I}} \ge 0, \quad \displaystyle\int_{\mathbf{R}^2} U^{\mathcal{I}}(x)dx \le 8\pi \quad (n = 2). \text{ [Biler etc '06]} \end{cases}$$

- There exist solutions blowing up at infinite time.

$$\limsup_{t \to \infty} \|U(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

($n = 2$ [Blanchet etc '08, Kavallavis-Souplet '09 (bounded domain)], $n \ge 11$ [S., '09])

Oscillating solutions $\sim \lambda = 8\pi$ and $n = 2 \sim$

Put $\quad \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)) = \Big\{ F \in C(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \ : \ \lim_{n\to\infty} t_n = \infty,$

$$\lim_{n\to\infty} \|U(\cdot, t_n) - F\|_{L^\infty(\mathbf{R}^2)} = 0 \quad \text{for some } \{t_n\} \subset (0,\infty) \Big\}.$$

Theorem 1.[Naito-S. preprint]

(1) For a and d with $0 < a < d$ there exists a radial solution (U, V) with $U(\cdot, 0) = U^{\mathcal{I}}$ satisfying

$$\{U_b\}_{b\in[a,d]} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \quad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi.$$

(2) For $\{b_j\}_{j=1}^{\infty} \subset (0,\infty)$ with $\lim_{j\to\infty} b_j = \infty$ there exists a radial solution (U, V) with $U(\cdot, 0) = U^{\mathcal{I}}$ satisfying

$$\{U_{b_j}\}_{j=1}^{\infty} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \quad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi$$

Remark

(1) For each $b \in [a, d]$ there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0, \infty)$ satisfying

$$\lim_{k \to \infty} \|U(\cdot, t_k) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k \to \infty} t_k = \infty.$$

(2) For each $j = 1, 2, 3, \cdots$ there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0, \infty)$ satisfying

$$\lim_{k \to \infty} \|U(\cdot, t_k) - U_{b_j}\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k \to \infty} t_k = \infty.$$

There exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0, \infty)$ satisfying

$$\lim_{k \to \infty} \|U(\cdot, t_k)\|_{L^\infty(\mathbf{R}^2)} = \infty, \quad \lim_{k \to \infty} t_k = \infty.$$

※ $\lim_{b \to \infty} U_b = 8\pi\delta_0$.

Tools for construction of oscillating solutions $\sim n = 2 \sim$

- Stability of stationary solutions for radial perturbation.

Proposition 1 [*Biler-Karch-Laurençot-Nadzieja*]
Let $U^{\mathcal{I}}$ *be nonnegative and radial,* $\|U^{\mathcal{I}}\|_{L^1(\mathbf{R}^2)} = 8\pi$ *and*

$$\sup_{x \in \mathbf{R}^2} (1 + |x|)^5 |U^{\mathcal{I}}(x) - U_b(x)| < \infty$$

with some $b > 0$. *Then,* $\lim_{t \to \infty} \|U(\cdot, t) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0$.

- Layer of stationary solutions.

$$\begin{cases} \lim_{a \to b} \|U_a - U_b\|_{L^\infty(\mathbf{R}^2)} = 0 \quad (b > 0) \\ \int_{|x|<r} U_a(x)dx \le \int_{|x|<r} U_b(x)dx \quad (r > 0), \qquad \text{if } a \le b. \end{cases}$$

- Arguments in [Poláčik and Yanagida '03]

Our result ∼ high dimensional case ∼

Theorem 1 *Let* $n \geq 11$. $\beta_\pm = \{n+2 \pm \sqrt{(n-2)(n-10)}\}/2 \in (2,n)$. *Suppose* $0 \leq U^{\mathcal{I}} \leq U_\infty$ *in* $\mathbf{R}^n$ *and*

$$\lim_{|x|\to\infty} (1+|x|)^{\beta_-}|U^{\mathcal{I}}(x) - U_\alpha(x)| = 0$$

with some $\alpha > 0$. *Then, the solution* (U,V) *to (PE) satisfies*

$$\lim_{t\to\infty} \|U(\cdot,t) - U_\alpha\|_{\beta_-,\mathbf{R}^n} = 0,$$

where $\|F\|_{\beta,\mathbf{R}^n} = \sup_{x\in\mathbf{R}^n}(1+|x|)^\beta |F(x)|$.

- Layer of stationary solutions. ($n \geq 11$)

$$\begin{cases} \lim_{a\to b} \|U_a - U_b\|_{\beta_-,\mathbf{R}^n} = 0 \quad (b>0) \\ U_b(x) = \dfrac{2(n-2)}{|x|^2} - \dfrac{A(b)}{|x|^{\beta_-}} \quad \text{as} |x| \to \infty. \\ A(b) > 0 \ \text{ is continous with respect to } b > 0. \\ U_a(x) \leq U_b(x) \ \text{ in } \mathbf{R}^n \quad \text{if } a \leq b. \end{cases}$$

Remark

Let W be a solution to

$$W_t = \Delta W + e^W \quad \text{in } \mathbf{R}^n \times (0,\infty) \qquad W(\cdot,0) = W^{\mathcal{I}} \quad \text{in } \mathbf{R}^n$$

and let W_α be a radial stationary solution satisfying $W_\alpha(0) = \alpha$.

Theorem 2 (Tell '06) *Let $n > 10$. For some $0 < \gamma < \gamma'$ suppose $W_\gamma \le W^{\mathcal{I}} \le W_{\gamma'}$ in $\mathbf{R}^n$ and*

$$\lim_{|x|\to\infty} (1+|x|)^{\beta_- - 2} |W^{\mathcal{I}}(x) - W_\alpha(x)| = 0$$

with some $\alpha > 0$. Then, $\lim_{t\to\infty} \|W(\cdot,t) - W_\alpha\|_{\beta_- - 2, \mathbf{R}^n} = 0$.

• Assumption "$\lim_{|x|\to\infty} (1+|x|)^{\beta} |U^{\mathcal{I}}(x) - U_\alpha(x)| = 0$" is optimal, since $\lim_{\gamma\to\alpha} \|U_\gamma - U_\alpha\|_{\beta_-, \mathbf{R}^n} = 0$ and

$$\lim_{|x|\to\infty} (1+|x|)^{\beta_-} |U_\alpha(x) - U_\gamma(x)| > 0 \qquad \text{if } \gamma \ne \alpha.$$

Functional spaces and ω-limit set

- For a non-negative constant β, put

$$C_\beta(\mathbf{R}^n) = \left\{ F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) : \sup_{x \in \mathbf{R}^n} (1 + |x|)^\beta |F(x)| < \infty \right\}.$$

- Let (U, V) be a solution to (PE) with initial data $U^{\mathcal{I}}$ satisfying $U \in C([0, \infty) : C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. We put

$$\omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) = \Big\{ F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \ : \ \lim_{n \to \infty} t_n = \infty,$$

$$\lim_{n \to \infty} \|U(\cdot, t_n) - F\|_{\beta, \mathbf{R}^n} = 0 \quad \text{for some } \{t_n\} \subset (0, \infty) \Big\}.$$

Application of Theorem 1 $\sim$ oscillating solutions $\sim$

Theorem 3 *Let* $n \geq 11$ *and let* Λ *be a set of* $[0, \infty)$*.*
Then, there exists a radial and continuous function $U^{\mathcal{I}}$ *such that*

$$0 \leq U^{\mathcal{I}} \leq U_\infty \equiv \frac{2(n-2)}{|x|^2} \qquad \textit{in } \mathbf{R}^n.$$

and

$$\{U_a\}_{a \in \Lambda} \subset \omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) \qquad \textit{for any } \beta \in [0, 2).$$

Moreover, suppose $\inf \Lambda > 0$*. Then, we can take* $\beta \in [0, \beta_-)$*.*

Remark. $U_a \to U_\infty$ as $a \to \infty$. Then, if $\sup \Lambda = \infty$,

$$\limsup_{t \to \infty} \|U(\cdot, t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

Proof of stability $\sim$ sub-solutions, super-solutions $\sim$

Lemma 1 (Tello '06, Gui-Ni-Wang '92) *Let* $n \geq 10$. *There exists a sequence of radial functions* $\{\overline{V}_\alpha^{(k)}, \underline{V}_\alpha^{(k)}\}_{k\geq 1}$ *satisfying the following properties.*

- $\Delta \overline{V}_\alpha^{(k)} + \alpha e^{\overline{V}_\alpha^{(k)}} < 0$ *in* $\mathbf{R}^n$ *(Super-solutons).*
- $\Delta \underline{V}_\alpha^{(k)} + \alpha e^{\underline{V}_\alpha^{(k)}} > 0$ *in* $\mathbf{R}^n$*(Sub-solutions).*
- $\overline{V}_\alpha^{(1)} > \overline{V}_\alpha^{(2)} > \cdots > V_\alpha > \cdots > \underline{V}_\alpha^{(2)} > \underline{V}_\alpha^{(1)}$.
- $\liminf_{|x|\to\infty}(1+|x|)^{\beta_+-2}|V(x) - V_\alpha(x)| > 0$, *where* $V = \overline{V}_\alpha^{(k)}$ *or* $\underline{V}_\alpha^{(k)}$.
- $\lim_{|x|\to\infty}(1+|x|)^{\beta_--2}|V(x) - V_\alpha(x)| = 0$, *where* $V = \overline{V}_\alpha^{(k)}$ *or* $\underline{V}_\alpha^{(k)}$.
- V_α *is a unique solutions to* $\Delta V + \alpha e^V = 0$ *in* $\mathbf{R}^n$ *with* $V(0) = 0$ *such that* $\overline{V}_\alpha^{(1)} > V_\alpha > \underline{V}_\alpha^{(1)}$.

Transformation and sub and super solutions

Let (U,V) be a solutions to (PE) and let $\omega_n = |S^{n-1}|$.

- Put $u(r,t) = \dfrac{1}{\omega_n r^n}\displaystyle\int_{|x|<r} U(x,t)dx$. u satisfies

$$\begin{cases} \mathcal{L}(u) = u_t - u_{rr} - \frac{n+1}{r}u_r - u\{ru_r + nu\} = 0 \quad (0 < r < \infty,\ t > 0), \\ u_r(0,t) = 0 \quad (t > 0), \\ u(x,0) = u^{\mathcal{I}} \quad (0 \le r < \infty). \end{cases}$$

- $u_\alpha(r) = \dfrac{1}{\omega_n r^n}\displaystyle\int_{|x|<r} U_\alpha(x)dx$ is a stationary solution to $\mathcal{L}(u) = 0$.
- $\overline{u}_\alpha^{(k)}(r) = \dfrac{1}{\omega_n r^n}\displaystyle\int_{|x|<r} \alpha e^{\overline{V}_\alpha^{(k)}(x)}dx$ is a super-solution to $\mathcal{L}(u) = 0$.
- $\underline{u}_\alpha^{(k)}(r) = \dfrac{1}{\omega_n r^n}\displaystyle\int_{|x|<r} \alpha e^{\underline{V}_\alpha^{(k)}(x)}dx$ is a sub-solution to $\mathcal{L}(u) = 0$.

Linearized equation

For a stationary solution u_α, let m be a solution to

$$\begin{cases} \mathcal{M}_\alpha(m) = m_t - m_{rr} - \frac{n+1}{r} m_r - m\left\{ r u_{\alpha r} + n u_\alpha \right\} - u_\alpha \left\{ r m_r + n m \right\} \\ \hfill (0 < r < \infty,\ t > 0), \\ m_r(0,t) = 0 \quad (t > 0), \\ m(x,0) = m^{\mathcal{I}} \quad (0 \le r < \infty). \end{cases}$$

Estimates of solutions

Lemma 2 *Let* $n \geq 11$*,* $\beta \in [\beta_-, \beta_+]$ *and* $\alpha \geq 0$*. Suppose*

$$m^{\mathcal{I}} \geq 0 \quad \textit{and} \quad rm^{\mathcal{I}}_r + \beta m^{\mathcal{I}} \geq 0 \qquad (0 < r < \infty).$$

Then, the solution m *to* $\mathcal{M}_\alpha(m) = 0$ *satisfies*

$$m \geq 0, \qquad rm_r + \beta m \geq 0 \qquad (0 < r < \infty,\ t > 0).$$

⟨ Idea of proof ⟩ "$m \geq 0$" comes from the comparison theorem. Multiplying $\mathcal{M}_\alpha(m) = 0$ by r^β and differentiating with respect to r, we have

$$[(r^\beta m)_r]_t = [(r^\beta m)_r]_{rr} + \text{“ terms of } [(r^\beta m)_r]\text{''} + \text{“ positive term ''}(r^\beta m)$$

and that "positive term" $> -2(\beta - \beta_-)(\beta - \beta_+)r^{-3}$ for $r > 0$. Then, the second claim comes from the comparison theorem.

Sub-solutions and super-solutions to $\mathcal{L}(u) = 0$

Lemma 3 *Let $n \geq 11$, $\beta \in [\beta_-, \beta_+]$, $\alpha > 0$ and let m be a solution to $\mathcal{M}_\alpha(m) = 0$. Suppose*

$$m^{\mathcal{I}} \geq 0, \quad r m_r^{\mathcal{I}} + \beta m^{\mathcal{I}} \geq 0 \quad (0 < r < \infty).$$

Then, $u_\alpha \pm m$ is a sub-solution to $\mathcal{L}(u) = 0$.

Lemma 4 *Let $n \geq 11$, $\beta \in [\beta_-, \beta_+]$, $\eta > \gamma > \alpha > 0$ and let m be a solution to $\mathcal{M}_\gamma(m) = 0$. Suppose*

$$m^{\mathcal{I}} \geq 0, \quad r m_r^{\mathcal{I}} + \beta m^{\mathcal{I}} \geq 0 \quad (0 < r < \infty)$$

and

$$m^{\mathcal{I}} \leq u_\eta - u_\gamma \leq u_\gamma - u_\alpha \quad (0 < r < \infty).$$

Then, $u_\alpha + m$ is a super-solution to $\mathcal{L}(u) = 0$.

$\langle$ Idea of proof of Lemma 3 $\rangle$

$$\mathcal{L}(u_\alpha \pm m) = \mathcal{L}(u_\alpha) \pm \mathcal{M}_\alpha(m) - m\{rm_r + nm\} \leq 0.$$

- $rm_r + nm \geq 0$ and $m \geq 0$ by Lemma 2.

$\langle$ Idea of proof of Lemma 4 $\rangle$

$$\begin{aligned}\mathcal{L}(u_\alpha + m) &= \mathcal{L}(u_\alpha) + \mathcal{M}_\gamma(m) + (u_\gamma - u_\alpha - m)\{rm_r + nm\} \\ &\quad + \{U_\gamma - U_\alpha\}\, m \geq 0.\end{aligned}$$

- $m \leq u_\eta - u_\gamma \leq u_\gamma - u_\alpha$ by Lemma 3 and assumptions.
- $rm_r + nm \geq 0$ and $m \geq 0$ by Lemma 2.
- $U_\gamma \geq U_\alpha$ in $\mathbf{R}^n$.

Then, $\mathcal{L}(u_\alpha + m) \geq 0$.

Decay of solutions to $\mathcal{M}_\alpha(m) = 0$

Lemma 5 *Suppose* $\lim_{r\to\infty}(1+r)^{\beta-}|m^{\mathcal{I}}(r)| = 0$. *Then, the solution* m *to* $\mathcal{M}_\alpha(m) = 0$ *satisfies*

$$\lim_{t\to\infty} \|m(\cdot,t)\|_{\beta-,[0,\infty)} = 0,$$

where $\|f\|_{\beta,[0,\infty)} = \sup_{0<r<\infty}(1+r)^\beta |f(r)|$.

Remark

- $\mathcal{L}(\overline{u}_\alpha^{(1)}) > 0, \quad \mathcal{L}(\underline{u}_\alpha^{(1)}) < 0.$
- $\lim_{r\to\infty}(1+r)^{\beta-}|u_\alpha(r) - u_\gamma(r)| > 0$ if $\alpha \neq \gamma$.

$\langle$ Idea of proof $\rangle$ Let $m^{\mathcal{I}} = m_+^{\mathcal{I}} - m_-^{\mathcal{I}}$ and $m_+^{\mathcal{I}} = m_+^{\mathcal{I}} \chi_{[0,R)} + m_+^{\mathcal{I}} \chi_{[R,\infty)}$.
For any $\varepsilon > 0$, there exists a $R > 0$ s.t. $u_\alpha + m_+^{\mathcal{I}} \chi_{[R,\infty)} \leq u_{\alpha+\varepsilon}$.
Let m_{+1} be a solutions to $\mathcal{M}_\alpha(m) = 0$ with $m_{+1}(\cdot, 0) = m_+^{\mathcal{I}} \chi_{[R,\infty)}$.
m_{+1} satisfies $u_\alpha + m_{+1} \leq u_{\alpha+\varepsilon}$.
On the other hand,
For some $\delta > 0$, $u_\alpha + \delta m_+^{\mathcal{I}} \chi_{[0,R)} \leq \overline{u}_\alpha^{(1)}$.
Let $\overline{u}$ be a solution to $\mathcal{L}(u) = 0$ with $\overline{u}(\cdot, 0) = \overline{u}_\alpha^{(1)}$ and let m_{+2} be a solutions to $\mathcal{M}_\alpha(m) = 0$ with $m_{+2}(\cdot, 0) = m_+^{\mathcal{I}} \chi_{[0,R)}$.
By $\overline{u}_t < 0$, $\lim_{t\to\infty} \|\overline{u}(\cdot, t) - u_\alpha\|_{\beta_-,[0,\infty)} = 0$ and $u_\alpha + \delta m_{+2} \leq \overline{u}$.
Then, $\limsup_{t\to\infty} \|m_+\|_{\beta_-,[0,\infty)} \leq \|u_{\alpha+\varepsilon} - u_\alpha\|_{\beta_-,[0,\infty)}$.
Here, $m_+ = m_{+1} + m_{+2}$ is a solution to $\mathcal{M}_\alpha(m) = 0$ with $m_+(\cdot, 0) = m_+^{\mathcal{I}}$.
ε is arbitrary.
Similarly, we can show the decay of solution m_- to $\mathcal{M}_\alpha(m) = 0$ with $m_-(\cdot, 0) = m_-^{\mathcal{I}}$.

Proof of stability of u_α

$\langle$ Simple case $\rangle$ $u^{\mathcal{I}} - u_\beta \le u_\eta - u_\gamma \le u_\gamma - u_\beta$ and $u_\alpha \le u^{\mathcal{I}} \le u_\gamma$.
Let $m^{\mathcal{I}} = u^{\mathcal{I}} - u_\beta$.
Let m_β be a solution to $\mathcal{M}_\beta(m) = 0$ with $m_\beta(\cdot, 0) = m^{\mathcal{I}}$.
Let m_γ be a solution to $\mathcal{M}_\gamma(m) = 0$ with $m_\gamma(\cdot, 0) = m^{\mathcal{I}}$.
By Lemmas 3 and 4,

$$u_\beta + m_\beta \le u \le u_\beta + m_\gamma.$$

By Lemma 5,

$$\lim_{t \to \infty} \|u(\cdot, t) - u_\beta\|_{\beta-,[0,\infty)} = 0.$$

Thus, stationary solutions u_β is stable.

$\langle$ Parabolic regularity method $\rangle$ Using the parabolic regularity method for the above solution, we establish the convergence of solutions to (PE).

Parabolic regularity argument. There exists a constant C such that

$$\|U(\cdot,t) - U_\alpha\|_{\beta,\mathbf{R}^n} \le C \max_{t-\frac{1}{2}\le s\le t+\frac{1}{2}} \|u(\cdot,s) - u_\alpha\|_{\beta,[0,\infty)}$$

for $t \ge 1$.