Finite-time blowup for a complex Ginzburg-Landau equation

Fred Weissler

Université Paris 13, Sorbonne Paris Cité

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Joint work with T. Cazenave (UPMC) and F. Dickstein (UFRJ)

The equations

- Ginzburg-Landau equation :

$$\begin{cases} u_t = e^{i\theta}\Delta u + e^{i\gamma}|u|^\alpha u, \\ u(0) = u_0, \end{cases} \tag{GL}$$

in $\mathbb{R}^N$, where $\alpha > 0$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and $\gamma \in \mathbb{R}$ (“$\alpha = p - 1$”).

- Special case $\theta = \gamma = 0$, nonlinear heat equation :

$$u_t = \Delta u + |u|^\alpha u \tag{NLH}$$

- Special case $\theta = \gamma = \pm\pi/2$, nonlinear Schrödinger equation :

$$\pm\, iu_t = \Delta u + |u|^\alpha u \tag{NLS}$$

well-posedness

- If $-\pi/2 < \theta < \pi/2$, the Cauchy problem (GL) is locally well-posed in $C_0(\mathbb{R}^N)$.
- We denote by $T_{max} = T_{max}(u_0, \theta, \gamma)$ the existence time of the maximal solution $u(t)$. If $T_{max} < \infty$, then $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T_{max}$.
- If $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, then $u(t) \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for all $0 \leq t < T_{max}$.
- If $\alpha < 4/N$, then the Cauchy problem (GL) is also well-posed in $L^2(\mathbb{R}^N)$, with the same blowup time. So if $T_{max} < \infty$, then also $\|u(t)\|_{L^2} \to \infty$ as $t \to T_{max}$.
- If $\theta = \pm\pi/2$, and $\alpha < 4/(N-2)$, then (GL) is locally well-posed in $H^1(\mathbb{R}^N)$.

QUESTION : When do there exist solutions with $T_{max} < \infty$?

Previous results on finite-time blowup

- Levine 1973, (NLH), if $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ has negative energy, i.e. $E(u_0) < 0$, where

$$E(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |w|^{\alpha+2},$$

  then $T_{max} < \infty$.

- Zakharov 1972, Glassey 1977 (NLS), $\alpha \geq 4/N$, $E(u_0) < 0$, and u_0 has finite variance, i.e. $\int_{\mathbb{R}^N} |x|^2 |u_0|^2 < \infty$, then $T_{max} < \infty$.

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- Zaag, 1998, "close to nonlinear heat equation", construction of initial value such that resulting solution blows up in finite time (at a single point, with precise profile), when
  - $\alpha < 4/(N-2)$ (subcritical Sobolev),
  - $\theta = 0$,
  - $|\gamma|$ small.

Previous results on finite-time blowup, continued

- Snoussi and Tayachi, 2001, "somewhat close to nonlinear heat equation", negative energy initial value, $E(u_0) < 0$, produces finite-time blowup when :
  - $\theta = \gamma$,
  - $\cos^2\theta > 2/(\alpha + 2)$ (calculations done in the case $\alpha = 2$).

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  - $\theta = \gamma$,
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- Masmoudi and Zaag, 2008, construction of initial value such that resulting solution blows up in finite time (at a single point, with precise profile), when $\tan^2\gamma + (\alpha + 2)\tan\gamma\tan\theta < \alpha + 3$.
  - If θ is close to $\pi/2$ (Schrödinger), then γ must be small.
  - If $\theta = \gamma$, condition becomes $\cos^2\theta > (\alpha + 3)/2(\alpha + 2)$. Improves ST (2001) for $\alpha > 1$.

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  - If $\theta = \gamma$, condition becomes $\cos^2\theta > (\alpha+3)/2(\alpha+2)$. Improves ST (2001) for $\alpha > 1$.
- Rottschäfer, 2008, "close to nonlinear Schrödinger equation", construction of radially symmetric, backwards self-similar solution when :
  - $\alpha = 2$
  - $N = 3$
  - $\theta < \pi/2$ with $\pi/2 - \theta$ small,
  - $\gamma > \pi/2$ with $\gamma - \pi/2$, small.

special case $\theta = \gamma$

In what follows, we consider the special case of the general Ginzburg-Landau equation :

$$\begin{cases} e^{-i\theta} u_t = \Delta u + |u|^\alpha u, \\ u(0) = u_0. \end{cases} \tag{GL$_\theta$}$$

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- Stationary solutions of (GL_θ) are the same, independent of $-\pi/2 \le \theta \le \pi/2$.

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- Solutions of (GL_θ) satisfy energy identities similar to those satisfied by solutions of (NLH) and (NLS).

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- Levine's argument (for a general domain, in particular $\mathbb{R}^N$), however, seems to require the Snoussi-Tayachi condition : $\cos^2\theta > 2/(\alpha + 2)$.

main results (Cazenave, Dickstein, W)

Theorem 1. Let $-\pi/2 < \theta < \pi/2$ and $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with $E(u_0) < 0$. It follows that the resulting solution u of (GL_θ) blows up in finite time. More precisely,

$$T_{max} \leq \frac{\|u_0\|_{L^2}^2}{\alpha(\alpha+2)(-E(u_0))\cos\theta}.$$

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Remark. In the case $\theta = \pi/2$ (NLS), and $\alpha < 4/N$, all solutions are global. Thus, in the case $-\pi/2 < \theta < \pi/2$, we expect the blowup time to get arbitrarily large, for a given initial value, as θ approaches $\pm\pi/2$.

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Theorem 2. In Theorem 1, suppose in addition that $\alpha < 4/N$. Then there exists a constant $c = c(N, \alpha, \|u_0\|_{L^2}, E(u_0)) > 0$ such that

$$T_{max}^{\theta} \ge \frac{c}{\cos\theta}.$$

main results, continued

Remark. In the case $\theta = \pi/2$ (NLS), and $\alpha \geq 4/N$, negative energy solutions (with finite variance) blowup in finite time. The same is true for radially symmetric solutions if $4/N \leq \alpha \leq 4$, without the condition of finite variance (Ogawa and Tsutsumi, 1991). Thus, in the case $-\pi/2 < \theta < \pi/2$, we expect the blowup time to stay bounded, for a given initial value, as θ approaches $\pm\pi/2$.

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Theorem 3. In Theorem 1 ($-\pi/2 < \theta < \pi/2$ and $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with $E(u_0) < 0$) suppose in addition that u_0 is radially symmetric and that $N \geq 2$ and $4/N \leq \alpha \leq 4$ (no assumption of finite variance). There exists $\overline{T}$ such that $T^{\theta}_{max} \leq \overline{T}$ for all $-\pi/2 < \theta < \pi/2$.

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Proof based on truncated variance identity (Ogawa and Tsutsumi) and requires the same condition $4/N \leq \alpha \leq 4$.

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Proof based on truncated variance identity (Ogawa and Tsutsumi) and requires the same condition $4/N \leq \alpha \leq 4$. If we assume finite variance and use the "real" variance identity (viriel), we still require radial symmetry and $4/N \leq \alpha \leq 4$, but for different reasons.

energy identities

Functionals :

$$E(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha+2}\int_{\mathbb{R}^N} |w|^{\alpha+2},$$

$$J(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - \int_{\mathbb{R}^N} |w|^{\alpha+2}.$$

Identities for solutions of (GL_θ), with $-\pi/2 < \theta < \pi/2$,

$$\frac{d}{dt}E(u(t)) = -\cos\theta \|u_t\|_{L^2}^2, \qquad \frac{d}{dt}\|u(t)\|_{L^2}^2 = -2\cos\theta J(u(t))$$

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = -2(\alpha+2)\cos\theta E(u(t)) + \alpha\cos\theta\|\nabla u(t)\|_{L^2}^2$$

$$= 2(\alpha+2)\cos^2\theta \int_0^t \|u_t\|_{L^2}^2 - 2(\alpha+2)\cos\theta E(u_0) + \alpha\cos\theta\|\nabla u(t)\|_{L^2}^2$$

Proof of Theorem 1 (based on Levine 1973)

Let $f(t) = \int_0^t \|u(t)\|_{L^2}^2 = \int_0^t \int_{\mathbb{R}^N} |u|^2$. Since $E(u_0) \le 0$, we have that

$$f''(t) = \frac{d}{dt}\|u(t)\|_{L^2}^2 \ge 2(\alpha+2)\cos^2\theta \int_0^t \int_{\mathbb{R}^N} |u_t|^2$$

$$\begin{aligned}\frac{1}{2(\alpha+2)\cos^2\theta} f(t)f''(t) &\ge \int_0^t \int_{\mathbb{R}^N} |u|^2 \int_0^t \int_{\mathbb{R}^N} |u_t|^2 \\ &\ge \Big(\int_0^t \int_{\mathbb{R}^N} |u||u_t|\Big)^2 \ge \frac{1}{4}\Big(\int_0^t \frac{d}{dt}\int_{\mathbb{R}^N} |u|^2\Big)^2 \\ &= \frac{1}{4}\Big(\int_0^t f''(t)\Big)^2 = \frac{1}{4}(f'(t)-f'(0))^2\end{aligned}$$

where we have used

$$0 < \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^2 = \Re \int_{\mathbb{R}^N} u_t \overline{u} \le \int_{\mathbb{R}^N} |u_t||u|$$

Proof of Theorem 1, continued

However, $\int_{\mathbb{R}^N} |u_t||u|$ is better estimated from below by

$$\int_{\mathbb{R}^N} |u_t||u| \geq \Big| \int_{\mathbb{R}^N} u_t \overline{u} \Big| = |-e^{i\theta} J(u)| = |J(u)| = \frac{1}{2\cos\theta} \frac{d}{dt} \|u(t)\|_{L^2}^2.$$

Indeed,

$$u_t \overline{u} = e^{i\theta} \big[(\Delta u)\overline{u} + |u|^{\alpha+2} \big],$$

and so

$$\Re \int_{\mathbb{R}^N} u_t \overline{u} \leq \cos\theta \int_{\mathbb{R}^N} |u_t||u|.$$

So we get

$$\frac{1}{2(\alpha+2)\cos^2\theta} f(t) f''(t) \geq \frac{1}{4\cos^2\theta} \Big(\int_0^t \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 \Big)^2$$

and finally

$$\frac{2}{\alpha+2} f(t) f''(t) \geq \Big(\int_0^t f''(t) \Big)^2 = (f'(t) - f'(0))^2$$

Proof of Theorem 1, continued

The inequality

$$\frac{2}{\alpha+2} f(t) f''(t) \geq \Big(\int_0^t f''(t) \Big)^2 = (f'(t) - f'(0))^2$$

shows finite time blowup since

$$f''(t) = \frac{d}{dt} \|u(t)\|_{L^2}^2 \geq 2(\alpha+2) \cos^2\theta \int_0^t \|u_t\|_{L^2}^2,$$

and so $f'(t) \to \infty$.

Proof of Theorem 1, continued

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and so $f'(t) \to \infty$.

We note for further reference that this inequality also shows $T_{max}^\theta \leq K_1 \tau^\theta$, where

$$\tau^\theta = \sup_{0 \leq t < T_{max}^\theta} \{ \|u(t)\|_{L^2} \leq K_2 \|u_0\|_{L^2} \},$$

where K_1 and K_2 are explicit constants depending only on α.

sketch of proof of Theorem 3, assuming finite variance

We have the following variance identity for solutions u of (GL_θ):

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^N}|x|^2|u|^2 = 2N\alpha E(u(t)) - (N\alpha-4)\int_{\mathbb{R}^N}|\nabla u|^2 + \cos\theta\frac{d}{dt}\int_{\mathbb{R}^N}\left\{-2|x|^2|\nabla u|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u|^{\alpha+2} + 2N|u|^2\right\} - 2\cos^2\theta\int_{\mathbb{R}^N}|x|^2|u_t|^2. \quad \text{(Var)}$$

Integrating twice , we see that there exists $C_1 > 0$, depending on u_0 but not on θ, such that for all $0 \le t < T^\theta_{max}$,

$$\int_{\mathbb{R}^N}|x|^2|u|^2 \le \int_{\mathbb{R}^N}|x|^2|u_0|^2 + C_1 t + N\alpha E(u_0)t^2 + 2\cos\theta\int_0^t\int_{\mathbb{R}^N}\left\{-2|x|^2|\nabla u|^2 + \frac{\alpha+4}{\alpha+2}|x|^2|u|^{\alpha+2} + 2N|u|^2\right\}.$$

Using $E(u_0) < 0$, we wish to show that the RHS above becomes negative before a certain time, independent of $-\pi/2 < \theta < \pi/2$.

sketch of proof of Theorem 3, continued

Control of $\|u(t)\|_{L^2}$: Using Levine's argument for finite time blowup, it can be shown that $T^{\theta}_{max} \le K_1 \tau^{\theta}$, where $\tau^{\theta} = \sup_{0 \le t < T^{\theta}_{max}} \{\|u(t)\|_{L^2} \le K_2 \|u_0\|_{L^2}\}$, where K_1 and K_2 are explicit constants depending only on α. Thus, it suffices to obtain a uniform bound on τ^{θ}.

For all $0 \le t \le \tau^{\theta}$ one has $\|u(t)\|_{L^2} \le K_2 \|u_0\|_{L^2}$.

Lemma. Suppose $N \ge 2$ and $4/N \le \alpha \le 4$. Given any $M > 0$, there exists a constant C such that

$$\int_{\mathbb{R}^N} |x|^2 |u|^{\alpha+2} \le \int_{\mathbb{R}^N} |x|^2 |\nabla u|^2 + C \int_{\mathbb{R}^N} |u|^{\alpha+2} + C, \qquad (*)$$

for all smooth, radially symmetric u such that $\|u\|_{L^2} \le M$.

Thus, for all $0 \le t \le \tau^{\theta}$, we have

$$\int_{\mathbb{R}^N} |x|^2 |u^{\theta}|^2 \le C_1 + C_2 t + N\alpha E(u_0) t^2 + C_3 \cos\theta \int_0^t \int_{\mathbb{R}^N} |u|^{\alpha+2}.$$

Control of $\cos\theta \int_0^t \int_{\mathbb{R}^N} |u|^{\alpha+2}$:

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{L^2}^2 &= -2\cos\theta J(u(t)) \\ &= -4\cos\theta E(u(t)) + \frac{2\alpha\cos\theta}{\alpha+2}\int_{\mathbb{R}^N}|u|^{\alpha+2} \\ &\geq \frac{2\alpha\cos\theta}{\alpha+2}\int_{\mathbb{R}^N}|u|^{\alpha+2}\end{aligned}$$

so that

$$\cos\theta\int_0^t\int_{\mathbb{R}^N}|u|^{\alpha+2} \leq \|u(t)\|_{L^2}^2 - \|u_0\|_{L^2}^2,$$

and thus is bounded for $0 \leq t \leq \tau^\theta$, independently of θ.

This finally gives, for all $0 \leq t \leq \tau^\theta$,

$$\int_{\mathbb{R}^N}|x|^2|u^\theta|^2 \leq C_1 + C_2 t + N\alpha E(u_0)t^2$$

which concludes the proof, since $T_{max}^\theta \leq K_1\tau^\theta$.

THANK YOU

for your *attention*!!!