Finite-time blowup for a complex Ginzburg-Landau equation

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Joint work with T. Cazenave (UPMC) and F. Dickstein (UFRJ)
The equations

- Ginzburg-Landau equation:
  \[
  \begin{aligned}
  &u_t = e^{i\theta} \Delta u + e^{i\gamma |u|^\alpha u}, \\
  &u(0) = u_0,
  \end{aligned}
  \tag{GL}
  \]
  in $\mathbb{R}^N$, where $\alpha > 0$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and $\gamma \in \mathbb{R}$ ("$\alpha = p - 1$").

- Special case $\theta = \gamma = 0$, nonlinear heat equation:
  \[
  u_t = \Delta u + |u|^\alpha u
  \tag{NLH}
  \]

- Special case $\theta = \gamma = \pm \pi/2$, nonlinear Schrödinger equation:
  \[
  \pm iu_t = \Delta u + |u|^\alpha u
  \tag{NLS}
  \]
well-posedness

• If $-\pi/2 < \theta < \pi/2$, the Cauchy problem (GL) is locally well-posed in $C_0(\mathbb{R}^N)$.

• We denote by $T_{\text{max}} = T_{\text{max}}(u_0, \theta, \gamma)$ the existence time of the maximal solution $u(t)$. If $T_{\text{max}} < \infty$, then $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T_{\text{max}}$.

• If $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, then $u(t) \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for all $0 \leq t < T_{\text{max}}$.

• If $\alpha < 4/N$, then the Cauchy problem (GL) is also well-posed in $L^2(\mathbb{R}^N)$, with the same blowup time. So if $T_{\text{max}} < \infty$, then also $\|u(t)\|_{L^2} \to \infty$ as $t \to T_{\text{max}}$.

• If $\theta = \pm \pi/2$, and $\alpha < 4/(N - 2)$, then (GL) is locally well-posed in $H^1(\mathbb{R}^N)$.

QUESTION : When do there exist solutions with $T_{\text{max}} < \infty$?
Previous results on finite-time blowup

- Levine 1973, (NLH), if \( u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) has negative energy, i.e. \( E(u_0) < 0 \), where
  \[
  E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha + 2} \int_{\mathbb{R}^N} |w|^\alpha + 2,
  \]
  then \( T_{\text{max}} < \infty \).

- Zakharov 1972, Glassey 1977 (NLS), \( \alpha \geq 4/N \), \( E(u_0) < 0 \), and \( u_0 \) has finite variance, i.e. \( \int_{\mathbb{R}^N} |x|^2 |u_0|^2 < \infty \), then \( T_{\text{max}} < \infty \).
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- Zaag, 1998, “close to nonlinear heat equation”, construction of initial value such that resulting solution blows up in finite time (at a single point, with precise profile), when
  - \( \alpha < 4/(N - 2) \) (subcritical Sobolev),
  - \( \theta = 0 \),
  - \( |\gamma| \) small.
Previous results on finite-time blowup, continued

- Snoussi and Tayachi, 2001, “somewhat close to nonlinear heat equation”, negative energy initial value, $E(u_0) < 0$, produces finite-time blowup when:
  - $\theta = \gamma$,
  - $\cos^2 \theta > 2/(\alpha + 2)$ (calculations done in the case $\alpha = 2$).

- Masmoudi and Zaag, 2008, construction of initial value such that resulting solution blows up in finite time (at a single point, with precise profile), when
  \[
  \tan^2 \gamma + (\alpha + 2) \tan \gamma \tan \theta < \alpha + 3.
  \]

- If $\theta$ is close to $\pi/2$ (Schrödinger), then $\gamma$ must be small.

- If $\theta = \gamma$, condition becomes $\cos^2 \theta > (\alpha + 3)/2(\alpha + 2)$.

- Improves ST (2001) for $\alpha > 1$.

- Rottschäfer, 2008, “close to nonlinear Schrödinger equation”, construction of radially symmetric, backwards self-similar solution when:
  - $\alpha = 2$
  - $N = 3$
  - $\theta < \pi/2$ with $\pi/2 - \theta$ small,
  - $\gamma > \pi/2$ with $\gamma - \pi/2$ small.
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special case $\theta = \gamma$

In what follows, we consider the special case of the general Ginzburg-Landau equation:

$$\begin{cases}
    e^{-i\theta} u_t = \Delta u + |u|^\alpha u, \\
    u(0) = u_0.
\end{cases} \quad (GL_\theta)$$
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- Stationary solutions of $(\text{GL}_\theta)$ are the same, independent of $-\pi/2 \leq \theta \leq \pi/2$. 

- Solutions of $(\text{GL}_\theta)$ satisfy energy identities similar to those satisfied by solutions of $(\text{NLH})$ and $(\text{NLS})$.

- With these identities, Ball’s blowup proof (1977) on a bounded domain goes over easily and shows that for $-\pi/2 < \theta < \pi/2$, if $E(u_0) < 0$, then the resulting solution blows up in finite time.

- Levine’s argument (for a general domain, in particular $\mathbb{R}^N$), however, seems to require the Snoussi-Tayachi condition: $\cos^2 \theta > \frac{2}{\alpha + 2}$. 
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Theorem 1. Let $-\pi/2 < \theta < \pi/2$ and $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with $E(u_0) < 0$. It follows that the resulting solution $u$ of $(GL_\theta)$ blows up in finite time. More precisely,

$$T_{\text{max}} \leq \frac{\|u_0\|^2_{L^2}}{\alpha(\alpha + 2)(-E(u_0)) \cos \theta}.$$ 

Remark. In the case $\theta = \pi/2$ (NLS), and $\alpha < 4/N$, all solutions are global. Thus, in the case $-\pi/2 < \theta < \pi/2$, we expect the blowup time to get arbitrarily large, for a given initial value, as $\theta$ approaches $\pm \pi/2$.

Theorem 2. In Theorem 1, suppose in addition that $\alpha < 4/N$. Then there exists a constant $c = c(N, \alpha, \|u_0\|_{L^2}, E(u_0)) > 0$ such that $T_{\theta_{\text{max}}} \geq c \cos \theta$. 
main results (Cazenave, Dickstein, W)

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T_{max} \leq \frac{\|u_0\|_{L^2}^2}{\alpha(\alpha + 2)(-E(u_0)) \cos \theta}.
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**Remark.** In the case $\theta = \pi/2$ (NLS), and $\alpha \geq 4/N$, negative energy solutions (with finite variance) blowup in finite time. The same is true for radially symmetric solutions if $4/N \leq \alpha \leq 4$, without the condition of finite variance (Ogawa and Tsutsumi, 1991). Thus, in the case $-\pi/2 < \theta < \pi/2$, we expect the blowup time to stay bounded, for a given initial value, as $\theta$ approaches $\pm\pi/2$. 
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**Theorem 3.** In Theorem 1 ($-\pi/2 < \theta < \pi/2$ and $u_0 \in C_0(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ with $E(u_0) < 0$) suppose in addition that $u_0$ is radially symmetric and that $N \geq 2$ and $4/N \leq \alpha \leq 4$ (no assumption of finite variance). There exists $\bar{T}$ such that $T_{\max}^{\theta} \leq \bar{T}$ for all $-\pi/2 < \theta < \pi/2$. 

Proof based on truncated variance identity (Ogawa and Tsutsumi) and requires the same condition $4/N \leq \alpha \leq 4$. If we assume finite variance and use the "real" variance identity (viriel), we still require radial symmetry and $4/N \leq \alpha \leq 4$, but for different reasons.
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energy identities

Functionals:

\[ E(w) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 - \frac{1}{\alpha + 2} \int_{\mathbb{R}^N} |w|^\alpha + 2, \]

\[ J(w) = \int_{\mathbb{R}^N} |\nabla w|^2 - \int_{\mathbb{R}^N} |w|^\alpha + 2. \]

Identities for solutions of \((GL_\theta)\), with \(-\pi/2 < \theta < \pi/2\),

\[ \frac{d}{dt} E(u(t)) = -\cos \theta \|u_t\|_{L^2}^2, \quad \frac{d}{dt} \|u(t)\|_{L^2}^2 = -2 \cos \theta J(u(t)) \]

\[ \frac{d}{dt} \|u(t)\|_{L^2}^2 = -2(\alpha + 2) \cos \theta E(u(t)) + \alpha \cos \theta \|\nabla u(t)\|_{L^2}^2 \]

\[ = 2(\alpha + 2) \cos^2 \theta \int_0^t \|u_t\|_{L^2}^2 - 2(\alpha + 2) \cos \theta E(u_0) + \alpha \cos \theta \|\nabla u(t)\|_{L^2}^2 \]
Proof of Theorem 1 (based on Levine 1973)

Let \( f(t) = \int_0^t \|u(t)\|^2_{L^2} = \int_0^t \int_{\mathbb{R}^N} |u|^2 \). Since \( E(u_0) \leq 0 \), we have that

\[
f''(t) = \frac{d}{dt} \|u(t)\|^2_{L^2} \geq 2(\alpha + 2) \cos^2 \theta \int_0^t \int_{\mathbb{R}^N} |u_t|^2
\]

\[
\frac{1}{2(\alpha + 2) \cos^2 \theta} f(t)f''(t) \geq \int_0^t \int_{\mathbb{R}^N} |u|^2 \int_0^t \int_{\mathbb{R}^N} |u_t|^2
\]

\[
\geq \left( \int_0^t \int_{\mathbb{R}^N} |u||u_t| \right)^2 \geq \frac{1}{4} \left( \int_0^t \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 \right)^2
\]

\[
= \frac{1}{4} \left( \int_0^t f''(t) \right)^2 = \frac{1}{4} \left( f'(t) - f'(0) \right)^2
\]

where we have used

\[
0 < \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 = \Re \int_{\mathbb{R}^N} u_t \bar{u} \leq \int_{\mathbb{R}^N} |u_t||u|
\]
Proof of Theorem 1, continued

However, \( \int_{\mathbb{R}^N} |u_t||u| \) is better estimated from below by

\[
\int_{\mathbb{R}^N} |u_t||u| \geq \left| \int_{\mathbb{R}^N} u_t \bar{u} \right| = \left| -e^{i\theta} J(u) \right| = |J(u)| = \frac{1}{2 \cos \theta} \frac{d}{dt} \|u(t)\|_{L^2}^2.
\]

Indeed,

\[
u_t \bar{u} = e^{i\theta} [(\Delta u) \bar{u} + |u|^{\alpha+2}],
\]

and so

\[
\mathcal{R} \int_{\mathbb{R}^N} u_t \bar{u} \leq \cos \theta \int_{\mathbb{R}^N} |u_t||u|.
\]

So we get

\[
\frac{1}{2(\alpha + 2) \cos^2 \theta} f(t) f''(t) \geq \frac{1}{4 \cos^2 \theta} \left( \int_0^t \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 \right)^2
\]

and finally

\[
\frac{2}{\alpha + 2} f(t) f''(t) \geq \left( \int_0^t f''(t) \right)^2 = (f'(t) - f'(0))^2
\]
Proof of Theorem 1, continued

The inequality

\[ \frac{2}{\alpha + 2} f(t) f''(t) \geq \left( \int_0^t f''(t) \right)^2 = (f'(t) - f'(0))^2 \]

shows finite time blowup since

\[ f''(t) = \frac{d}{dt} \|u(t)\|_{L^2}^2 \geq 2(\alpha + 2) \cos^2 \theta \int_0^t \|u_t\|_{L^2}^2, \]

and so \( f'(t) \to \infty \).
Proof of Theorem 1, continued

The inequality
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We note for further reference that this inequality also shows \( T_{\theta}^{\max} \leq K_1 \tau^{\theta} \), where
\[ \tau^{\theta} = \sup_{0 \leq t < T_{\theta}^{\max}} \left\{ \| u(t) \|_{L^2} \leq K_2 \| u_0 \|_{L^2} \right\}, \]
where \( K_1 \) and \( K_2 \) are explicit constants depending only on \( \alpha \).
sketch of proof of Theorem 3, assuming finite variance

We have the following variance identity for solutions $u$ of (GL$_\theta$):

$$\frac{1}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^N} |x|^2 |u|^2 = 2N\alpha E(u(t)) - (N\alpha - 4) \int_{\mathbb{R}^N} |\nabla u|^2$$

$$+ \cos\theta \frac{d}{dt} \int_{\mathbb{R}^N} \left\{-2|\nabla u|^2 + \frac{\alpha + 4}{\alpha + 2}|x|^2 |u|^\alpha + 2 \right\}$$

$$- 2\cos^2\theta \int_{\mathbb{R}^N} |x|^2 |u_t|^2.$$ (Var)

Integrating twice, we see that there exists $C_1 > 0$, depending on $u_0$ but not on $\theta$, such that for all $0 \leq t < T_{\theta \text{max}}$,

$$\int_{\mathbb{R}^N} |x|^2 |u|^2 \leq \int_{\mathbb{R}^N} |x|^2 |u_0|^2 + C_1 t + N\alpha E(u_0)t^2$$

$$+ 2 \cos\theta \int_0^t \int_{\mathbb{R}^N} \left\{-2|\nabla u|^2 + \frac{\alpha + 4}{\alpha + 2}|x|^2 |u|^\alpha + 2 \right\}.$$}

Using $E(u_0) < 0$, we wish to show that the RHS above becomes negative before a certain time, independent of $-\pi/2 < \theta < \pi/2$. 
Control of $\|u(t)\|_{L^2}$: Using Levine's argument for finite time blowup, it can be shown that $T_{\text{max}}^\theta \leq K_1 \tau^\theta$, where 
$\tau^\theta = \sup_{0 \leq t < T_{\text{max}}^\theta} \{\|u(t)\|_{L^2} \leq K_2 \|u_0\|_{L^2}\}$, where $K_1$ and $K_2$ are explicit constants depending only on $\alpha$. Thus, it suffices to obtain a uniform bound on $\tau^\theta$.

For all $0 \leq t \leq \tau^\theta$ one has $\|u(t)\|_{L^2} \leq K_2 \|u_0\|_{L^2}$.

**Lemma.** Suppose $N \geq 2$ and $4/N \leq \alpha \leq 4$. Given any $M > 0$, there exists a constant $C$ such that

$$
\int_{\mathbb{R}^N} |x|^2 |u|^\alpha + 2 \leq \int_{\mathbb{R}^N} |x|^2 |\nabla u|^2 + C \int_{\mathbb{R}^N} |u|^\alpha + 2 + C, \quad (*)
$$

for all smooth, radially symmetric $u$ such that $\|u\|_{L^2} \leq M$.

Thus, for all $0 \leq t \leq \tau^\theta$, we have

$$
\int_{\mathbb{R}^N} |x|^2 |u^\theta|^2 \leq C_1 + C_2 t + N\alpha E(u_0)t^2 + C_3 \cos \theta \int_0^t \int_{\mathbb{R}^N} |u|^\alpha + 2.
$$
Control of $\cos \theta \int_0^t \int_{\mathbb{R}^N} |u|^{\alpha+2}$:

$$
\frac{d}{dt} \|u(t)\|_{L^2}^2 = -2 \cos \theta J(u(t))
$$

$$
= -4 \cos \theta E(u(t)) + \frac{2\alpha \cos \theta}{\alpha + 2} \int_{\mathbb{R}^N} |u|^{\alpha+2}
\geq \frac{2\alpha \cos \theta}{\alpha + 2} \int_{\mathbb{R}^N} |u|^{\alpha+2}
\]

so that

$$
\cos \theta \int_0^t \int_{\mathbb{R}^N} |u|^{\alpha+2} \leq \|u(t)\|_{L^2}^2 - \|u_0\|_{L^2}^2,
$$

and thus is bounded for $0 \leq t \leq \tau^\theta$, independently of $\theta$.

This finally gives, for all $0 \leq t \leq \tau^\theta$,

$$
\int_{\mathbb{R}^N} |x|^2 |u^\theta|^2 \leq C_1 + C_2 t + N\alpha E(u_0) t^2
\]

which concludes the proof, since $T_{\max}^\theta \leq K_1 \tau^\theta$. 
THANK YOU
for your attention!!!