Finite-time blow-up in the higher-dimensional Keller-Segel system

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September 10, 2012

Chemotaxis

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- Cells partially orient their movement toward increasing signal concentration

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Chemotactic movement plays a key role in many processes of communication bewteen cells, e.g. in

- formation of aggregates such as in populations of *Dictyostelium discoideum* or *Escherichia coli*
- tumor cell migration
- organization of cell positioning during embryonic development
- ...

The Keller-Segel model

KELLER/SEGEL 1970:

- $u = u(x,t)$: Density of cell population
- $v = v(x,t)$: Concentration of signal

$$\boxed{\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v), & x\in\Omega,\ t>0, \\ v_t &= \Delta v - v + u, & x\in\Omega,\ t>0. \end{aligned}} \qquad \text{(KS)}$$

We consider (KS) along with homogeneous Neumann boundary conditions $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$ and initial conditions

$$u(x,0) = u_0(x) \geq 0, \qquad v(x,0) = v_0(x) \geq 0, \qquad x \in \Omega,$$

with smooth u_0, v_0, in bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary.

The Keller-Segel model

Appropriate to describe aggregation?

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$$\int_\Omega u(x,t)dx \equiv \int_\Omega u_0(x)dx$$

Hence, the mass of an unbounded solution should essentially concentrate near blow-up points.

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Goal: Identify circumstances under which solutions (KS) blow up.

Aggregation

A first blow-up result

Theorem (Herrero/Velázquez 1997). If $\Omega \subset \mathbb{R}^2$ is a disk, then there exists at least one (u_0, v_0) such that (u, v) blows up in finite time.

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Do there exist *more* unbounded solutions?

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- $\underline{n = 2}$: If
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then (u, v) is global and bounded (NAGAI/SENBA/YOSHIDA 1997).
- $\underline{n \geq 3}$: Let $\delta > 0$. Then there exists $\varepsilon(\delta) > 0$ such that whenever
$$\|u_0\|_{L^{\frac{n}{2}+\delta}(\Omega)} \leq \varepsilon(\delta) \qquad \text{and} \qquad \|v_0\|_{W^{1,n+\delta}(\Omega)} \leq \varepsilon(\delta),$$
the solution (u, v) is global and bounded (W. 2010).

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Many boundedness results are available for related systems, involving e.g. nonlinear diffusion or variants in cross-diffusion, signal production,...(Senba, Suzuki, Cieślak, Blanchet, Laurençot, Wrzosek, Corrias, Perthame, Tao, Tello, Friedman, Sugiyama, Hillen, Painter, Ishida, Yokota, Carrillo, Calvez, Mimura, Naito...)

Aggregation

The challenge of detecting blow-up

Theorem 1 (HORSTMANN/WANG 2001). If $\Omega \subset \mathbb{R}^2$ is simply connected, then for almost every $m > 4\pi$ there exist initial data (u_0, v_0) such that $\int_\Omega u_0 = m$, and such that (u, v) blows up either in finite or infinite time.

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Theorem 2 (W. 2010). If $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ is a ball, then for all $m > 0$ one can find radial (u_0, v_0) with mass $\int_\Omega u_0 = m$ such that (u, v) blows up either in finite or infinite time.

Strategy I

Large negative energy enforces blow-up

Underlying strategy, e.g. in Theorem 2:

Step 1. Energy inequality

$$\frac{d}{dt}\mathcal{F}(u(t),v(t)) \le -\mathcal{D}(u(t),v(t)) \qquad \text{for } t \in (0,T_{max}),$$

where $T_{max} \le \infty$ denotes the maximal existence time of (u,v),

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u$$

and

$$\begin{aligned}\mathcal{D}(u,v) &:= \int_\Omega v_t^2 + \int_\Omega u\left|\frac{\nabla u}{u} - \nabla v\right|^2 \\ &\equiv \int_\Omega |\Delta v - v + u|^2 + \int_\Omega \left|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\right|^2.\end{aligned}$$

Strategy I

Large negative energy enforces blow-up

Underlying strategy, e.g. in Theorem 2:

Step 2. If (u, v) is global and bounded, then along some $t_k \to \infty$, $(u(t_k), v(t_k))$ approaches a solution (u_∞, v_∞) of

$$(S) \quad \begin{cases} 0 = \frac{\nabla u_\infty}{u_\infty} - \nabla v_\infty, & x \in \Omega, \\ 0 = \Delta v_\infty - v_\infty + u_\infty, & x \in \Omega, \\ 0 = \frac{\partial v_\infty}{\partial \nu}, & x \in \partial\Omega, \\ \int_\Omega u_\infty = \int_\Omega v_\infty = m \equiv \int_\Omega u_0. \end{cases}$$

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Step 3. There exists $K(m) > 0$ such that

$$\mathcal{F}(u, v) \geq -K(m) \qquad \text{for all radial solutions of (S)}$$

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Step 4. There exist radially symmetric (u_0, v_0) such that $\int_\Omega u_0 = m$ and

$$\mathcal{F}(u_0, v_0) < -K(m).$$

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By Step 1-Step 3: The corresponding solution cannot be global and bounded.

Finite-time blow-up?

Drawback of the above strategy: Only unboundedness is proved.

Questions:

- Does finite-time blow-up occur in the case $n \geq 3$?
- Is finite-time blow-up a rarely occurring phenomenon?

Finite-time blow-up !

Theorem 3 (W. 2011) Let $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$. Then for all $m > 0$ there exist (u_0, v_0) with mass $\int_\Omega u_0 = m$ such that (u, v) blows up in finite time.

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Theorem 3 (W. 2011) Let $n \geq 3$, $R > 0$ and $\Omega = B_R(0) \subset \mathbb{R}^n$. Then for all $m > 0$ there exist (u_0, v_0) with mass $\int_\Omega u_0 = m$ such that (u, v) blows up in finite time.

Moreover, for any positive radial (u_0, v_0) one can find smooth positive radial u_{0k} and v_{0k} such that $(u_{0k}, v_{0k}) \to (u_0, v_0)$ in $L^p(\Omega) \times W^{1,2}(\Omega)$ for all $p > \frac{2n}{n+2}$, but such that the corresponding solutions (u_k, v_k) blow up in finite time for each k.

In particular, all the constant steady states $(u, v) \equiv (m, m)$ are unstable in this sense.

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Deriving a superlinear ODI for the energy

Goal: Use largeness of dissipation rate $\mathcal{D}(u, v)$ when (u, v) is far from equilibrium.

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Step 2': Show

$$\mathcal{F}(u,v) \ge -C_0\Big(\mathcal{D}^\theta(u,v)+1\Big) \qquad (1)$$

with $\theta \in (0,1)$ and C_0 'sufficiently independent of (u_0,v_0)'.

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$$\frac{d}{dt}\big(-\mathcal{F}(u,v)\big) = \mathcal{D}(u,v) \ge \Big(\frac{-\mathcal{F}(u,v)}{C_0}-1\Big)_+^{\frac{1}{\theta}}.$$

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Since $\frac{1}{\theta} > 1$: If

$$\mathcal{F}(u_0,v_0) < -C_0, \tag{2}$$

then (u,v) blows up.

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Step 3': Explicit construction: (2) is possible for – many! – intitial data.

Strategy II

Outline of Step 2'

Prove more than necessary: Show

$$\mathcal{F}(u,v) \geq -C_0\Big(\mathcal{D}^\theta(u,v)+1\Big)$$

for all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$, where

$$\mathcal{S}(m,M,B,\kappa) := \Big\{(u,v) \in C^1(\overline{\Omega}) \times C^2(\overline{\Omega}) \;\Big|\; (u,v) \text{ is radial and positive}, \; \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0, \; \int_\Omega u = m, \; \int_\Omega v \leq M, \; v(r) \leq Br^{-\kappa} \text{ for } r \in (0,R)\Big\}$$

for fixed $m > 0, M > 0, B > 0$ and $\kappa > n-2$.

Strategy II

Outline of Step 2'

Lemma 1 Fix $\kappa > n-2$. Given any smooth positive radial (u_0, v_0), let

$$m := \int_\Omega u_0 \qquad \text{and} \qquad M := \max\left\{1, \int_\Omega v_0\right\}.$$

Then there exists $B = B(m, M) > 0$ such that

$$(u(t), v(t)) \in \mathcal{S}(m, M, B, \kappa)$$

for all t; in particular,

$$v(r,t) \le B(m,M) r^{-\kappa} \qquad \text{for all } r, t.$$

Proof: Smoothing properties of the Neumann heat semigroup.

Strategy II

Outline of Step 2'

Observe: Since $\xi \ln \xi \geq -\frac{1}{e}$ for $\xi > 0$, have

$$\begin{aligned}
\mathcal{F}(u,v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u \\
&\geq -\int_\Omega uv - \frac{|\Omega|}{e}.
\end{aligned}$$

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Thus the main step is:

Lemma 2 Let $n \geq 3$, $m > 0, M > 0, B > 0$ and $\kappa > n-2$. Then there exist $\theta \in (\frac{1}{2}, 1)$ and $C > 0$ such that

$$\int_\Omega uv \leq C(m,M,B,\kappa) \cdot \left\{ \left\| \Delta v - v + u \right\|_{L^2(\Omega)}^{2\theta} + \left\| \frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v \right\|_{L^2(\Omega)} + 1 \right\}$$

for all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$.

Strategy II

Proof of Lemma 2

First reformulate: Let $f := -\Delta v + v - u$ and $g := \frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v$.
Then our goal is to derive the a priori estimate

$$\int_\Omega uv \leq C\Big\{\|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)} + 1\Big\}$$

for solutions $(u, v) \in \mathcal{S}(m, M, B, \kappa)$ of elliptic-hyperbolic system

$$\begin{cases} -\Delta v + v = u + f, & x \in \Omega, \\ \frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v = g, & x \in \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

Strategy II

Proof of Lemma 2

i) Multiply $-\Delta v + v = u + f$ by v and integrate:

$$\int_\Omega uv \le 2\int_\Omega |\nabla v|^2 + C\Big\{\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1\Big\}.$$

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ii) Test $-\Delta v + v = u + f$ by v^α for $\alpha \in (0,1)$ and use $v(r) \le Br^{-\kappa}$:

$$\int_{\Omega\setminus B_{r_0}(0)} |\nabla v|^2 \le \varepsilon \int_\Omega uv + \varepsilon \int_\Omega |\nabla v|^2 + C_\varepsilon\Big\{r_0^{-\beta} + \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}\Big\}$$

with some $\beta > 0$ and $C_\varepsilon > 0$ independent of r_0.

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with some $\beta > 0$ and $C_\varepsilon > 0$ independent of r_0.

iii) Use radial symmetry and $n \ge 3$ in deriving

$$\int_{B_{r_0}(0)} |\nabla v|^2 \le C\cdot\Big\{r_0\cdot\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)} + 1\Big\}$$

with C independent of r_0.

Strategy II

Proof of Lemma 2

Collect:

$$\int_\Omega uv \leq 2\int_\Omega |\nabla v|^2 + C\Big\{\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1\Big\}, \tag{3}$$

$$\int_{\Omega\setminus B_{r_0}(0)} |\nabla v|^2 \leq \varepsilon\int_\Omega uv + \varepsilon\int_\Omega |\nabla v|^2 + C_\varepsilon\Big\{r_0^{-\beta} + \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}\Big\}, \tag{4}$$

$$\int_{B_{r_0}(0)} |\nabla v|^2 \leq C\cdot\Big\{r_0\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)} + 1\Big\}. \tag{5}$$

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iv) In (5), let $r_0 \sim \|f\|_{L^2(\Omega)}^{-\gamma}$ with appropriate $\gamma > 0$. Then (4)-(5) yield

$$\int_\Omega |\nabla v|^2 \le 2\varepsilon\int_\Omega uv + C_\varepsilon\Big\{\int_\Omega \|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)} + 1\Big\}$$

with some $\theta \in (\frac{1}{2}, 1)$.

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with some $\theta \in (\frac{1}{2}, 1)$.

Combined with (3), this finally shows

$$\int_\Omega uv \le C\Big\{\int_\Omega \|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)} + 1\Big\}.$$

Strategy II

Completion of blow-up proof

Hence, for $(u,v) \in \mathcal{S}(m,M,B,\kappa)$,

$$\int_\Omega uv \le C(m,M,B,\kappa)\cdot\left\{\left\|\Delta v - v + u\right\|_{L^2(\Omega)}^{2\theta} + \left\|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\right\|_{L^2(\Omega)} + 1\right\}.$$

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In particular: For some $C_0 = C_0(m,M,B,\kappa) > 0$ and all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$,

$$\mathcal{F}(u,v) \ \ge \ -\int_\Omega uv - \frac{|\Omega|}{e} \ \ge \ -C_0 \Big\{ \mathcal{D}^\theta(u,v) + 1 \Big\}.$$

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Since given $m > 0$ and $M > 0$ it is possible to construct (u_0, v_0) with $\int_\Omega u_0 = m$ and $\int_\Omega v \le M$ such that $\mathcal{F}(u_0,v_0) < -2C_0(m,M,B,\kappa)$ and $(u(t),v(t)) \in \mathcal{S}(m,M,B,\kappa)$ for all t, we thus have

$$\frac{d}{dt}\Big(-\mathcal{F}(u(t),v(t))\Big) \le \Big(\frac{-\mathcal{F}(u(t),v(t))}{C_0} - 1\Big)_+^{\frac{1}{\theta}} \ge \Big(\frac{-\mathcal{F}(u(t),v(t))}{2C_0}\Big)^{\frac{1}{\theta}}$$

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$$\frac{d}{dt}\Big(-\mathcal{F}(u(t),v(t))\Big) \le \Big(\frac{-\mathcal{F}(u(t),v(t))}{C_0} - 1\Big)_+^{\frac{1}{\theta}} \ge \Big(\frac{-\mathcal{F}(u(t),v(t))}{2C_0}\Big)^{\frac{1}{\theta}},$$

implying blow-up in finite time.

Conclusion

Blow-up in chemotaxis systems - future challenges

We have seen:

- Finite-time blow-up in the Keller-Segel system occurs for **many** solutions
  - when $n \geq 3$ and
  - the spatial setting is radial.

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- blow-up in more general models, including nonlinear diffusion and/or cross-diffusion (see also CIEŚLAK/STINNER 2012, CIEŚLAK/LAURENÇOT 2010)

Conclusion

Blow-up in chemotaxis systems - future challenges

We have seen:

- Finite-time blow-up in the Keller-Segel system occurs for **many** solutions
  - when $n \geq 3$ and
  - the spatial setting is radial.

Some open problems concern

- finite-time blow-up in the two-dimensional case (for supercriticial mass)
- asymptotic behavior near blow-up (MIZOGUCHI/SOUPLET 2012)
- explosions in nonradial frameworks
- blow-up in more general models, including nonlinear diffusion and/or cross-diffusion (see also CIEŚLAK/STINNER 2012, CIEŚLAK/LAURENÇOT 2010)

Thank you very much!