Construction of a multi-soliton blow-up solution to the semilinear wave equation in one space dimension

Hatem ZAAG
CNRS & LAGA, Université Paris 13

Fifth Euro-Japanese workshop on blow-up
CIRM, Marseille, September 10-14, 2012

Joint work with

Raphaël Côte, CNRS Polytechnique & University of Chicago,
Frank Merle, Université de Cergy-Pontoise & IHES

The equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1} u, \\ \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases}$$

where $p > 1$, $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in H^1(\mathbb{R})$ and $u_1 \in L^2(\mathbb{R})$.

Earlier work: Levine 1974, Caffarelli and Friedman 1985, Ginibre, Soffer and Velo 1992, Kichenassamy and Littman 1993, Alinhac 1995, Lindblad and Sogge 1995, Shatah and Struwe 1998.

Remark: All the results extend to the radial case outside the origin:

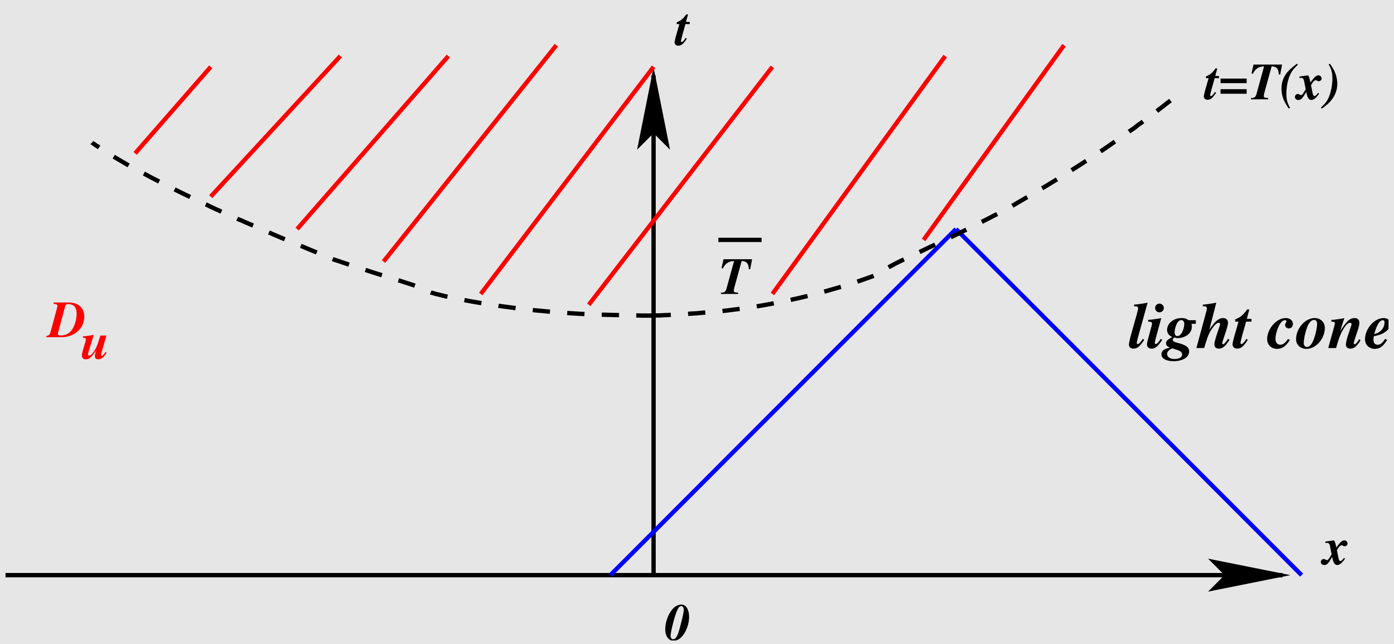
$$\partial_t^2 u = \partial_r^2 u + \frac{(N-1)}{r} \partial_r u + |u|^{p-1} u.$$

Singular solutions: the maximal influence domain

We consider an arbitrary blow-up solution $u(x,t)$.
From the finite speed of propagation, its domain of definition is

$$D_u = \{(x,t) \mid 0 \le T(x)\}$$

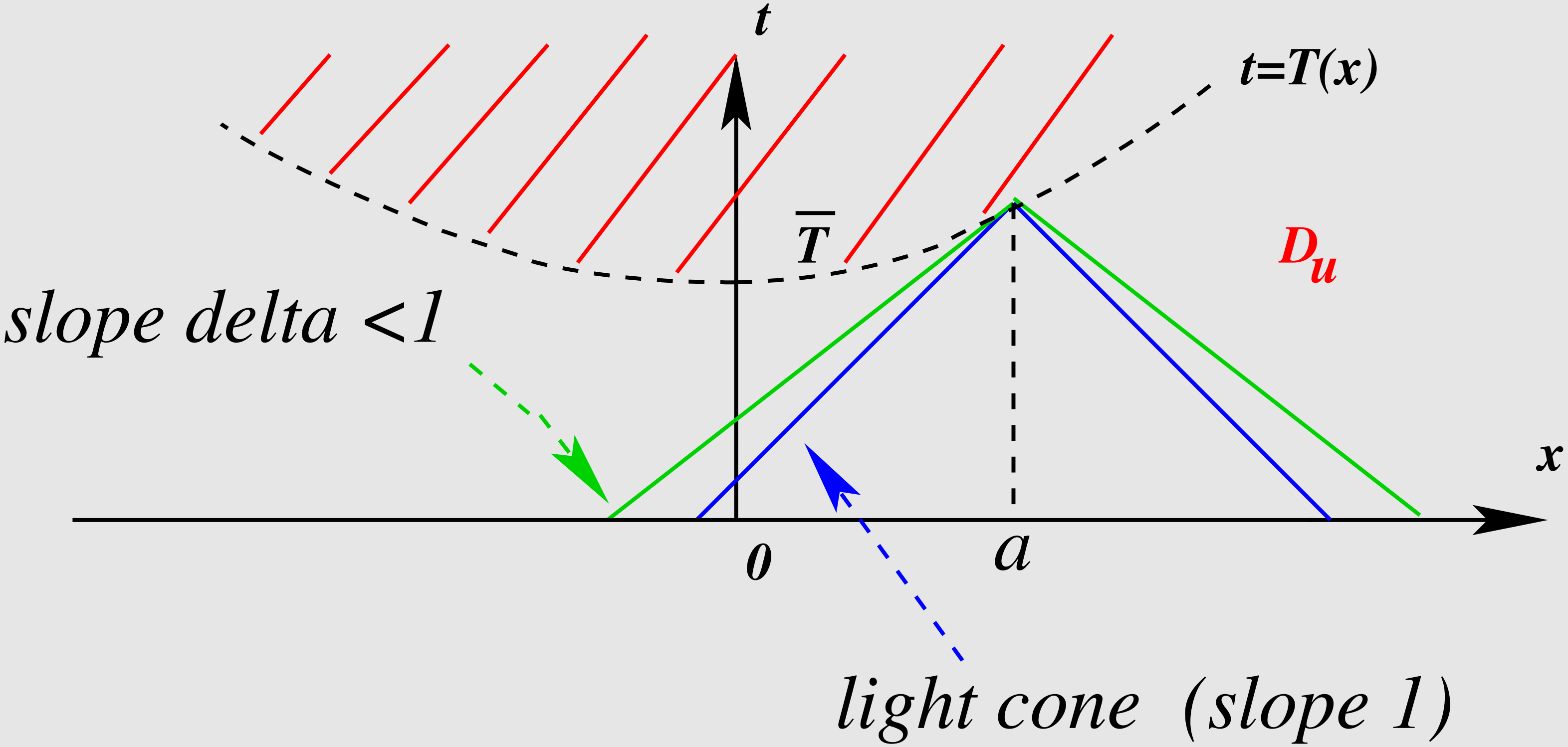
where $x \mapsto T(x)$ is 1-Lipschitz.



Remark: For all $x \in \mathbb{R}$, there exists a “local” blow-up time $T(x)$.

Definition: Non characteristic points and characteristic points

A point a is said *non characteristic* if the domain contains a cone with vertex $(a, T(a))$ and slope $\delta < 1$.



The point is said *characteristic* if not.

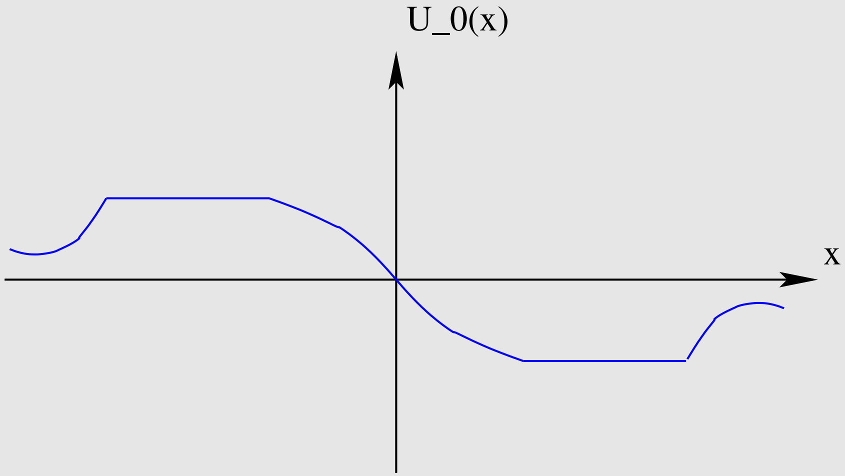
- Notation: $\mathcal{R} \subset \mathbb{R}$ is the set of all *non* characteristic points.
- Notation: $\mathcal{S} \subset \mathbb{R}$ is the set of all characteristic points ($\mathcal{S} \cup \mathcal{R} = \mathbb{R}$).

Part 1: Existence of characteristic points

We recall: Any *solution to the Cauchy problem has (at least) a non characteristic point (the minimum of the blow-up set).*

Th. (Merle-Z.) There exist *initial data which give solutions with a characteristic point.*

Example: We take odd initial data, with two large plateaus of different signs. Then, the solution blows up, and the origin is a characteristic point with $\forall t < T(0),\ u(0,t) = 0$.



Th. (Merle-Z.) If we perturb the constructed initial data, then the new solution blows up and has a characteristic point.

Part 2: Regularity of the blow-up set

- Near a non characteristic point:
  Th. (Merle-Z.) *The set of non characteristic points* $\mathcal{R}$ **is open** *and* $T(x)$ **is of class** C^1 *on this set*.

- Near a characteristic point:
  Th. (Merle-Z.) *The set of characteristic points* $\mathcal{S}$ *is made of* **isolated points**.
  If $a \in \mathcal{S}$, *then* $T_l'(a) = 1$ *and* $T_r'(a) = -1$.

  Cor. *There is no solution with* $a \in \mathcal{S}$ *and* $T'(a) = 1$.

— LaTeX prosper —

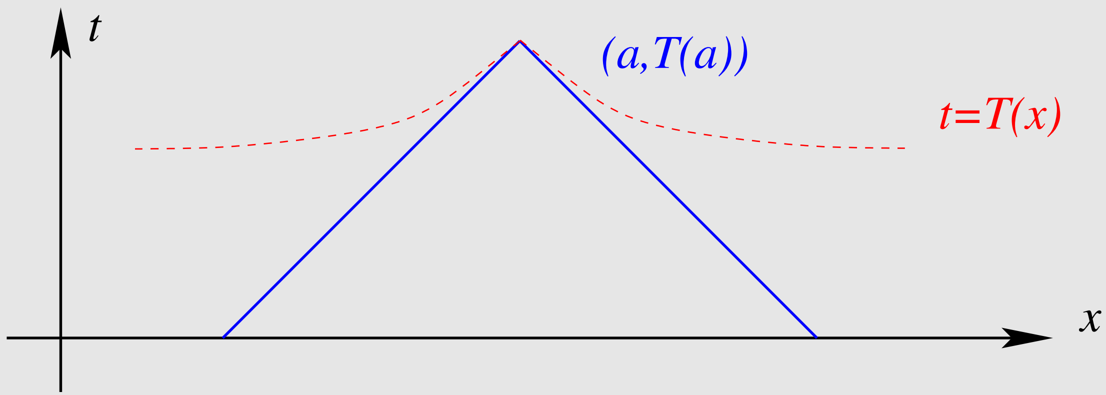
Part 2: The corner property near a *characteristic* point

Th. (Merle-Z. refined in Côte-Z.) *If $a \in \mathcal{S}$, then for all x near a,*

$$T(x) - T(a) + |x-a| \sim \frac{C_0|x-a|}{|\log|x-a||^{\gamma(a)}} \text{ as } x \to a, \qquad (1)$$

where $C_0 = C_0(sgn(x-a)) > 0$ with $C_0(-1) = C(p)/C_0(1)$,

$$\gamma(a) = \frac{(k(a)-1)(p-1)}{2} \text{ with } k(a) \in \mathbb{N}, k(a) \geq 2.$$



— LATEX prosper —

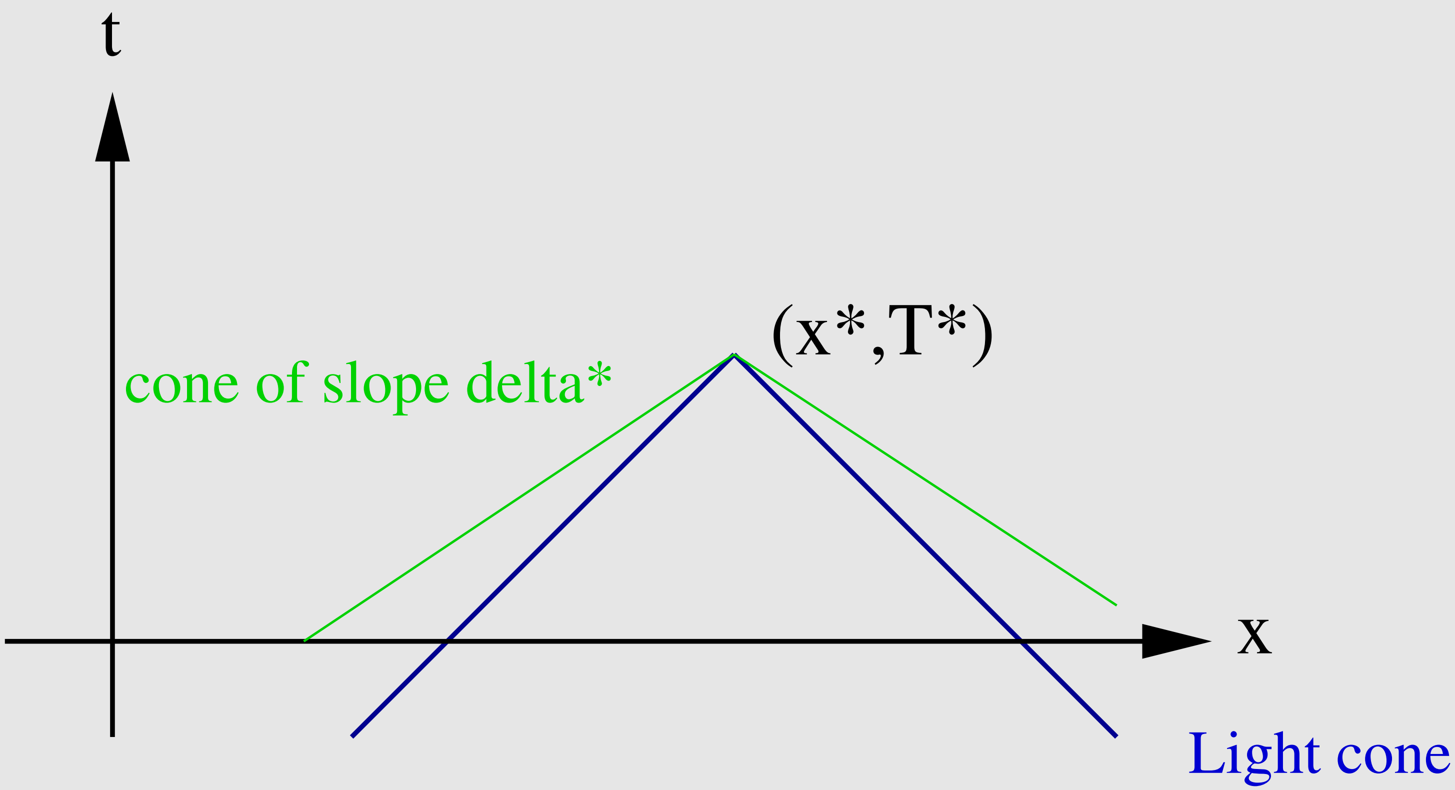
Comments

Idea of the proof:
The techniques are based on

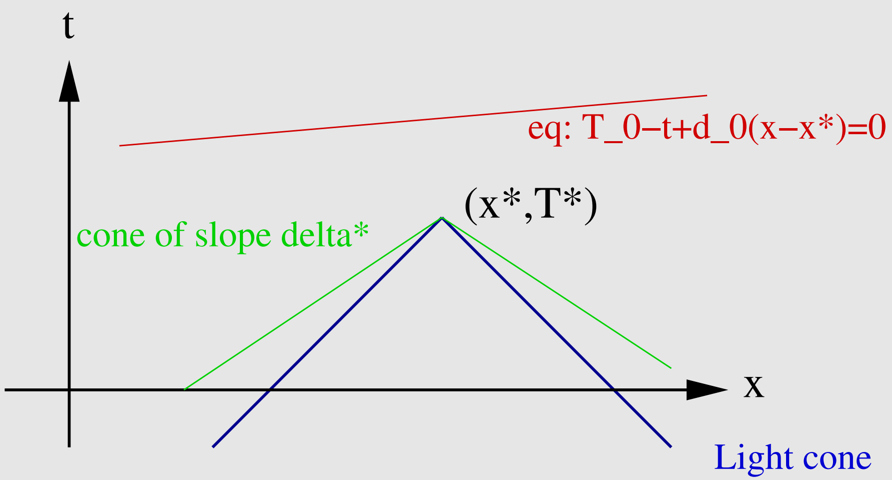
- ▷ - a very good understanding of the **behavior of the solution in selfsimilar variables in the energy space** related to the selfsimilar variable.
- ▷ - a **Liouville Theorem.** (see next slide).

A Liouville Theorem

Th. (Merle-Z.) Consider $u(x,t)$ a solution of $u_{tt} = u_{xx} + |u|^{p-1}u$ such that:
- u is defined in the *infinite* green cone,
- u is less than $(T^* - t)^{-\frac{2}{p-1}}$ (in L^2 average).



A Liouville Theorem



Then,
- either $u \equiv 0$,
- or there exists $T_0 \geq T^*$, $d_0 \in [-\delta_*, \delta_*]$ and $\theta_0 = \pm 1$ such that u is actually defined below the red line by

$$u(x,t) = \theta_0 \kappa_0(p) \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x-x^*))^{\frac{2}{p-1}}}.$$

Remark: u blows up on the red line.

Comments

- ▷ The limiting case $\delta^* = 1$ is still open.

The proof:

- ▷ The proof has a completely different structure from the proof for the heat equation.
- ▷ The proof is based on various energy arguments and on a dynamical result.

Part 3: Selfsimilar transformation for all $x_0 \in \mathbb{R}$

$$w_{x_0}(y,s) = (T(x_0)-t)^{\frac{2}{p-1}}u(x,t),\ y = \frac{x-x_0}{T(x_0)-t},\ s = -\log(T(x_0)-t).$$

(x,t) in the light cone of vertex $(x_0, T(x_0)) \Longleftrightarrow (y,s) \in (-1,1)\times[-\log T(x_0), \infty)$.

Equation on $w = w_{x_0}$: For all $(y,s) \in (-1,1)\times[-\log T(x_0), \infty)$:

$$\partial^2_{ss} w - \frac{1}{\rho}\partial_y(\rho(1-y^2)\partial_y w) + \frac{2(p+1)}{(p-1)^2}w - |w|^{p-1}w$$

$$= -\frac{p+3}{p-1}\partial_s w - 2y\partial^2_{sy} w$$

where $\rho(y) = (1-|y|^2)^{\frac{2}{p-1}}$

A Lyapunov functional (Antonini-Merle)

$$E(w) = \int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2(1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1} \right) \rho dy,$$

Thanks to a Hardy-Sobolev inequality, $E = E(w, \partial_s w)$ is well defined in the energy space

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(B) \ \mid \ \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(\partial_y q_1\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

Properties of the Lyapunov functional E

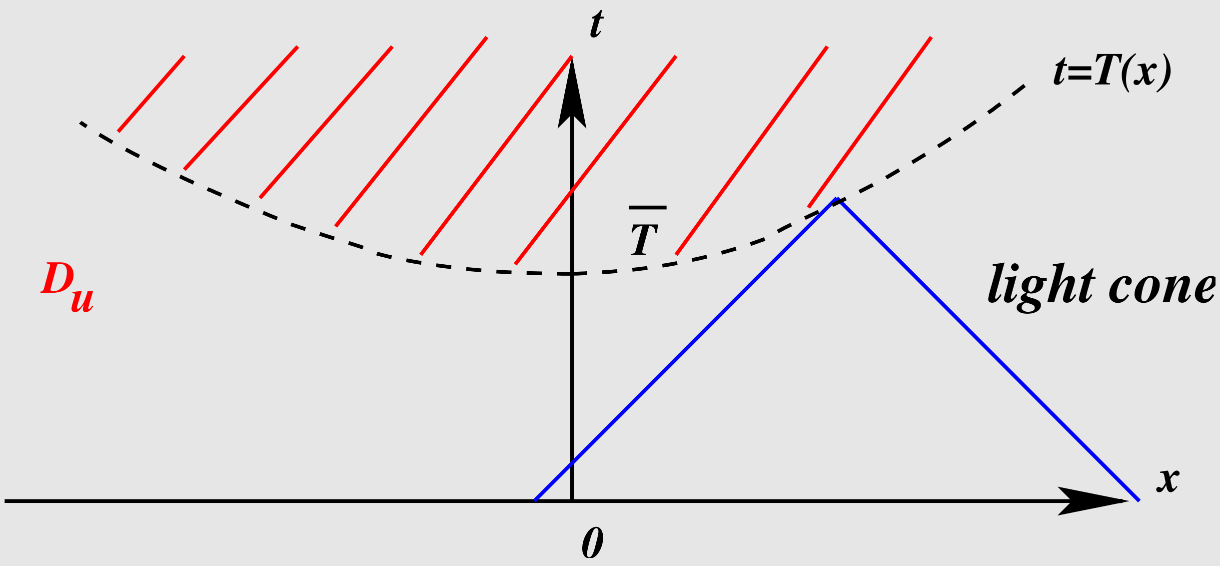
Lemma 1 (Monotonicity (Antonini-Merle)) *For all s_1 and s_2:*

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1}\int_{s_1}^{s_2}\int_{-1}^{1}(\partial_s w)^2(1-|y|^2)^{\frac{2}{p-1}-1}dyds.$$

Lemma 2 (A blow-up criterion) *Consider a solution W such that $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S > s_0$.*

The blow-up rate

We look for a *local* **blow-up rate** near the singular surface (i.e. near every local blow-up time, $t \to T(x_0)$), in $H^1 \times L^2$ **of the section of the light cone.**



Hint: Is the rate given by the associated ODE $v'' = v^p$?

An upper bound on the blow-up rate in selfsimilar variables

Th. (Merle-Z.) For all $x_0 \in \mathbb{R}$ and $s \geq -\log T(x_0) + 1$,

$$\int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2 (1 - |y|^2) + \frac{(p+1)}{(p-1)^2} w^2 + \frac{1}{p+1} |w|^{p+1} \right) \rho dy \leq K$$

where the constant K depends only on p and an upper bound on $T(x_0)$, $1/T(x_0)$ and $\|(u_0, u_1)\|$.

Idea of the proof of the upper bound

- ▷ Selfsimilar transformation and existence of a Lyapunov functional
- ▷ Interpolation to gain regularity
- ▷ Gagliardo-Nirenberg estimates.

— LaTeX prosper —

Part 4: Asymptotic behavior at a *non characteristic* point

Take $x_0 \in \mathbb{R}$ **non characteristic.** Using a covering argument for x near x_0, we obtain that $\|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-1,1)}$ is bounded.

Question: Does $w_{x_0}(y,s)$ have a limit or not, as $s \to \infty$ (that is as $t \to T(x_0)$).

Remark: In the context of Hamiltonian systems, **this question is delicate**, and there is no natural reason for such a convergence, since **the wave equation is time reversible.**

See for similar difficulty and approach, results for

- the **critical KdV** (Martel and Merle),
- **NLS** (Merle and Raphaël).

Stationary solutions.

We look for solutions of

$$\frac{1}{\rho}\left(\rho(1-y^2)w'\right)' - \frac{2(p+1)}{(p-1)^2}w + |w|^{p-1}w = 0.$$

We work in $\mathcal{H}_0$, the (**stationary energy space**) defined by

$$\mathcal{H}_0 = \{r \in H^1_{loc}(-1,1) \mid \|r\|^2_{\mathcal{H}_0} \equiv \int_{-1}^{1}\left(r'^2(1-y^2)+r^2\right)\rho dy < +\infty\}.$$

Prop. Consider a stationary solution in $\mathcal{H}_0$. Then, either $w \equiv 0$ or there exist $d \in (-1,1)$ and $e = \pm 1$ such that $w(y) = e\kappa(d,y)$ where

$$\forall (d,y) \in (-1,1)^2,\ \ \kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ and } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}.$$

Remark: We have 3 connected components. $E(0) = 0 < E(\pm\kappa(d)) = E(\kappa_0)$.

Proof : A new set of variables

Introducing

$$\bar{w}(\xi,s) = (1-y^2)^{\frac{1}{p-1}} w(y,s) \text{ where } y = \tanh\xi,$$

we have

$$\bar{w} \in H^1(\mathbb{R}) \text{ and } \bar{w}'' - \frac{4}{(p-1)^2}\bar{w} + |\bar{w}|^{p-1}\bar{w} = 0.$$

The solutions are known:

$$\bar{w} \equiv 0 \text{ or } \bar{w}(\xi) = \pm\kappa_0 \cosh^{-\frac{2}{p-1}}(\xi-\zeta)$$

where $\zeta \in \mathbb{R}$. Putting $d = -\tanh\zeta$, we find

$$w \equiv 0 \text{ or } w(y) = \pm\kappa(d,y) = \pm\kappa_0(p)\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}}.$$

Remark: We have 3 connected components.

Blow-up profile near a *non-characteristic* point

Th. (Merle-Z.) *There exist* $C_0 > 0$ *and* $\mu_0 > 0$ *such that if* x_0 *is* **non characteristic**, *then there exist* $d(x_0) \in (-1,1)$, $e(x_0) = \pm 1$ *and* $s^*(x_0) \geq -\log T(x_0)$ *such that:*
(i) For all $s \geq s^*(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \begin{pmatrix} \kappa(d(x_0),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}$$

and $E(w_{x_0}(s) \to E(\kappa_0)$ *where the energy space*

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \;\middle|\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1')^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}.$$

(ii) $d(x_0) = T'(x_0)$.
Rk. We have exp. fast convergence (hence, C^{1,μ_0} regularity of $\mathcal{R}$, see Nouaili).
Rk. $\|w_{x_0}(y,s) - e(x_0)\kappa(d(x_0),y)\|_{L^\infty(-1,1)} \to 0$.

In the ξ variable

For s large enough,

$$\|\bar{w}_{x_0}(\xi,s) - e(x_0)\kappa_0 \cosh^{-\frac{2}{p-1}}(\xi - \zeta(x_0))\|_{H^1\cap L^\infty(\mathbb{R})} \le C_0 e^{-\mu_0(s-s^*)}$$

where

$$\bar{w}_{x_0}(\xi,s) = (1-y^2)^{\frac{1}{p-1}} w_{x_0}(y,s) \text{ with } y = \tanh\xi \text{ and } \zeta(x_0) = -\tanh^{-1} d(x_0).$$

Difficulties of the proof of convergence

- The set of non zero stationary solutions is made up of non isolated solutions (one parameter family):
  $\longrightarrow$ we need **a modulation technique**.
- The linearized operator around a non zero stationary solution **is non self-adjoint**:
  $\longrightarrow$ we need to use dispersive properties coming from the Lyapunov functional to control the negative part of the spectrum.

Part 5: Asymptotic behavior at a *characteristic* point

Th. (Merle-Z. refined in Côte-Z.) If $x_0 \in \mathbb{R}$ is **characteristic**, then, there exist $k(x_0) \geq 2$, $e(x_0) = \pm 1$ and continuous $d_i(s) = -\tanh \zeta_i(s)$ for $i = 1, ..., k$ such that:
(i)

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

(ii) Introducing

$$\bar{w}_{x_0}(\xi, s) = (1 - y^2)^{\frac{1}{p-1}} w_{x_0}(y, s) \text{ with } y = \tanh \xi \text{ and } \zeta_i(s) = -\tanh^{-1} d_i(s),$$

we get

$$\|\bar{w}_{x_0}(\xi, s) - e(x_0) \sum_{i=1}^{k(x_0)} (-1)^i \kappa_0 \cosh^{-\frac{2}{p-1}} (\xi - \zeta_i(s))\|_{H^1 \cap L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,$$

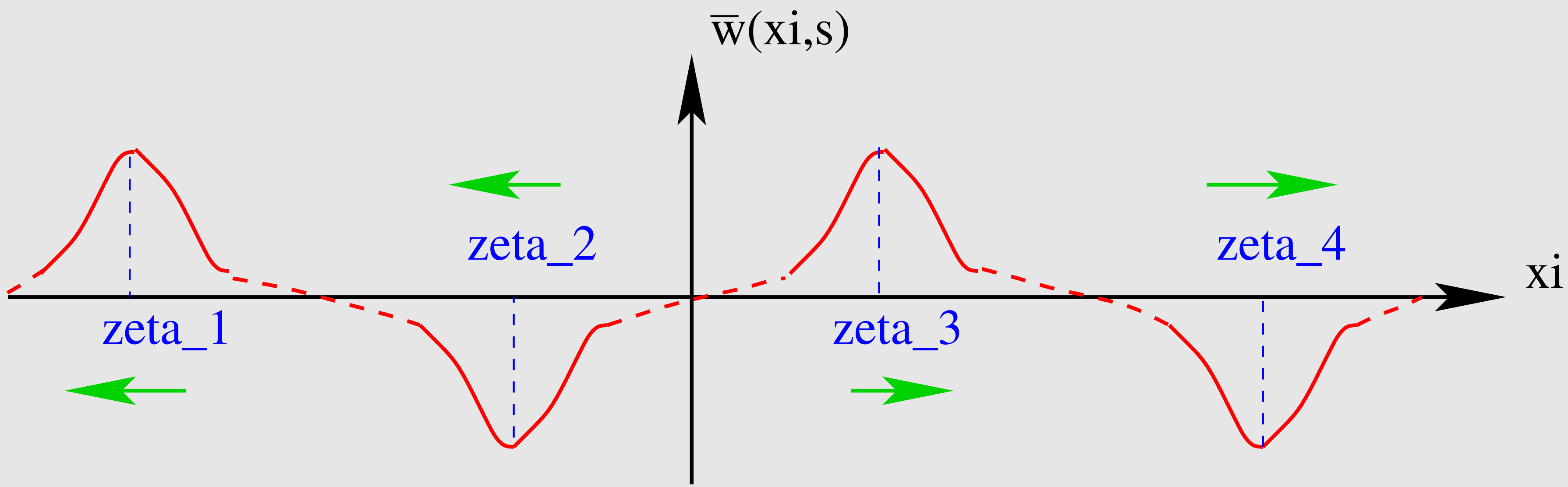
Part 5: Asymptotic behavior at a *characteristic* point (cont.)

(iii) For all $i = 1, ..., k(x_0)$ and $s > 0$

$$\frac{1}{c_1}\zeta_i'(s) = e^{-\frac{2}{p-1}(\zeta_i - \zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1} - \zeta_i)} \text{ and } \zeta_i(s) = \left(i - \frac{(k+1)}{2}\right)\frac{(p-1)}{2}\log s + C_i$$

(a one parameter family, indexed by the barycenter of ζ_i, which is conserved).

(iv) $E(w_{x_0}(s)) \to k(x_0)E(\kappa_0)$ as $s \to \infty$.



Part 5: Asymptotic behavior at a *characteristic* point (cont.)

Rk.
- As $s \to \infty$, w_{x_0} becomes like a **decoupled** sum of *equidistant* stationary solutions ("solitons"), with *alternate* signs.
- In the ξ variable, half of the solitons go to $-\infty$, and the other half to $+\infty$.
- The main difficulty in the proof is to prove that $k(x_0) \geq 2$ (the case $k(x_0) = 0$ is harder to eliminate).

The energy behavior

Defining

$$k(x_0) = 1 \text{ when } x_0 \in \mathcal{R},$$

we get the following:

Cor.
(i) *For all* $x_0 \in \mathbb{R}$ *and* $s \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s)) \geq k(x_0) E(\kappa_0).$$

(ii) **(An energy criterion for non characteristic points)** *If for some* $x_0 \in \mathbb{R}$ *and* $s_0 \geq -\log T(x_0)$, *we have*

$$E(w_{x_0}(s_0)) < 2E(\kappa_0),$$

then $x_0 \in \mathcal{R}$.

Blow-up speed or the L^∞ norm behavior

Cor.
(i) **(Case of non-characteristic points)** *If* $x_0 \in \mathcal{R}$, *then*

$$\frac{(T(x_0)-t)^{-\frac{2}{p-1}}}{C} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le C(T(x_0)-t)^{-\frac{2}{p-1}}$$

(i) **(Case of characteristic points)** *If* $x_0 \in \mathcal{S}$, *then*

$$\frac{|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{C(T(x_0)-t)^{\frac{2}{p-1}}} \le \sup_{|x-x_0|<T(x_0)-t} |u(x,t)| \le \frac{C|\log(T(x_0)-t)|^{\frac{k(x_0)-1}{2}}}{(T(x_0)-t)^{\frac{2}{p-1}}}.$$

where $k(x_0) \ge 2$ *is the solitons' number in the decomposition of* w_{x_0}.

Rk.
When $x_0 \in \mathcal{R}$, the speed is given by the associated ODE $u'' = u^p$.
When $x_0 \in \mathcal{S}$, the speed is higher. It has a $\log$ correction depending on the number of solitons.

Part 6: Construction of blow-up modalities in the non-characteristic case

Given $d_0 \in (-1,1)$, we have the following explicit solution

$$u(x,t) = \kappa_0(p)\frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0 - t + d_0(x-x^*))^{\frac{2}{p-1}}}.$$

which blows-up on the line

$$T(a) = T_0 - t + d_0(a - x^*) \text{ satisfying } T'(a) = d_0, \text{ hence } \mathcal{R} = \mathbb{R}. \qquad (2)$$

Moreover,

$$\forall a \in \mathbb{R}, \quad w_a(y,s) = \kappa(d_0, y), \qquad (3)$$

which is the desired profile.

However, u is not a solution of the Cauchy problem at $t = 0$ (unless $d_0 = 0$).

By truncation and from the finite speed of propagation, we get a solution of the Cauchy problem satisfying (2) and (3) for all $|a| < R$.

— LaTeX prosper —

Part 6: Construction of blow-up modalities in the characteristic case

Th. (Côte-Z.): Given $k \geq 2$ and $\zeta_0 \in \mathbb{R}$, there exists a solution $u(x,t)$ such that $0 \in \mathcal{S}$ and

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_0(s) \end{pmatrix} - \sum_{i=1}^{k} (-1)^i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty,$$

with

$$d_i(s) = -\tanh \zeta_i(s), \;\; \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0$$

and $(\bar{\zeta}_i(s))_i$ is THE solution of

$$\frac{1}{c_1} \bar{\zeta}_i'(s) = e^{-\frac{2}{p-1}(\bar{\zeta}_i - \bar{\zeta}_{i-1})} - e^{-\frac{2}{p-1}(\bar{\zeta}_{i+1} - \bar{\zeta}_i)} \text{ with } \bar{\zeta}_1(s) + \dots + \bar{\zeta}_k(s) \equiv 0.$$

Remark: We have $\frac{1}{k}(\zeta_1(s) + \dots + \zeta_k(s)) \equiv \zeta_0$.

Idea of the proof

It relies on the proof of the *stability* of the decomposition into a sum of (generalized) solitons.
More precisely, we need the following steps:

- ▷ Linearization around the sum of solitons (which is *not* a solution): the linearized operator is not self-adjoint;
- ▷ Control of the negative part thanks to a modified version of the Lyapunov functional;
- ▷ Control of the zero eigenvalues thanks to a modulation technique;
- ▷ Control of the positive eigenvalues thanks to a contradiction argument based on Brouwer Theorem.

Prescribing more characteristic points

From the finite speed of propagation, we have:

Cor (Côte-Z.) Let $\{x_n\}$ be a discrete subset of $\mathbb{R}$, $\{T_n\} \subset \mathbb{R}^+$ be the set of blow-up times, $\{k_n\}$ a set of integers ≥ 2, $\{e_n\}$ a set in $\{-1, 1\}$ and $\{\zeta_{0,n}\} \subset \mathbb{R}$. Assume

$$\forall n, \quad x_n + T_n < x_{n+1} + T_{n+1}.$$

Then there exist a blow-up solution $u(t,x)$ such that $\{x_n\} \subset \mathcal{S}$, $T(x_n) = T_n$ and for all n,

$$\left\| \begin{pmatrix} w_{x_n,T(x_n)}(s) \\ \partial_s w_{x_n,T(x_n)}(s) \end{pmatrix} - e(x_n) \sum_{i=1}^{k_n} (-1)^i \begin{pmatrix} \kappa(d_{i,n}(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to +\infty.$$

with

$$d_{i,n}(s) = -\tanh \zeta_{i,n}(s), \quad \zeta_{i,n}(s) = \left(i - \frac{k_n + 1}{2} \right) \frac{(p-1)}{2} \ln s + \alpha_i + \zeta_{0,n}.$$

— LaTeX prosper —