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ON ONE-PARAMETER FAMILIES OF DIFFEOMORPHISMS

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This paper is concerned with diffeomorphisms of manifolds, depending on a parameter. This means that we shall consider mappings $f: P \times M \longrightarrow M$, where P is a 1-dimensional C^r $(1 < r < \infty)$ manifold, M is an n-dimensional C^r manifold, f is C^r and such that for every $p \in P$, the mapping $f_p : M \rightarrow M$ given by $f_p(m) = f(p, m)$ is a diffeomorphism. Given P, M, we denote by $\mathcal{F}$ the set of all mappings f with the above properties, endowed with the C^r Whitney topology. We shall be interested in the generic behavior of the periodic points of f_p, (i.e. fixed points of f_p and its iterates) if p is varied.

We say that a property is generic in $\mathcal{F}$ if it is valid for every f from a residual subset of $\mathcal{F}$.

The first part of our results (§ 1) concerns the case of arbitrary n, the second (§ 2) takes place for $n = 2$.

The problems studied in this paper are to a great extent motivated by differential equations, where problems of dependence of critical points and periodic trajecto-

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ries on a parameter are frequent.

The present research has been stimulated by the work of K.R. Meyer [1] on two dimensional symplectic diffeomorphisms, to whom the author is indebted for valuable discussions. Similar problems have been studied by J. Sotomayor [2] whose work deals with two-dimensional flows. His setting of the problem and results are of a somewhat different character.

§ 1

Denote by $Z_k = Z_k(f) \subset P \times M$ the set of all k-periodic points of f, i.e. $Z_k = \{(p, m) \mid f_p^k(m) = m,$ $f_p^j(m) \neq m$ for $0 < j < k\}$. In this section, we shall study the sets Z_k. k will be called the prime period of a point $(p, m) \in Z_k$.

A closed subset Q of $P \times M$ will be called invariant, if $\{(p, f(p,m)) \mid (p,m) \in Q\} \subset Q$ and $\{(p, f_p^{-1}(m)) \mid$ $\mid (p,m) \in Q\} \subset Q$. By the orbit of a point (p, m) we shall understand the set of all points $(p, f_p^k(m))$, k integer.

Lemma 1. For every f from a certain open and dense subset $\mathcal{F}_1'$ of $\mathcal{F}$, Z_1 is a closed one-dimensional submanifold of $P \times M$.

Proof. It is obvious that Z_1 is closed. Associate with every $f \in \mathcal{F}$ a mapping $F : P \times M \to P \times M$ given by $F(p, m) = (m, f(p, m))$. Then, $Z_1 = F^{-1}(\Delta)$ where Δ is the diagonal in $M \times M$ and by the transversality theorems 18.2, 19.1 of [3], the set of f's for which F meets Δ transversally, is open and dense in $\mathcal{F}$. The statement of the lemma follows by the implicit function theorem.

Denote by X_1 the set of those points $(p,m) \in Z_1$ for which $df_p(m) - id$ (or, $dF(p,m)$) is singular (i.e. at least one eigenvalue of $df_p(m)$ is equal 1). Further, denote by $j = j_p \times j_M$ the imbedding of Z_1 into $P \times M$. From the implicit function theorem it follows that X_1 is exactly the set of those points $x \in Z_1$ for which $Tj_p(x)$ meets the submanifold $(TP)_0$ of those points from TP satisfying $dp = 0$.

<u>Lemma 2.</u> For every f from an open and dense subset $\mathcal{F}_1''$ of $\mathcal{F}_1'$, $Tj_p(x)$ meets $(TP)_0$ transversally.

<u>Corollary 1</u>. For $f \in \mathcal{F}_1''$, if $(p,m) \in X_1$, then there is a coordinate neighbourhood $(W, \mu \times x)$, $W = U \times V$, of (p,m) such that $\mu \times x\,(p,m) = (0,0)$, $Z_1 \cap W$ can be parametrized by x_1, i.e. $(\mu \times x)(Z_1 \cap W) = \{(\mu, x) \mid \mu = \varphi_0(x_1),$ $x_i = \varphi_i(x_1),\ 2 \leq i \leq n,\ x_1 \in \mathcal{J}\}$ where φ is C^k, $0 \in \mathcal{J}$, $\mathcal{J}$ is an interval, and $\dfrac{d^2\varphi_0}{dx_1}(0) > 0$. (The last inequality is the coordinate representation of the transversality condition of Lemma 2.)

Based upon this corollary, we shall call the points of X_1 collapsation (fixed) points. Namely, there are exactly two points in $Z_1 \cap W$ with fixed $\mu > 0$ small enough; these points collapse at $\mu = 0$ and disappear for $\mu < 0$.

Corollary 2. For every $f \in \mathcal{F}_1''$, the fixed points of f_p are isolated for every $p \in P$.

Corollary 3. For $f \in \mathcal{F}_1''$, X_1 is discrete.

Proof of Lemma 2. Openness. Assume $f \in \mathcal{F}_1'$. We cover Z_1 by a countable number of coordinate neighbourhoods $(U_\alpha \times V_\alpha, \mu_\alpha \times x_\alpha)$. Using the implicit function formula for second derivatives, we can express the transversality condition of Lemma 2 by inequalities $\pi_\alpha \neq 0$, where π_α are polynomials in $(\mu_\alpha \times x_\alpha) \circ f \circ (\mu_\alpha \times x_\alpha)^{-1}$ and its first and second derivatives. Restricting suitably the coordinate neighbourhoods, we can assume that $|\pi_\alpha|$ are bounded away from zero by positive constants ε_α . If $\tilde{f}$ is close enough to f (in the C^r Whitney topology), $Z_1(\tilde{f})$ will be contained in $\bigcup_\alpha (U_\alpha \times V_\alpha)$ and $\pi_\alpha(\tilde{f})$ will be non zero on $U_\alpha \times V_\alpha$. Consequently, $Z_1(\tilde{f})$ will satisfy the transversality condition.

For the proof of density, we first prove the following lemma:

Lemma 3. Denote $B_q(\varepsilon) = \{x \in R^q \mid |x| < \varepsilon\}$, $|\cdot|$ being the Euclidean norm. Let $f \in \mathcal{F}_1'$ and let $(W, \mu \times x)$, $W = U \times V$ be a coordinate neighbourhood in $P \times M$ such that $\mu(U) = B_1(1)$, $x(V) = B_n(1)$ and $W \cap Z_1$ is connected. Denote $W_i = U_i \times V_i = (\mu \times x)^{-1}[B_1(i/3) \times B_n(i/3)]$, $i = 1, 2$. Then, in any neighbourhood Q of f in $\mathcal{F}_1'$ there is an $\tilde{f}$ which coincides with f outside W and such that $T_p(Z_1(\tilde{f}) \cap W_1)$ meets $(TP)_0$ transversally, $T_p(\cdot)$ being the projection of $T(\cdot)$ into TP .

Proof. Denote by $\mathcal{G}$ the set of all C^r maps of $Z_1 \cap W$ into $\mathcal{U}$, $\hat{\mathcal{G}} = \{Tg \mid g \in \mathcal{G}\}$. We consider $\hat{\mathcal{G}}$ as a submanifold of the Banach manifold $\mathcal{G}$ of all C^r maps $T(Z_1 \cap W) \to TP$. By Theorem 19.1 of [3], there is a $\gamma \in \mathcal{G}$, arbitrary C^r-close to j_p such that $T\gamma$ meets $(TP)_0$ transversally. In particular, γ can be chosen so that $|\mu \circ \gamma - \mu \circ j_p| \leq 1/4$. Let φ be a C^r bump function such that $\varphi = 1$ on W_1, $\varphi = 0$ on $W \setminus W_2$. Define $g(x) = \mu^{-1}(\mu \circ j_p + \varphi \circ (\gamma \times j_M) \cdot (\mu \circ \gamma - \mu \circ j_p))$. Then, g meets $(TP)_0$ transversally in W_1 and coincides with j_p outside W_2.

Since W is isomorphic with a subset of R^{n+1} and $(\mu \times x)(Z_1 \cap W)$ is a C^r curve in R^{n+1}, there is a C^r tubular neighbourhood of $Z_1 \cap W$, $h : Z_1 \cap W \times B_n(1) \to W$ such that $h(x, 0) = j(x)$ (for the concept of tubular neighbourhood cf.[4]). This tubular neighbourhood can be constructed e.g. so that $(\mu \times x) \circ h(x, B_n(1))$ lies in the n-hyperplane passing through $(\mu \times x)(x)$ and orthogonal to the tangent to $(\mu \times x)(Z \cap W)$ at $(0,0)$.

Denote π_1, π_2 the natural projections of $Z_1 \cap W \times B_n(1)$ into $Z_1 \cap W$ and $B_n(1)$ respectively, $\psi : R^n \to R$ a C^r bump function such that $\psi = 1$ on $B_n(1/2)$ and $\psi = 0$ outside $B_n(1)$. We define

$$\tilde{f}(p,m) = f(\mu^{-1}(\mu(p,m) + \psi \pi_2 h^{-1}(p,m) \cdot [\mu g \pi_1 h^{-1}(p,m) - \mu j_p \pi_1 h^{-1}(p,m)], m) \text{ for } (p,m) \in h[(Z_1 \cap W) \times B_n(1)],$$

$\tilde{f}(p,m) = f(p,m)$ elsewhere.

Then, $Z_1(\tilde{f}) \cap W = (g \times j_M)(Z_1(f) \cap W)$, $\tilde{f}$ coincides with f outside $\mathcal{U}$ and $\tilde{f}$ can be made arbitrary close to f by choosing g sufficiently close to j_p. This proves the lemma.

To prove the density part of Lemma 2, we find a countable family of coordinate neighbourhoods $(W_\alpha, \mu_\alpha \times j_\alpha)$ in such a way that every $(W_\alpha, \mu_\alpha \times j_\alpha)$ satisfies the assumptions of Lemma 3 and $Z_1(f) \subset \bigcup_\alpha W_{\alpha 1}$ (the subscript 1 used as in Lemma 3). Then, we apply Lemma 3 stepwise for every α and choose the approximation of f at every step so close that the transversality condition is not destroyed in $\bigcup_{\beta<\alpha} \mathcal{U}_{\beta 1} \cap \mathcal{U}_{\alpha 1}$. This is possible due to the first part of the proof.

The next lemma examines the behaviour of f in the neighbourhood of a collapsation point.

<u>Lemma 4</u>. For every f from an open and dense subset $\mathcal{F}_1'''$ of $\mathcal{F}_1''$, the following is true:

(a) for every $(p_0, m_0) \in X_1$, one eigenvalue of $df_{p_0}(m_0)$ is 1, the moduli of the others being different from 1,

(b) locally, (p_0, m_0) divides $Z_1 \setminus \{(p_0, m_0)\}$ into two components and the number of eigenvalues of df_p with modulus 1 at points from different components of $Z_1 \setminus \{(p_0, m_0)\}$ differs by 1.

(c) There is a neighbourhood W of (p_0, m_0) such that

$W \setminus Z_1$ contains no invariant set.

Proof. Since $(p_0, m_0) \in X_1$, $df_{p_0}(m_0)$ has 1 as an eigenvalue. This eigenvalue is simple because of Lemma 1.

If $(p_0, m_0) \in X_1$ and $f \in \mathcal{F}_1''$, then there is a coordinate neighbourhood $(W, \mu \times x)$ of (p_0, m_0), $W = U \times V$ such that $(\mu \times x)(p_0, m_0) = (0, 0)$ and f can be in these coordinates represented by

$$(1) \quad x_1' = x_1 + \alpha\mu + \beta x_1^2 + \omega(\mu, x_1, y) ,$$

$$(2) \quad y' = Ay + \chi(\mu, x_1, y)$$

where $y = (x_2, \dots, x_m)$, the primed coordinates are those of the images, $\alpha < 0$,

$$(3) \quad \chi(0,0,0) = 0, \omega(\mu, x, 0) = \sigma(|\mu| + x_1^2) .$$

Note that from the form of (2) it follows that every fixed point in W satisfies $y = 0$ (W possibly restricted).

We denote by $\mathcal{F}_1'''$ the set of all $f \in \mathcal{F}_1''$, in the representation (1),(2) of which (i) $\beta \neq 0$ and (ii) the eigenvalues of A have moduli $\neq 1$. It is obvious that the meaning of these conditions is independent of the choice of coordinates. Also, (ii) is equivalent with (a). We show that $\mathcal{F}_1'''$ is open dense.

Openness follows easily from the continuous dependence of the eigenvalues on f. To prove density, we note that there is a real σ arbitrarily small in absolute value such that $\beta + \sigma \neq 0$ and for any eigenvalue λ of $df_{p_0}(m_0)$, $|\lambda + \sigma| \neq 1$. We change f into $\tilde{f}$ by changing the terms Ay and βx_1^2 in the representation (1),(2) of f into $(A + \psi(\mu, x)\sigma E)y$ and $(\beta + \psi(\mu,x)\sigma)x_1^2$ (E being the unity matrix) respectively, where $\psi(\mu, x)$ is a C^r bump function vanishing outside W, and equal 1 at $(0,0)$. By the choice of a sufficiently small σ, $\tilde{f}$ can be made sufficiently close to f. $d\tilde{f}_p(m)$ will then satisfy (a) and we do not introduce any new fixed points. Since X_1 is discrete for $f \in \mathcal{F}_1'$, this proves the density of $\mathcal{F}_1'''$.

To prove (b) we note that if f satisfies (a), only one eigenvalue can cross the unit circle at (p_0, m_0) and this eigenvalue is the eigenvalue of the restriction of df_p to the manifold $y = 0$, $df_p|_{y=0}$. This mapping is represented by (1) with $y = 0$.

Assume $\beta > 0$ (in the other case we change the sign of x_1). To prove (c), we note first that A is similar to a matrix $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, i.e. there is a nonsingular matrix Q such that $Q^{-1}AQ = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where the moduli of eigenvalues of B and C are < 1 and > 1 respectively. Applying first the linear coordinate transformation $y = Q\binom{u}{z}$ and then $z = w^{*+}(x_1, u) + \zeta$
$u = w^{*-}(x_1, \zeta) + \eta$ where $z = w^{*+}(x_1, u)$ and $u = w^{*-}(x_1, \zeta)$ (w^{*+}, w^{*-} being C^r) are the equations of the center-stable and center-unstable mani-

folds respectively (cf.[3], Appendix C [(1)], (1) and (2) is transformed into

(4) $\xi' = \xi + \alpha\mu + \beta\xi^2 + \Xi(\mu, \xi, \eta, \zeta)$,

(5) $\eta' = B\eta + \Theta(\mu, \xi, \eta, \zeta)$,

(6) $\zeta' = C\zeta + \Omega(\mu, \xi, \eta, \zeta)$

where $\alpha < 0, \Xi, \Theta, \Omega$ are C^k and

(7) $\Theta(\mu, \xi, 0, \zeta) = 0, \Omega(\mu, \xi, \eta, 0) = 0$,

$$\Xi(\mu, \xi, \eta, \zeta) = o(|\mu| + \xi^2), d\Xi(0,0,0,0) =$$
$$= 0, d\Theta(0,0,0,0) = 0, d\Omega(0,0,0,0) = 0 .$$

From (5) and (7) it follows that the orbit of every point (p, m) which is contained entirely in some sufficiently small neighbourhood of (p_0, m_0) satisfies $\eta(f_p^k(m)) \to 0$ for $k \to \infty$ and $\zeta(f_p^k(m)) \to 0$ for $k \to -\infty$. Thus, if there is an invariant set contained in this neighbourhood, it must be a part of the manifold $\eta = 0$, $\zeta = 0$. In particular, this implies

(8) $\eta(Z_1 \cap W) = 0$ $\zeta(Z_1 \cap W) = 0$

(W possibly restricted).

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(1) Actually, Appendix C in [3] deals with flows rather than mappings. Therefore, in order to use its results directly, we have to construct a flow from f as in [5] and then return to f by considering the cross-section mapping.

We therefore consider the restriction of f to the center manifold $\eta = 0$, $\zeta = 0$, the representation of which is given by

(9) $\xi' = \xi + \alpha\mu + \beta\xi^2 + \Xi(\mu, \xi, 0, 0)$.

It follows from Corollary 1 and (8) that for $\mu > 0$ fixed, $Z_1 \cap W$ consists of two points $(\mu, \xi_1(\mu), 0, 0)$, $(\mu, \xi_2(\mu), 0, 0)$ satisfying $\xi_1(\mu) < 0$, $\xi_2(\mu) > 0$ and

(10) $\qquad \varkappa_1 \mu^{1/2} \leq |\xi_i(\mu)| \leq \varkappa_2 \mu^{1/2} \qquad i = 1, 2$

for some positive constants $\varkappa_1$, $\varkappa_2$. From (9) and (10) it follows

(11) $\qquad \xi' - \xi > 0 \quad$ for $\mu \leq 0$,

(12) $\qquad \xi_1(\mu) < \xi' < 0 \quad$ for $\mu > 0$, $\xi = 0$,

(13) $\qquad \xi' - \xi > 0 \quad$ for $\mu > 0$, $(-4\alpha\beta^{-1}\mu)^{1/2} < |\xi| < \delta$.

Since $\xi' - \xi$ can change its sign only at fixed points, for $\mu > 0$ from (12),(13) we conclude $\xi_1(\mu) < \xi' < \xi$ for $\xi_1(\mu) < \xi < \xi_2(\mu)$, $\xi' - \xi > 0$ for $\xi > \xi_2(\mu)$. This, together with (11), proves (c).

To prove (b) we note that if $f \in \mathcal{F}_1'''$, then only one eigenvalue of df_p can cross the unit circle at (p_0, m_0) and this eigenvalue is the eigenvalue of the restriction of df_p to the manifold $\eta = 0$, $\zeta = 0$, which is represented by (9). From (13) it follows

$\frac{d\xi'}{d\xi}(\mu, \xi_i(\mu)) = 1 + 2\beta\xi_i + o(\xi_i)$ which implies

$\frac{d\xi'}{d\xi}(\mu, \xi_1(\mu)) < 1, \frac{d\xi'}{d\xi}(\mu, \xi_2(\mu)) > 1$ for small $\mu > 0$.

This completes the proof.

We summarize the results of Lemmas 1 - 4 together with their generalization for periodic points with higher prime period in the following theorem.

Denote $X_k = Z_k \cap X_1(f^k)$.

Theorem 1. For every f from a residual subset $\mathcal{F}_1 \subset \mathcal{F}$:

(i) Z_k are 1-dimensional submanifolds of $P \times M$; Z_1 is closed;

(ii) for fixed p , the k-periodic points of f_p are isolated;

(iii) X_k is discrete;

(iv) for every $(p, m) \in Z_k \setminus X_k$, there is a neighborhood $W = U \times V$ of (p, m) and a C^r function $\varphi :$ $: U \to V$ such that $Z_k \cap W$ is the graph of φ ;

(v) for every $(p_0, m_0) \in X_k$, there is a coordinate neighbourhood $(W; \mu \times x)$ of (p_0, m_0), $(\mu \times x)(p_0, m_0) =$ $= (0, 0)$ such that

(a) there is a C^r function $\psi : U_1 \to W$, $U_1 \subset R$ open, such that $Z_k \cap W = \{\psi(x_1) \mid x_1 \in U_1\}$, $x_1 \circ \psi = id$, $\frac{d^2 \mu \circ \psi}{d x_1^2}(0) > 0$;

(b) $df_p^k(m)$ has one eigenvalue 1, the others having moduli different from 1; the number of eigenvalues with moduli > 1 in the components $x_1 > 0$, and $x_1 < 0$ of $Z_k \cap W$ is constant and differ by one;

(c) $W \setminus Z_k$ contains no invariant set.

Proof. The statement for $k = 1$ is proven in Lemmas 1 - 4. To prove the rest, we denote by $\mathcal{F}_{1\ell}(U)$ the set of all $f \in \mathcal{F}$ such that $f|_U$ satisfies (i) - (v) for $1 \leq k \leq \ell$.

Let d be a C^r Riemannian metric on $P \times M$, $\{K_\sigma\}$ an increasing sequence of compact sets, $\bigcup_\sigma K_\sigma = P \times M$. Denote $B(N, \delta) = \{(p,m) \mid d(N,(p,m)) < \delta\}$ for $N \subset P \times M$. We show that the sets $\hat{\mathcal{F}}_{j\ell} = \mathcal{F}_{1j}(K_\ell \setminus B(\bigcup_{k<j} Z_k, \ell^{-1}))$ are open and dense. Since $\mathcal{F}_1 = \bigcap_{\ell,j} \hat{\mathcal{F}}_{j\ell}$, this will complete the proof.

To prove density, we cover $Z_1 \cap K_\ell \setminus B(\bigcup_{k<j} Z_k, \ell^{-1})$ by a countable family $\{W_i\}$ of open sets such that $\overline{W}_i \cap f(\overline{W}_i) \cap \ldots \cap f^{j-1}(\overline{W}_i) = \emptyset$ and $W_i \cap Z_k = \emptyset$, $k < j$. Using Lemmas 1 - 4 we find that f^j can be arbitrarily closely approximated by a map h such that $h \in \mathcal{F}_1(W_i)$ and h coincides with f^j outside W. We denote

$$\tilde{f} = \begin{cases} f^{1-j} h & \text{on } W_i, \\ f & \text{outside } W_i. \end{cases}$$

Then, if h is close enough to f^j, $W_i \cap \tilde{f}(W_i) \cap \ldots \cap \tilde{f}^{j-1}(W_i) = \emptyset$, $\tilde{f}^j = h$ and, therefore, $\tilde{f} \in \mathcal{F}_{1\ell}(W_i)$.

Repeating this for every i and taking into account the openness of $\mathcal{F}_1(W_i)$, one concludes the proof of density of $\hat{\mathcal{F}}_{j\ell}$.

For the proof of openness we note that since $K_\ell \setminus B(\bigcup_{k<j} Z_k, \ell^{-1})$ is compact, from $f \in \hat{\mathcal{F}}_{j\ell}$ it follows $f \in \mathcal{F}_{1j}(K_\ell \setminus B(\bigcup_{k<j} Z_k, \ell^{-1} - \sigma))$ for some small $\sigma > 0$.

If $\tilde{f}$ is close enough to f, $\bigcup_{k<j} Z_k(\tilde{f}) \in B(\bigcup_{k<j} Z_k(f), \sigma)$.

Thus,

(14) $B(\bigcup_{k<j} Z_k(\tilde{f}), \ell^{-1}) \supset \overline{B(\bigcup_{k<j} Z_k(f), \ell^{-1} - \sigma)}$.

The openness of $\hat{\mathcal{F}}_{j\ell}$ follows now from (14), Lemmas 1 - 4 and the fact that $\tilde{f}^j$ is arbitrarily close to f^j if $\tilde{f}$ is close enough to f.

Remarks. 1. In case $n = 2$, the points of one component of $Z_k \cap W \setminus \{(p_0, m_0)\}$ are saddles, the points of the other are either sources or sinks.

2. The set $\mathcal{F}_{11}$ of those $f \in \mathcal{F}$ satisfying (i) - (v) of Theorem 1 for $k = 1$ is open dense in $\mathcal{F}$.

§ 2.

The sets Z_k for $k > 1$ are not closed in general. A point from $\overline{Z}_k \setminus Z_k$ is also a periodic point, its prime period being a divisor of k. We shall call the points of $\overline{Z}_k \setminus Z_k$ branching (ℓ-periodic, according to their prime period) points. In this section, we shall study the behaviour of f in the neighbourhood of branching points in the case $n = 2$ which allows us to obtain some information about the sets $\overline{Z}_k$.

If $f \in \mathcal{F}_1$, a k-periodic point (p, m) can be a branching point only if $df_p^k(m)$ has some root of unity different from 1 as an eigenvalue. For, if $df_p^k(m)$ has no root of unity as an eigenvalue, $df_p^k(m) - id$ is regular for every $\nu > 0$ and by the implicit function theorem there is a unique C^r 1-dimensional submanifold of periodic points with (not necessarily prime) period νk, $\nu > 0$; thus, this manifold coincides with Z_k for every $\nu > 0$. The case of 1 being an eigenvalue is covered by Theorem 1.

Therefore, we need first to know how the eigenvalues cross the unit circle if p is changed, in the generic case.

Henceforth we shall assume $n = 2$ without repeating it. Let $f \in \mathcal{F}_1$ and denote $D_k = \{(p, m) \in Z_k \mid df_p(m)$ has double eigenvalues$\}$.

From the implicit function theorem it follows that the eigenvalues $\lambda_1^{(k)}$, $\lambda_2^{(k)}$ of $df_p^k(m)$ are C^r functions on $Z_k \setminus D_k$.

Denote by S the unit circle in the complex plane.

Theorem 2. For a residual subset $\mathcal{F}_2$ of $\mathcal{F}$, $\mathcal{F}_2 \subset \mathcal{F}_1$:

(i) $\lambda_i^{(k)}(D_k) \cap S = \emptyset$, $i = 1, 2$,

(ii) $\lambda_i^{(k)}$, $i = 1, 2$ meet S transversally.

(iii) If, for some $(p,m) \in Z_k$, $\lambda_1^{(k)}(p,m) \in S$, then either $\lambda_2^{(k)} \notin S$ or $\lambda_1^{(k)}(p,m)$ is not a root of 1.

Corollary. Generically, (p,m) can be a branching point only if one of the eigenvalues of $df_p^k(m)$ is -1, the other being real $\neq 1$. We denote by Y_k the set of such points.

Proof of Theorem 2. We prove the statement of the theorem for $k = 1$ (fixed points), the generalization to the case $k > 1$ being similar as in the proof of Theorem 1.

From Theorem 1, (vc) and its proof it follows that for every $f \in \mathcal{F}_1$, if some eigenvalue meets S at 1, it is single and meets S transversally. Therefore, we can restrict our attention to $S \setminus \{1\}$.

Let $f \in \mathcal{F}_{11}$, where $\mathcal{F}_{11}$ is defined at the end of § 1, $(p,m) \in Z_1 \setminus X_1$. Then, according to Theorem 1, (iv), there is a coordinate neighbourhood $(W, \mu \times x)$, $W = U \times V$ such that $\mu(p) = 0$, $x(m) = 0$ and the representation of f in these coordinates is given by

$$x' = A(\mu)x + \Omega(\mu, x) ,$$

where $\Omega(\mu, 0) = 0$, $d\Omega(0,0) = 0$.

The subset of matrices with both eigenvalues on the unit circle is a submanifold $\mathcal{U}$ of co-dimension 1 in GL (2) (it is the set of matrices A such that $\det A = 1$). Further, the set of all 2×2 matrices with eigenvalues being ℓ -th roots of unity (the unity matrix E excluded), $\mathcal{M}_\ell$ is a 2-dimensional submanifold of GL (2), given by $\det A = 1$, $\operatorname{tr} A = \alpha_j + \alpha_j^{-1}$ for ℓ odd, and a union of the 2-dimensional manifold given as for ℓ odd and the isolated matrix $-E$ for ℓ even, where α_j are the ℓ -th roots of unity, lying in the open upper complex halfplane.

Using the elementary transversality theorem, we can approximate the function $A : \mu(\mathcal{U}) \longrightarrow GL(2)$ arbitrarily closely by $\tilde{A} : \mu(\mathcal{U}) \longrightarrow GL(2)$ so that

$\tilde{A}$ coincides with A outside $\mathcal{U}_1$, $\overline{\mathcal{U}}_1 \subset \mu(\mathcal{U})$,
$\tilde{A}$ meets $\mathcal{M}$ transversally and does not meet $\mathcal{M}_\ell$ at all for $\mu \in \mathcal{U}_2$, $\mathcal{U}_2$ open, $\overline{\mathcal{U}}_2 \subset \mathcal{U}_1$. As a consequence we obtain that $\tilde{A}(\mu)$ does not have -1 as double eigenvalue for any $\mu \in \mathcal{U}_2$. This implies that the eigenvalues λ_1, λ_2 are C^r functions of matrices in the neighbourhood of any $A(\mu)$, some eigenvalue of which is -1.

Therefore, in the neighbourhood of the values of $\tilde{A}(\mu)$,

$\mu \in \mathcal{U}_2$, the subsets of GL (2), given by $\lambda_1 = -1$ and $\lambda_2 = -1$ are submanifolds of co-dimension 1. Thus, we can use the transversality theorem again (for $\tilde{A}$ and $\mathcal{M}_\ell$) to obtain that arbitrarily near $\tilde{A}$ (and, thus, A) there is a function $\tilde{\tilde{A}} : \mu(\mathcal{U}) \longrightarrow GL(2)$ such that for $\mu \in \mathcal{U}_2$, -1 is not double eigenvalue of $\tilde{\tilde{A}}(\mu)$ and the eigenvalues λ_1 and λ_2 cross the unit circle transversally at the points which are not ℓ -th roots of unity.

Let V_2, V_1 be open, $\overline{V}_2 \subset V_1$, $\overline{V}_1 \subset V$, let $\varphi(\mu, x)$ be a bump function such that $\varphi(\mu, x) = 1$ for $(\mu, x) \in U_2 \times V_2$, $\varphi(\mu, x) = 0$ outside $U_1 \times V_1$. We denote by f the map that coincides with f outside $U \times V$ and is given in W by its coordinate representation

$$x' = [A(\mu) + \varphi(\mu, x)(\tilde{A}(\mu) - A(\mu))]\ x + \Omega(\mu, x) \ .$$

Then, if A is chosen close enough to A, f is arbitrarily close to f, satisfies (i),(ii) and

(iii$_\ell$) if $\lambda_1 \in S$, then either $\lambda_2 \notin S$, or λ_1 is not an ℓ-th root of unity, in $U_2 \times V_2$.

As usual, we can prove that f can be approximated by a function $\tilde{f}$ having Properties (i),(ii),(iii$_\ell$) over all $Z_1 \setminus X_1$ by covering $Z_1 \setminus X_1$ by a countable family of coordinate neighbourhoods. It is obvious that the set of f's, having Properties (i),(ii),(iii) is open.

Since the subset $\mathcal{F}_{21} \subset \mathcal{F}$ of maps, having Properties (i),(ii),(iii) for $k = 1$ is the intersection of the sets $\mathcal{F}_{21\ell} \subset \mathcal{F}$, satisfying (i),(ii),(iii$_\ell$), the proof of Theorem 2 for $k = 1$ is completed.

<u>Remark</u>. Note that the subset $\mathcal{F}_{2k\ell} \subset \mathcal{F}$ of maps, all iterates up to order k of which satisfy (iii$_\ell$), is open dense in $\mathcal{F}$.

We shall now study the behaviour of f in the neighbourhood of a branching point.

__Theorem 3__. Assume $r \geq 3$. Then, for a residual subset $\mathcal{F}_3$ of $\mathcal{F}$, $\mathcal{F}_3 \subset \mathcal{F}_2$, the following is valid:

(i) Y_k coincides with the set of k-periodic branching points.

(ii) For every $(p_o, m_o) \in Y_k$ there is a coordinate neighbourhood $(W, \mu \times x)$, $W = \mathcal{U} \times V$ of (p_o, m_o) such that $\mu(p_o) = 0$, $x(m_o) = 0$, $Z_k \cap W = \mathcal{U} \times \{0\}$ and

(a) $Z_{2k} \cap W$ consists of two components, separated by (p_o, m_o); all points of $Z_{2k} \cap W$ satisfy $\mu > 0$ and $Z_{2k} \cap W \cup \{(p_o, m_o)\}$ is a C^1 (but not C^2) submanifold of W.

(b) Either the points of $Z_k \cap W$ are sinks for $\mu > 0$ saddles for $\mu \leq 0$ (degenerated for $\mu = 0$), and the points of $Z_{2k} \cap W$ are saddles, or the same is true with sink replaced by saddle and conversely, or one of the above cases is true for the inverse of f.

(c) $W \setminus (Z_k \cup Z_{2k})$ contains no invariant set of f_p^k.

__Proof__. We again prove the theorem for $k = 1$, the generalization for $k > 1$ being similar as in the proof of Theorem 1.

Assume $f \in \mathcal{F}_{21}$. Then, one eigenvalue of $df_{p_o}(m_o)$ is -1, the other, λ, is not on S. We can assume $|\lambda| < 1$, in the other case we consider the inverse of f. As in the proof of Theorem 1, using [3], Appendix C, we find that there is a coordinate neighbourhood

$(W, \mu \times x)$, $W = U \times V$, $(\mu \times x)(p_0, m_0) = (0,0)$ such that the local representation of f in the coordinates μ, x is given by

$$(15) \quad x_1' = -x_1 + \alpha\mu x_1 + \beta x_1^2 + \gamma x_1^3 + \omega(\mu, x_1, x_2),$$

$$(16) \quad x_2' = \lambda x_2 + \vartheta(\mu, x_1, x_2),$$

where ω, ϑ are C^r and

$$(17) \quad \vartheta(\mu, x_1, 0) = 0, \; d\vartheta(0,0,0) = 0, \omega(\mu, x_1, x_2) = \\ = (|x_1^3| + |\mu x_1| + |x_2|) \; .$$

Similarly, as in the proof of Lemma 4, it can be shown that every f can be arbitrarily closely approximated in $\mathcal{F}_{21}$ by a map the local representation of which satisfies $\beta^2 + \gamma \neq 0$ at every point from Y_1. We denote $\mathcal{F}_{31}$ the set of such maps. The openness of $\mathcal{F}_{31}$ is obvious.

We prove that if $f \in \mathcal{F}_{31}$ then f satisfies (i), (ii), of this theorem for $k = 1$. We shall analyze the case $\alpha > 0$, $\beta^2 + \gamma < 0$. The other cases can be transformed to the above case by a suitable change of coordinates or lead to other cases of (ii b), which can be analyzed similarly.

From (15),(16) we obtain the representation of the second iterate of $f|_{x_2=0}$

$$(18) \quad x_1'' = x_1 - 2\alpha\mu x_1 - 2(\beta^2 + \gamma)x_1^3 + \omega_2(\mu, x_1),$$

where $\omega_2(\mu, x) = (|\mu x_1| + |x_1^3|)$. By a change of variables $x_1 = \nu^2 \xi$, $\mu = \nu^2$ for $\mu > 0$, (18) is transformed into

$$(19) \quad \xi'' = \xi - 2\nu^2 [\alpha \xi + (\beta^2 + \gamma)\xi^3] + \chi(\nu, \xi) \ ,$$

where $\chi(\nu, \xi) = \nu^{-1}\omega_2(\nu^2, \nu\xi)$ is C^{k-1} for $\nu > 0$ and satisfies

$$(20) \qquad \chi(\nu, \xi) = o(\nu^2) \ .$$

ξ is a 2-periodic point of $f_\mu|_{x_2=0}$ for $\nu > 0$ if ξ satisfies

$$(21) \qquad \alpha\xi + (\beta^2 + \gamma)\xi^3 - \chi_1(\nu, \xi) = 0 \ ,$$

where $\chi_1(\nu, \xi) = \nu^2 \chi(\nu, \xi)$. From (20) it follows that if we define $\chi_1(0, \xi) = 0$,

then χ_1 is C^{k-3} for $\nu \geq 0$ and, in the case $k = 3$, that $\dfrac{\partial \chi_1}{\partial \xi}$ is continuous.

For $\nu = 0$, (21) has two non-zero solutions $\xi_1(0) = -[-\alpha(\beta^2+\gamma)^{-1}]^{1/2}$, $\xi_2(0) = [-\alpha(\beta^2+\gamma)^{-1}]^{1/2}$. Using the implicit function theorem of [6] and returning to the coordinates μ, x_1 we obtain that for $\mu > 0$ sufficiently small there are two 2-periodic points (1 orbit) of $f_\mu|_{x_2=0}$ with coordinates

$$(22) \quad x_{11}(\mu) = -[-\alpha(\beta^2 + \gamma)^{-1}\mu]^{1/2} + \psi_1(\mu) \ ,$$

$x_{12}(\mu) = [-\alpha(\beta^2+\gamma)^{-1}\mu]^{1/2} + \psi_2(\mu)$,

where ψ_1, ψ_2 are C^{k-3} and satisfy $\psi_i(\mu) = o(\mu^{1/2})$; the eigenvalue of $df_\mu^2|_{x_2=0}$ at the points x_{11}, x_{12} is equal $1 + 4\alpha\mu + o(\mu)$. Since from (16) it follows that the other eigenvalue of df_μ^2 at the points $(\mu, x_{11}(\mu), 0), (\mu, x_{12}(\mu), 0)$ is of modulus less than one, this proves that the points $(\mu, x_{11}(\mu), 0), (\mu, x_{12}(\mu), 0)$ are saddles for small μ. From (15),(16) it follows further that for small $|\mu|$, the points of Z_1 are sinks for $\mu > 0$ and saddles for $\mu < 0$. This proves (ii b) if we show that $Z_2 \cap W$ (W possibly restricted) does not contain other points except of the points $(\mu, x_{1i}(\mu), 0)$, $i = 1, 2$.

From (16),(17) it follows that every orbit that remains in $|x| < \delta$ (δ sufficiently small independent of μ for $|\mu|$ small), approaches the submanifold $x_2 = 0$ (in the positive sense). Therefore, in order to prove (ii c) and thus also to complete the proof of (ii b) it suffices to prove that for sufficiently small μ the only periodic points of $f_\mu|_{x_2=0}$ for $|x_2| < \delta_1 < \delta$, δ_1 sufficiently small, are the points $x_{1i}(\mu)$, $i = 1, 2$, and 0.

From (17) it follows that

(23) $x_1'' - x_1 < 0$ for $\mu \leq 0$, $x_1 < 0$,

(24) $x_2'' - x_1 > 0$ for $\mu \leq 0,\ x_1 > 0$,

(25) $x_1'' - x_1 > 0$ for $\mu > 0$,

$$x_1 > [-4\alpha(\gamma+\beta^2)^{-1}\mu]^{1/2},$$

(26) $x_1'' - x_1 < 0$ for $\mu > 0$,

$$x_1 < -[-4\alpha(\gamma+\beta^2)^{-1}\mu]^{1/2},$$

and $|\mu| < \delta_2$, $|x_1| < \delta_2$, δ_2 being sufficiently small. From (23),(24), it follows that the orbit of every point with $0 > \mu > -\delta_2$, $|x_1| < \delta_2$ leaves $|x_1| < \delta_2$. From (22),(23),(24) and the implicit function argument used after (21) it follows that there are no periodic points with $|x_1| < [-4\alpha(\beta^2+\gamma)^{-1}\mu]^{1/2}$ except of the points $x_{11}(\mu)$, $x_{12}(\mu)$. From this, (25), (26) and (19) it follows $x_1'' - x_1 < 0$ for $\delta_2 < x_1 < x_{11}(\mu)$ or $0 < x_1 < x_{12}(\mu)$ and $x_1'' - x_1 > 0$ for $x_{11}(\mu) < x_1 < 0$ or $x_{12}(\mu) < x_1 < \delta_2$, $\mu > 0$, so that every orbit both in the positive and negative sense tends to one of the points 0, $x_{11}(\mu)$, $x_{12}(\mu)$. This completes the proof of (iv c).

To complete the proof of (ii a), we denote by $\varphi(x_1)$ the real function, defined as the inverse of the functions $x_1 = x_{11}(\mu)$ for $x_1 < 0$ and $x_1 = x_{12}(\mu)$ for $x_1 > 0$. From (22) it follows

(27) $$\lim_{x_1\to 0_-}\varphi(x_1) = \lim_{x_1\to 0_+}\varphi(x_1) = \frac{d\varphi^+}{dx_1}(0) = \frac{d\varphi^-}{dx_1}(0) = 0.$$

Further, from the fact that the points $(\mu, x_{11}(\mu), 0)$ $(\mu, x_{12}(\mu), 0)$ are nondegenerated for $\mu > 0$ it follows that φ is C^{κ}. Using (22) and the implicit function theorem we obtain

$$(28)\quad \frac{d\varphi}{dx_1} = -[\alpha^{-1}(\beta^2+\gamma)\mu]^{1/2} + \sigma(\mu^{1/2}) \quad \text{for } x_1 < 0,$$

$$\frac{d\varphi}{dx_1} = [-\alpha^{-1}(\beta^2+\gamma)\mu]^{1/2} + \sigma(\mu^{1/2}) \quad \text{for } x_1 > 0.$$

This, together with (27) shows that φ can be completed into a C^1 function (which is not C^2) in some neighbourhood of 0 by defining $\varphi(0) = 0$.

As a corollary of Theorem 1 and 3 we obtain

Theorem 4. Let $\kappa > 2$. Then for every $f \in \mathcal{F}_3$:

(i) for k odd, Z_k is a closed submanifold of $P \times M$,

(ii) for k even, $\overline{Z}_k$ is a closed C^1 (but not C^2) submanifold of $P \times M$; $\overline{Z}_k \setminus Z_k$ is discrete and coincides with $Y_{k/2}$.

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