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ON CNE-PARAMETER FAMILIES OF DIFFEOMORPHISMS II: GENERIC BRANCHING IN HIGHER DIMENSIONS

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§ 1

In [1], we have studied the generic nature of the loci of periodic points of a diffeomorphism of a finite dimensional manifold M, depending on a parameter with values in a one dimensional manifold P, in $P \times M$. A part of the results (those concerning the branching of periodic points), we have proved for two dimensional M only. It is the purpose of this paper to extend these results for M of arbitrary finite dimension.

Since this paper is a direct continuation of [1], we shall frequently refer to [1] for results of technical character as well as techniques of proof. Nevertheless, for the sake of the reader's convenience, we re-introduce those concepts and results of [1] which are necessary for the understanding of this paper, in the rest of this section. The main results of this paper and their proofs are given in § 3. § 2 has an auxiliary character; it establishes certain generic properties of maps of an interval into the

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set of matrices.

Denote $\mathcal{F}$ the space of C^r mappings $(1 < r \leq \infty)$ x) $f : P \times M \to M$, where P, M are C^r second countable manifolds of dimension 1, $m < \infty$ respectively, such that for every $p \in P$ the map $f_p : M \to M$, given by $f_p(m) = f(p, m)$ is a diffeomorphism, endowed with the C^r Whitney topology.

Let us note that, although this topology is not metrizable, it has the property that a residual set in $\mathcal{F}$ (i.e. a countable intersection of open dense sets) is dense in $\mathcal{F}$ (this can be proved similarly as the analogous statement for vector fields is proved in [2], using the openness of $\mathcal{F}$ in the set of all C^r mappings $P \times M \to M$).

Denote by $Z_k = Z_k(f)$ the set of k-periodic points of f, i.e. $Z_k(f) = \{(p, m) \mid f_p^k(m) = m$, $f_p^j(m) \neq m$ for $0 < j < k\}$. In [1, Theorem 1] a residual subset $\mathcal{F}_1$ of $\mathcal{F}$ was defined and it was shown that for every $f \in \mathcal{F}_1$, Z_k are one dimensional submanifolds of $P \times M$ (Z_1 being closed) and, if an eigenvalue of $df_p^k(m)$ at some point $(p, m) \in Z_k$ is 1 (we denote the set of such points by X_k), then it meets the unit circle S in the complex plain transversally at (p, m) (in the sense of Remark 3) and the remaining eigenvalues of $df_p^k(m)$ do not lie on S. Also, it was shown that the subset $\mathcal{F}_h$ of maps from $\mathcal{F}$, having the

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x) In [1] we have assumed $1 < r < \infty$, but Theorems 1 - 4 of [1] are trivially true for the C^∞ case.

above properties for $1 \leq k \leq n$, is open dense in $\mathcal{F}$.

§ 2

Denote by $\mathcal{M}$ the set of all $n \times n$ matrices with the differential structure induced by its natural identification with R^{n^2} . Further, denote by $\mathcal{M}_1$ the set of matrices having an eigenvalue of multiplicity ≥ 2 on S, $\mathcal{F}_{2\ell}$ the set of matrices having an ℓ-th root of unity different from ± 1 as eigenvalue, $\mathcal{M}_2 = \bigcup_{\ell=3}^{\infty} \mathcal{M}_{2\ell}$.

Let I be a closed interval on R . Denote by Φ the space of all C^r mappings $I \longrightarrow \mathcal{M}$ endowed with the C^r uniform topology.

<u>Proposition 1.</u> Let $J \subset I$ be a closed interval, $J \subset int\ I$. Then, for every $\ell = 3, 4, \ldots$ the set $\Psi_\ell(J)$ of all $F \in \Phi$ such that $F(J) \cap (\mathcal{M}_1 \cup \mathcal{M}_2) = \emptyset$ is open dense in Φ .

<u>Corollary 1.</u> Given J as in Proposiion 1, the set $\Psi(J)$ of all $F \in \Phi$ such that $F(J) \cap (\mathcal{M}_1 \cap \mathcal{M}_2) = \emptyset$ is residual in Φ .

For the proof of Proposition 1 we shall need to prove several lemmas.

Consider the sets $\widetilde{\mathcal{M}}_1 = \{(A, \lambda_1, \lambda_2) \in \mathcal{M} \times R^2 \mid P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = P_1'(\lambda_1, \lambda_2) = P_2'(\lambda_1, \lambda_2) = 0, \lambda_1^2 + \lambda_2^2 = 1\}$ and $\mathcal{M}_2(\lambda_{10}, \lambda_{20}) = \{(A, \lambda_1, \lambda_2) \mid P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = 0, \lambda_1 = \lambda_{10}, \lambda_2 = \lambda_{20}\}$, where $P(\lambda_1) = P_1(Re\ \lambda, Im\ \lambda) + i P_2(Re\ \lambda, Im\ \lambda)$ is the characteristic polynomial of

$A, P_1' + i P_2' = P' = \frac{\partial P}{\partial \lambda}$.

Being defined by polynomial equalities, $\widetilde{\mathcal{O}\!\mathcal{L}}_1$ and $\widetilde{\mathcal{O}\!\mathcal{L}}_2(\lambda_{10}, \lambda_{20})$ are real algebraic varieties and the sets $\mathcal{O}\!\mathcal{L}_1, \mathcal{O}\!\mathcal{L}_{2\ell}$ are the projections of $\widetilde{\mathcal{O}\!\mathcal{L}}_1$ and $\bigcup \widetilde{\mathcal{O}\!\mathcal{L}}_2(\lambda_{10}, \lambda_{20})$ into $\mathcal{O}\!\mathcal{L}$ respectively, where the union is taken over all $\lambda_{10}, \lambda_{20}$ such that $(\lambda_{10} + i\lambda_{20})^{\ell} = 1$ and $\lambda_{20} \neq 0$.

By [3, splitting (b) of § 11)], $\widetilde{\mathcal{O}\!\mathcal{L}}_1$ and $\widetilde{\mathcal{O}\!\mathcal{L}}_2$ can be written as a finite disjoint union of submanifolds of strictly decreasing dimensions, $\widetilde{\mathcal{O}\!\mathcal{L}}_1 = \bigcup_{j=1}^{r} \mathcal{M}_j$, $\widetilde{\mathcal{O}\!\mathcal{L}}_2(\lambda_{10}, \lambda_{20}) = \bigcup_{j=1}^{s} \mathcal{N}_j$ such that $\bigcup_{j=\varrho}^{r} \mathcal{M}_j$, $\bigcup_{j=\sigma}^{s} \mathcal{N}_j$ is closed for all $0 < \varrho \leq r$, $0 < \sigma \leq s$.

<u>Lemma 1</u>. $\operatorname{codim} \mathcal{M}_j \geq 4$ for all j .

For the proof of this lemma we need some more lemmas.

<u>Lemma 2</u>. For any $A \in \mathcal{O}\!\mathcal{L}$, the set of all matrices similar to A is an immersed submanifold of $\mathcal{O}\!\mathcal{L}$ of codimension $\geq m$.

<u>Proof</u>. Consider the group $GL(m)$, whose action ψ on $\mathcal{O}\!\mathcal{L}$ is given by $\psi(T,A) = T^{-1}AT$ for $T \in GL(m)$, $A \in \mathcal{O}\!\mathcal{L}$. The set of matrices similar to $\mathcal{O}\!\mathcal{L}$ is the orbit of A under this group action and, according to [4,2.2, Proposition 2], is an immersed submanifold of $\mathcal{O}\!\mathcal{L}$ of codimention equal to the dimension of the closed Lie subgroup $H = \{ T \in GL(m) \mid \psi(T, A) = A \}$. It is easy to show that H is identical with the subset of $GL(m)$ of matrices that commute with A . It follows from [5,VIII, §2, Theorem 2] that H has the dimension $\geq m$, q.e.d.

Corollary 2. Denote by p the map $\mathcal{M} \to R^n$ assigning to every matrix from $\mathcal{M}$ the n-tuple of coefficients of its characteristic polynomial and $\tilde{p} : \tilde{\mathcal{M}} \to R^{n+2}$ as $\tilde{p} = p \times id$. Then, for any point $x \in R^{n+2}$, $p^{-1}(x)$ is a finite disjoint union of immersed submanifolds of $\tilde{\mathcal{M}}$ of codimension $\geq n$.

Denote by $V \subset R^{n+2}$ the set of points $(\alpha_1, \dots, \alpha_n, \lambda_1, \lambda_2)$ such that $\lambda = \lambda_1 + i\lambda_2 \in S$ and is a root of the polynomial $P(\lambda) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n$ of multiplicity ≥ 2. Obviously, $\tilde{p}(\tilde{\mathcal{M}}_1) = V$.

Lemma 3. The map $\tilde{p}|_{\tilde{\mathcal{M}}_1} : \tilde{\mathcal{M}}_1 \to V$ is open (in the topologies on $\tilde{\mathcal{M}}_1$, V induced by their imbedding into $\tilde{\mathcal{M}}$, R^{n+2} respectively).

Proof. Obviously, it suffices to prove that $p|_{\mathcal{M}_1} : \mathcal{M}_1 \to \hat{V}$, where $\hat{V}$ is the projection ($R^n \times R^2 \to R^n$) of V into R^n, is open. That is, we have to prove that given a neighbourhood $\mathcal{U}$ of $A \in \mathcal{M}_1$, for any $P \in \hat{V}$ sufficiently close to $p(A)$, there is a $B \in \mathcal{U}$ such that $p(B) = P$.

This statement is obvious if A has the real canonical form; its extension for A not in canonical form follows from $p(A) = p(T^{-1}AT)$ for $T \in GL(n)$.

Proof of Lemma 1. V is án algebraic variety in R^{n+2}, defined by the polynomial identities $P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = P_1'(\lambda_1, \lambda_2) = P_2'(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 - 1 = 0$, where $P_1(\lambda_1, \lambda_2) = \mathrm{Re}\, P(\lambda_1 + i\lambda_2)$ etc. Therefore, it can be written as a finite disjoint union of submanifolds of R^{n+2} of decreasing dimension, $V = \bigcup_{i=1}^{q} V_i$.

We prove $\dim V_1 \leq n-2$. To do this, we note that $\operatorname{codim} V_1 \geq \operatorname{rank}_x V$ for any $x \in V_1$ (cf. [3]), where $\operatorname{rank}_x V$ is the dimension of the linear space spanned by the differentials at x of the polynomials of the ideal associated with V . Since V_1 is open in V it suffices to prove that the set of those x for which $\operatorname{rank}_x V \geq 4$ is dense in V .

For $x \in V$, $x = (\alpha_1, \dots, \alpha_n, \lambda_1, \lambda_2)$ we have

$$dP_1 = (\dots, \lambda_1, 1, \quad 0, \quad 0),$$

$$(1) \quad dP_1' = \left(\dots, \quad 1, 0, \frac{\partial P_1'}{\partial \lambda_1}, \frac{\partial P_1'}{\partial \lambda_2}\right),$$

$$dP_2' = \left(\dots, \quad 0, 0, \frac{\partial P_2'}{\partial \lambda_1}, \frac{\partial P_2'}{\partial \lambda_2}\right),$$

$$d(\lambda_1^2 + \lambda_2^2 - 1) = (\dots, 0, 0, 2\lambda_1, 2\lambda_2),$$

and, since

$$-\det \begin{pmatrix} \lambda_1, & 1, & 0, & 0 \\ 1, & 0, & \frac{\partial P_1'}{\partial \lambda_1}, & \frac{\partial P_1'}{\partial \lambda_2} \\ 0, & 0, & \frac{\partial P_2'}{\partial \lambda_1}, & \frac{\partial P_2'}{\partial \lambda_2} \\ 0, & 0, & 2\lambda_1, & 2\lambda_2 \end{pmatrix} = 2\left[\lambda_2 \frac{\partial P_2'}{\partial \lambda_1} - \lambda_1 \frac{\partial P_2'}{\partial \lambda_2}\right] =$$

$$= 2\left[\lambda_2 \frac{\partial P_2'}{\partial \lambda_1} + \lambda_1 \frac{\partial P_1'}{\partial \lambda_1}\right] = 2 \operatorname{Re}(\lambda^{-1} P''(\lambda)) .$$

Thus, it suffices to prove that for a dense subset of V , $\operatorname{Re}(\lambda^{-1} P''(\lambda)) \neq 0$.

It is obvious that the set of those $x \in V$ for which $P''(\lambda) \neq 0$ is dense in V . If λ is real and $\lambda \in S$,

$P''(\lambda) \neq 0$, then also $\lambda^{-1}P''(\lambda) = \operatorname{Re}\lambda^{-1}P''(\lambda) \neq 0$.

Assume that λ is not real, $\lambda \in S$ and $P''(\lambda) \neq 0$. Then $\lambda^{-1}P''(\lambda) = \bar{\lambda} P''(\lambda) = \bar{\lambda}(\lambda - \bar{\lambda})^2 R(\lambda)$,

where $R(\mu)$ is real for μ real. For ε real denote $P_\varepsilon(\mu) = (\mu-\lambda)^2(\mu-\bar{\lambda})^2[R(\mu)+\varepsilon] = \mu^n + \alpha_{1\varepsilon}\mu^{n-1} + \dots + \alpha_{n\varepsilon}$.

$P_\varepsilon(\mu)$ is real for μ real and $(\alpha_{1\varepsilon}, \dots, \alpha_{n\varepsilon}, \lambda_1, \lambda_2) \in V$. We have $\operatorname{Re}(\bar{\lambda} P_\varepsilon''(\lambda)) - \operatorname{Re}(\bar{\lambda}P''(\lambda)) = \varepsilon \operatorname{Re}[\bar{\lambda}(\lambda - \bar{\lambda})^2] =$

$= -4\varepsilon\lambda_1\lambda_2$. Since both $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, there is an $\varepsilon > 0$ arbitrarily small such that $\operatorname{Re}[\bar{\lambda}P_\varepsilon''(\lambda)] \neq$ $\neq 0$. This proves the density in V of the set of points x for which $\operatorname{Re}(\lambda^{-1}P''(\lambda)) \neq 0$.

Let i be such that $\tilde{p}(\mathcal{M}_1) \cap V_i \neq \emptyset$, $\tilde{p}(\mathcal{M}_1) \cap$ $\cap V_j = \emptyset$ for $j < i$. Since $\bigcup_{j=1}^{i} V_j$ is open, $\mathcal{M} =$ $= \tilde{p}^{-1}(V_i) = \tilde{p}^{-1}(\bigcup_{j=1}^{i} V_j)$ is open in $\mathcal{M}_1$ and, by Lemma 3, $p(\mathcal{M}_0)$ is open in V_i. From this and the Sard's theorem ([6, Theorem 15.1]) it follows that there is a point $\tilde{A} \in \mathcal{M}_0$ at which $\tilde{p}$ is regular. Thus, locally $\tilde{p}^{-1}(\tilde{p}(\tilde{A}))$ is an imbedded submanifold of the dimension $\dim \mathcal{M}_1 - \dim V_i \geq \dim \mathcal{M}_1 - n + 2$. On the other hand, from Corollary 2 it follows $\dim \tilde{p}^{-1}(\tilde{p}(\tilde{A})) \leq$ $\leq n^2 - n$. Consequently, $\dim \mathcal{M}_1 \leq n^2 - 2$, q.e.d.

<u>Lemma 4</u>. If $\lambda_{20} \neq 0$, then $\operatorname{codim} \mathcal{N}_1 \geq 4$.

The proof of this lemma is similar to that of Lemma 1, with V replaced by the set $W \subset R^{n+2}$ of points $(\alpha_1, \dots, \alpha_n, \lambda_{10}, \lambda_{20})$ for which $\lambda_0 = \lambda_{10} + i\lambda_{20}$ is a root of $P(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n$.

This is again an algebraic variety defined by the equations $\lambda_1 - \lambda_{10} = \lambda_2 - \lambda_{20} = 0, P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = 0$.
The differentials of the polynomials at the points of W are

$$\begin{aligned}
dP_1 &= (\dots, \lambda_{10}, 1, \frac{\partial P_1}{\partial \lambda_1}, \frac{\partial P_1}{\partial \lambda_2}) \ , \\
dP_2 &= (\dots, \lambda_{20}, 0, \frac{\partial P_2}{\partial \lambda_1}, \frac{\partial P_2}{\partial \lambda_2}) \ , \\
d(\lambda_1 - \lambda_{10}) &= (\dots, 0, 0, 1, 0) \ , \\
d(\lambda_2 - \lambda_{20}) &= (\dots, 0, 0, 0, 1) \ .
\end{aligned}$$

Obviously, they are independent if $\lambda_{20} \neq 0$. The rest of the proof is analogous to the proof of Lemma 1.

<u>Proof of Proposition 1.</u> Openness follows from the fact that both $\mathcal{O}\!\iota_1$ and $\mathcal{O}\!\iota_2$ are closed.

For the proof of density we consider the sets $\widetilde{\mathcal{O}\!\iota}_1, \widetilde{\mathcal{O}\!\iota}_2(\lambda_{10}, \lambda_{20})$ with $\lambda_{20} \neq 0$ and the space $\widetilde{\Phi}$ of maps $F : \mathit{int}\, I \times R^2 \longrightarrow \widetilde{\mathcal{O}\!\iota}$, defined by $\widetilde{F} = F|_{\mathit{int}\, I} \times id$, $F \in \Phi$, endowed with the C^n uniform topology. Further, we denote by $\widetilde{\Psi}_i = \{\widetilde{F} \mid \widetilde{F}(I) \cap \cap \bigcup_{j=n-i+1}^{n} \mathcal{M}_i = \emptyset\}$ for $1 \leq i \leq n$, $\widetilde{\Psi}_{n+i} = \{\widetilde{F} \mid \widetilde{F}(I) \cap \cap \widetilde{\mathcal{O}\!\iota}_1 \cap \bigcup_{j=s-i+1}^{s} \mathcal{N}_i = \emptyset\}$ for $1 \leq i \leq s$. Since Ψ_ℓ is the intersection of the projections of $\widetilde{\Psi}_{n+s}$ taken over all nonreal ℓ-th roots of unity, it suffices to prove that $\widetilde{\Psi}_{n+s}$ is dense in $\widetilde{\Phi}$. We prove this by induction showing that every $\widetilde{F} \in \widetilde{\Psi}_i$ can be approximated arbitrarily closely by an $\widetilde{F}' \in \widetilde{\Psi}_{i+1}$. Without loss of generality we assume $1 < i < n$.

The map $\varphi : \Phi \longrightarrow \widetilde{\Phi}$ given by $\varphi(F) = \widetilde{F}$ is a C^n-representation (here and further in this proof we use the terminology of [6]) and the evaluation map meets $\mathcal{M}_{n-i}$ transversally. Due to the dimension estimates of Lemma 1 and Lemma 4, the existence of the approximation of F not intersecting $\mathcal{M}_{n-i}$ follows from the transversality theorem [6, Theorem 19.1] and the openness of $\widetilde{\Psi}_i$, q.e.d.

Denote $\mathcal{U}_3$ the subset of $\mathcal{U}$ consisting of matrices having an eigenvalue on S . Again, we associate with $\mathcal{U}_3$ the algebraic variety $\widetilde{\mathcal{U}}_3$ in $\widetilde{\mathcal{U}}$, defined by
$\widetilde{\mathcal{U}}_3 = \{(A, \lambda_1, \lambda_2) | P_1(\lambda_1, \lambda_2) = P_2(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2 - 1 = 0\}$
whose projection is $\mathcal{U}_3$. Thus, $\widetilde{\mathcal{U}}_3 = \bigcup_{i=1}^{n} \mathcal{K}_i$, where $\mathcal{K}_i$ are mutually disjoint manifolds of decreasing dimension and $\bigcup_{j=i}^{n} \mathcal{K}_j$ is closed in $\widetilde{\mathcal{U}}_3$ for every i .

<u>Lemma 5</u>. $\operatorname{codim} \mathcal{K}_1 = 3$.

<u>Proof</u>. The proof of the inequality $\dim \mathcal{K}_1 \geq 3$ is analogous to that of Lemma 1. We only note that the differentials of the defining polynomials $P_1, P_2, \lambda_1^2 + \lambda_2^2 - 1$ of $\tilde{\pi}(\widetilde{\mathcal{U}}_3) \subset R^{n+2}$ ($\tilde{\pi}$ defined as in Corollary 2) are independent if $\operatorname{Re}(\lambda P'(\lambda)) \neq 0$; it can be shown similarly as in the proof of Lemma 1 that this is true for a dense subset of $\tilde{\pi}(\widetilde{\mathcal{U}}_3)$.

To prove the opposite inequality assume $I = [0, 2]$ and consider the map $F(t) = \operatorname{diag}\{t, 0, \dots, 0\}$. If

$\operatorname{codim} \mathcal{K}_1 < 3$ then it would follow from the transversality argument used in the proof of Proposition 1 that there should exist a small C^r perturbation $\hat{F}$ of F no value of which would have an eigenvalue on $\mathcal{S}$. This, however, is obviously impossible.

Proposition 2. Let $J \subset I$ be a closed interval, $J \subset \operatorname{int} I$. Then, for every $\ell > 2$ the subset $\Psi^o_\ell(J) \subset \Psi_\ell(J)$ of all F such that F meets $\tilde{\mathcal{U}}_3$ transversally (i.e. F meets transversally $\mathcal{K}_1$ and does not meet $\mathcal{K}_i$ for $i > 1$ at all) is open dense in $\Psi_\ell(J)$, and, thus, in Φ.

The proof is analogous to that of Proposition 1.

Corollary 3. Given J as in Proposition 2, the set $\Psi^o(J)$ of maps $F \in \Phi$ such that $F(J) \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = 0$ and F meets $\tilde{\mathcal{U}}_3$ transversally over J is residual in Φ.

Lemma 6. Let $F \in \Phi$ and let λ_0 be a simple eigenvalue of $F(t_0)$, where $t_0 \in I$. Then there is a neighbourhood N of t_0 in I and a unique function $\lambda : N \longrightarrow C$ such that $\lambda(t_0) = \lambda_0$ and $\lambda(t)$ is an eigenvalue of $F(t)$ for $t \in N$. Further, there is a nonsingular C^r matrix $C(t)$ on N such that $C^{-1}FC = B$, where the first column of $B(t)$ is the transpose of $(\lambda(t), 0, \ldots, 0)$.

Proof. Without loss of generality we may assume that $F(t_0)$ is in the Jordan canonical form with λ_0 in the first column. Choose $C(t_0) = E$ (the unity matrix) and $C(t) = (c_1(t), \ldots, c_m(t))$, $\lambda(t)$ as the solution of the set of equations $F(t)c_1(t) = \lambda(t)c_1(t)$, $c_i(t) = c_i(t_0), i > 1, |c_1(t)| = 1$ ($|.|$ being the Euc.idean norm). It is easy to check that the Jacobian of this set of equations at t_0 is not zero. The implicit function theorem completes the proof.

Remar 1. Under the assumptions of Lemma 6, for λ_0 not real, starting from the real canonical form of $F(t_0)$, one can similarly prove that there is a C^n real matrix $C(t)$ in some neighbourhood of t_0 in I that brings $F(t)$ into the form

$$\begin{pmatrix} B_1(t), & B_2(t) \\ 0 & , B_3(t) \end{pmatrix}, \text{ where } B_1(t) = \begin{pmatrix} \mathrm{Re}\,\lambda(t), & \mathrm{Im}\,\lambda(t) \\ -\mathrm{Im}\,\lambda(t), & \mathrm{Re}\,\lambda(t) \end{pmatrix}.$$

Corollary 4. Let $F \in \Phi$, $t_0 \in I$ and let $\lambda_{10}, \ldots, \lambda_{k0}$ be simple eigenvalues of $F(t_0)$. Then, there is a neighbourhood N of t_0 in I and unique C^n functions $\lambda_i : N \longrightarrow C$ such that $\lambda_i(t_0) = \lambda_{i0}$ and $\lambda_i(t)$ are eigenvalues of $F(t)$ for $t \in N$. Further, there is a C^n matrix $C(t)$ on N such that $C^{-1}AC = B$, where B has the form $\begin{pmatrix} B_1, & B_2 \\ 0, & B_3 \end{pmatrix}$ and B_1 is triangular with $\lambda_1, \ldots, \lambda_k$ on the diagonal. Also, there is a real C^n matrix $\hat{C}(t)$ on N that brings $F(t)$ into the form $\begin{pmatrix} \hat{B}_1(t), & \hat{B}_2(t) \\ 0 & , \hat{B}_3(t) \end{pmatrix}$, where $\hat{B}_1(t)$ is block diagonal with blocks as in Remark 1.

Proposition 3. Let $F \in \Psi^0_\ell(J)$ for some $\ell > 2$. Then, the eigenvalues of F meet S transversally.

By this proposition we mean that the functions λ, defined in Lemma 6 for $\lambda_0 \in S$ (note that such λ_0 are simple) meet S transversally.

Proof. Let $\lambda(t_0) \in S$ be an eigenvalue of $F(t_0)$. By Lemma 6, there is a nonsingular C^n matrix $C(t)$ such that $C^{-1}(t)F(t)C(t) = B(t)$, where $B(t)$ has the form specified in Lemma 6. Denote $B(t, \mu)$ the matrix obtained from $B(t)$ by replacing in the first column $\lambda(t)$ by μ. Denote by $\mu(t)$ the orthogonal projection of $\lambda(t)$ on S, φ the Euclidean distance. Since $C(t)B(t,\mu(t))C^{-1}(t) \in \mathcal{U}_3$ and $\mathcal{K}_1$ is open in $\tilde{\mathcal{U}}_3$, $(C(t)B(t,\mu(t))C^{-1}(t), \mu_1(t), \mu_2(t)) \in \mathcal{K}_1$, for t sufficiently close to t_0, where $\mu = \mu_1 + i\mu_2$. We have

$$|\lambda(t)| - 1 = |\lambda(t) - \mu(t)| = \varphi(B(t), B(t,\mu(t))) \geq |C(t)|^{-1},$$

$$|C(t)^{-1}|^{-1}\varphi(F(t), C(t)B(t,\mu(t))C^{-1}(t)) \geq k_1\,\varphi(\tilde{F}(t), \mathcal{K}_1),$$

where $k_1 > 0$ is a suitable constant. If $\tilde{F}$ meets $\mathcal{K}_1$ transversally, then obviously $\varphi(\tilde{F}(t), \mathcal{K}_1) \geq k_2 |t - t_0|$ for some $k_2 > 0$. Consequently, $\left.\dfrac{d|\lambda(t)|}{dt}\right|_{t=t_0} \neq 0$, q.e.d.

Corollary 5. The number of such $t \in J$ for which an eigenvalue of $F(t)$ is on S, is finite for every $F \in \Psi_\ell^0(J)$.

Theorem 1. Let $J \subset \operatorname{int} I$ be a closed interval. Then, the set $\Phi_{1\ell}(J)$ of those $F \in \Phi$, satisfying

(i) $F(t)$ has no double eigenvalue on S,

(ii) $F(t)$ has no non-real ℓ-th root of unity as eigenvalue,

(iii) the eigenvalues of $F(t)$ meet $\mathcal{S}$ transversally,

(iv) if an eigenvalue of $F(t)$ lies on $\mathcal{S}$, then no other eigenvalue of $F(t)$ lies on $\mathcal{S}$ except of its complex conjugate,

for every $t \in J$, is open dense in Φ.

Corollary 6. The set $\Phi_1(J)$ of those $F \in \Phi$ satisfying (i),(iii),(iv) of Theorem 1 and such that for every $t \in J$, $F(t)$ has no non-real root of unity as eigenvalue, is residual in Φ.

Proof. Openness is obvious. From Propositions 1 - 3 it follows that the set of maps from Φ, satisfying (i) - (iii) (i.e. the set $\Psi_{\ell}^{o}(J)$), is open dense in Φ. Therefore, it suffices to prove that every $F \in \Psi_{\ell}^{o}(J)$ can be arbitrarily closely approximated by an $\hat{F} \in \Psi_{\ell}^{o}(J)$ satisfying (iv). In virtue of Corollary 4 it suffices to show that if for some t_0 (iv) is not satisfied it is possible to perturb F in an arbitrary small neighbourhood $\mathcal{N}$ of t_0 by an arbitrary small perturbation, without changing it outside $\mathcal{N}$, in such a way that (i) - (iv) will be true for the perturbation of F for every $t \in \mathcal{N}$.

Assume that for some $t_0 \in J$, k pairs of conjugate eigenvalues λ_j^0, $\overline{\lambda_j^0}$, $j = 1, \dots, k$ lie on $\mathcal{S}$ (the modification of the proof for the case of some eigenvalue being real is straightforward). Let α be so small that the functions λ_j, defined by λ_j^0, t_0 as in Lemma 6 exist and do not meet $\mathcal{S}$ except at t_0 and no other eigenvalue of $F(t)$ lies on $\mathcal{S}$ on $K \cap J$, where

$K = [t_0 - \alpha, t_0 + \alpha]$, and that there is a C^n matrix C such that $C^{-1}(t)F(t)C(t) = B(t)$ has the form

$$B = diag\left\{\begin{pmatrix} \lambda_{11}, \lambda_{12} \\ -\lambda_{21}, \lambda_{22}\end{pmatrix}, \dots, \begin{pmatrix} \lambda_{k1}, \lambda_{k2} \\ -\lambda_{k2}, \lambda_{k1}\end{pmatrix}, B_1\right\}$$

where $\lambda_j = \lambda_{j1} + i\lambda_{j2}$ (cf. Remark 1). Choose an $\varepsilon < \frac{\alpha}{2}$, k real mutually distinct numbers τ_j, $j = 1,\dots,k$ such that $|\tau_j| < \varepsilon$ and a bump function $\chi : N \to R$ such that $\chi(t) = 0$ outside K, $\chi(t) = 1$ for $t \in K_0 = [t_0 - \frac{\alpha}{2}, t_0 + \frac{\alpha}{2}]$, $\hat{\lambda}_j(t) = \lambda_j(t + \tau_j\chi(t))$,

$$\hat{B}(t) = diag\left\{\begin{pmatrix} \hat{\lambda}_{11}(t)\,\hat{\lambda}_{12}(t) \\ -\hat{\lambda}_{21}(t)\,\hat{\lambda}_{11}(t)\end{pmatrix}, \dots, \begin{pmatrix} \hat{\lambda}_{k1}(t), \hat{\lambda}_{k2}(t) \\ -\hat{\lambda}_{k2}(t), \hat{\lambda}_{k1}(t)\end{pmatrix}, B_1(t)\right\},$$

$$F(t) = \begin{cases} F(t) \text{ for } t \notin K, \\ C(t)\hat{B}(t)C^{-1}(t) \text{ for } t \in K. \end{cases}$$

It is obvious that $\hat{F} \in \Psi^0_\ell$ and, in $K \cap J$, $\hat{\lambda}_j$ meets S exclusively at the point $t_0 - \tau_j$. If τ_j are chosen small enough, F will be arbitrarily close to F, q.e.d.

§ 3

In [1, § 2] it was shown that for $f \in \mathcal{F}_1$, each point of $\bar{Z} \setminus Z_k$ (such points have been called branching points) is contained in some set Z_ℓ with ℓ being a divisor of k and that some eigenvalue of df_p^{ℓ} at such point has to be a root of unity different from 1.

Theorem 2. There is a subset $\mathcal{F}_2$ of $\mathcal{F}_1$, residual in $\mathcal{F}$ such that for every $f \in \mathcal{F}_2$, the following is true for every $(p_0, m_0) \in Z_k(f)$, $k \geq 1$:

(i) $df_{p_0}{}^{k}(m_0)$ has no double eigenvalue on S,

(ii) $df_{p_0}{}^{k}(m_0)$ has no non-real root of 1 as an eigenvalue.

(iii) The eigenvalues of $df_p{}^{k}(m)$ meet S transversally at (p_0, m_0).

(iv) If an eigenvalue of $df_{p_0}{}^{k}(m_0)$ lies on S, then there is no other eigenvalue of $df_{p_0}{}^{k}(m_0)$ on S except of its complex conjugate.

Corollary 7. For $f \in \mathcal{F}_2$, $(p, m) \in Z_k(f)$ can be a branching point only if one of the eigenvalues of $df_p(m)$ is -1, the other being outside S.

Remark 2. Denote $\mathcal{F}_{2h\ell}$ the subset of $\mathcal{F}_{1h}$ of those mappings, satisfying (i),(iii),(iv) for $1 \leq k \leq h$ and (ii) with "roots" replaced by "ℓ-th roots" for $1 \leq k \leq h$. Then, $\mathcal{F}_{2h\ell}$ is open dense in $\mathcal{F}$.

Remark 3. (iii) should be understood as follows: If an eigenvalue λ_0 of $df_{p_0}{}^{k}(m_0)$ is on S, then in some neighbourhood N of (p_0, m_0) in Z_k, there is a unique C^{r} function $\lambda : N \longrightarrow C$ such that $\lambda(p, m)$ is an eigenvalue of $df_p^{\,k}(m)$ for $(p,m)\in N$ and

$\lambda(p_0,m_0)=\lambda_0$. This λ meets S transversally.

Proof. It suffices to prove Remark 2, from which Theorem 2 follows. We carry out the proof for $h=1$, i.e. we prove that $\mathcal{F}_{21\ell}$ is open dense for any ℓ; the extension for $h>1$ is similar as in the proof of [1, Theorem 1].

The openness of $\mathcal{F}_{21}$ is obvious. To prove density, assume $f\in\mathcal{F}_{11}$. Then, by [1, Theorem 1], there is an open set $\mathcal{U}$ containing $X_1(f)$ such that for every $(p_0,m_0)\in\mathcal{U}$, (i) - (iv) is trivially satisfied. $Z_1\setminus\mathcal{U}$ can be covered locally finitely by a countable family $(W_\alpha,\mu_\alpha\times x_\alpha)$, $W_\alpha=\mathcal{U}_\alpha\times V_\alpha$ of coordinate neighbourhoods in such a way that for any $K\in P\times M$ compact, $W_\alpha\cap K\neq\emptyset$ for a finite number of α's only and $(W_\alpha,\mu_\alpha\times x_\alpha)$ satisfy (iv) of [1, Theorem 1] (i.e. $W_\alpha\cap Z_1$ is the graph of a C^k function $\varphi_\alpha:\mathcal{U}\to V$). We show how for any open W'_α, $\overline{W}'_\alpha\subset\overline{W}'_\alpha=\mathcal{U}'_\alpha\times V'_\alpha$, f can be approximated by $\hat{f}$ such that $\hat{f}$ coincides with f outside W_α and satisfies (i) - (iv) of Theorem 2 for every $(p_0,m)\in Z_1\cap W_\alpha$. The construction of an approximation of f satisfying (i) - (iv) for any $(p_0,m_0)\in Z_1$ is then standard. In the rest of the proof we drop the subscript α.

In the coordinates $(p,m)\mapsto(\mu,y)$, $y=x-x_0\,\varphi(p)$, f can be represented by

$$y'=A(\mu)y+\Upsilon(\mu,y)$$

where the primed coordinates are those of the image,

$Y(\mu, 0) = 0, \; dY(\mu, 0) = 0.$

By Theorem 1, we can approximate $A : \mu(\mathcal{U}) \longrightarrow \mathfrak{A}$ by a map $\hat{A} : \mu(\mathcal{U}) \longrightarrow \mathfrak{A}$ such that A satisfies (i) - (iv) of Theorem 1 on $\mathcal{U}$.

Let $\psi : (\mu \times x)(W) \longrightarrow \mathbb{R}$ be a C^r bump function such that $\psi = 1$ on $(\mu \times x)(\overline{W'})$ and $\psi = 0$ outside $(\mu \times x)(W)$. Denote by $\hat{f}$ the map which coincides with f outside W and is given in W by the coordinate representation

$$y' = [A(\mu) + \psi(\mu, y)(\hat{A}(\mu) - A(\mu))]y + Y(\mu, y).$$

If we choose A sufficiently close to A, $\hat{f}$ will be arbitrarily close to f and will satisfy (i) - (iv) for every $(p_0, m_0) \in W'$.

Denote by Y_k the set of points $(p, m) \in Z_k$ for which one eigenvalue of $df_p{}^k(m)$ is -1. For $(p, m) \in$ $\in Z_k$ denote $h(p, m)$ the number of eigenvalues of $df_p{}^k(m)$ with modulus less than 1.

<u>Theorem 3</u>. Assume $r > 2$. Then, there is a subset $\mathcal{F}_3$ of $\mathcal{F}_2$, residual in $\mathcal{F}$, such that every $f \in \mathcal{F}_3$ has the following properties:

(i) Y_k coincides with the set of k-periodic branching points,

(ii) for every $(p_0, m_0) \in Y_k$, there is a coordinate neighbourhood $(W, \mu \times x)$, $W = \mathcal{U} \times V$ of (p_0, m_0) such that $\mu(p_0) = 0$, $x(m_0) = 0$, $Z_k \cap W = \mathcal{U} \times \{0\}$ and

(a) $Z_{2k} \cap W$ consists of two components, separated by (p_0, m_0); all points $(p, m) \in Z_{2k} \cap W$ satisfy $\mu(p) > 0$ and $Z_{2k} \cap W \cup \{(p_0, m_0)\}$ is a C^1 (but not C^2) submanifold of W.

(b) No eigenvalue of $[(Z_k \cup Z_{2k}) \cap W] \setminus \{(p_0, m_0)\}$ is on S; either $h(p, m) = h(p', m') = h(p'', m'') + 1$ or $h(p, m) = h(p', m') = h(p'', m'') - 1$ for any $(p, m) \in Z_k \cap W$, $\mu(p) < 0$, $(p', m') \in Z_{2k} \cap W$, $(p'', m'') \in Z_k \cap W$, $\mu(p'') > 0$,

(c) $W \setminus (Z_k \cup Z_{2k})$ contains no invariant set.

Proof. Again, we carry out the proof for $k = 1$, the proof of its extension for $k > 1$ being as in [1, Theorem 1].

Let $f \in \mathcal{F}_{21\ell}$. Then, $Y_1(f)$ is discrete and, if $(p_0, m_0) \in Y_1$, one eigenvalue of $df_{p_0}(m_0)$ is -1 and the remaining ones can be divided into two groups according to whether their moduli are < 1 or > 1, the number of the former ones being $h(p_0, m_0)$. Thus, using [6, Appendix 3] as in [1, Lemma 4], it follows that we can choose the coordinates (μ, x) in such a way that $x = (x_1, y, z)$, $\dim x_1 = 1$, $\dim y = h(p_0, m_0)$ and the coordinate representation of f in these coordinates is as follows:

$$
\begin{aligned}
&x_1 = -x_1 + \alpha\mu x_1 + \beta x_1^2 + \gamma x_1^3 + \omega(\mu, x_1, y, z), \\
(3)\quad &y = Ay + Y(\mu, x_1, y, z), \\
&z = Cz + Z(\mu, x_1, y, z),
\end{aligned}
$$

where ω, Y, Z are C^r and

ω, Y, Z are C^r and $Y(\mu, x_1, 0, z) = 0, Z(\mu, x_1, y, 0) = 0$,

$\omega(\mu, x_1, y, z) = O(|x_1^3| + |\mu x_1| + |y| + |z|)$,

$d\omega(0,0,0,0) = 0$,

$dY(0,0,0,0) = 0, dZ(0,0,0,0) = 0$.

We denote by $\mathcal{F}_{31}$ the subset of $\mathcal{F}_{11}$ of those maps in the coordinate representation (3) of which $\beta^2 + \gamma \neq 0$ for every $(p_0, m_0) \in Y_1(f)$. The definition of $\mathcal{F}_{31}$ does not depend on the choice of particular coordinates and the set $\mathcal{F}_{31}$ is open dense in $\mathcal{F}$. The proof of this as well as the proof that the maps of $\mathcal{F}_{31}$ satisfy (i),(ii) for $k = 1$ does not differ from the corresponding part of the proof of [1, Theorem 3], except of the proof of (ii)(c), where, because of the possible presence of the eigenvalues of moduli both < 1 and > 1 one has to use the argumentation of the proof of [1, Lemma 4].

As a corollary of [1, Theorem 1] and Theorem 3 we obtain

Theorem 4. Assume $r > 2$. Then, for every $f \in \mathcal{F}_3$:

(i) for k odd, Z_k is a closed submanifold of $P \times M$,

(ii) for k even, either Z_k is closed and $Y_{k/2}$ is empty, or Z_k is a C^1 (but not C^2) submanifold of $P \times M$ and $\bar{Z}_k \setminus Z_k$ is discrete and coincides with $Y_{k/2}$.

Remark 4. This theorem corrects the erroneous formulation of its two dimensional version [1, Theorem 4], in which the possibility of Z_k being closed was omitted.

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