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On the Structure of Optimal Feedback Systems

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The basic optimal control problem is given by a system

$$\dot{x} = f(x, u), \quad x \in R^n, \quad u \in R^m, \tag{1}$$

a control domain

$$U \subset R^m \tag{2}$$

a performance index

$$J(u) = \int_0^T f^0(x, u)\, dt \quad (f^0 \colon R^n \times R^m \to R), \tag{3}$$

initial and target states x_0, x_1 respectively. By an admissible control we understand a piecewise continuous function, defined on some interval of the real line with values in U. Under suitable regularity conditions on f^0, f every admissible control $u\colon [0, T] \to U$ when substituted into (1) defines a unique solution $x(t, u)$ starting at x_0 for $t=0$ (called the response of u). Substituting the control and its response into (3) for u, x respectively, gives a real value to J. One is interested in finding and studying the properties of the optimal control which steers the system from x_0 to x_1 (i.e. its response $x(t)$ called the optimal trajectory satisfies $x(T)=x_1$) for some $T>0$ and minimizes the performance index J.

From the very beginning of the optimal control theory one of the approaches to study this problem has been to imbed it in a family of problems with a varying initial state x_0. This approach is based on the simple observation (frequently called Bellman's optimality principle) that if u is an optimal control on $[0, T]$, then its restriction to any interval $[t_0, T]$, $t_0 \geq 0$, is an optimal control for the initial state $x(t_0, u)$. If for each initial state x in some region G the optimal control u_x (and, consequently, its response ξ_x starting at x) is unique, from the optimality principle we obtain immediately that the optimal control can be expressed, independently of the initial state, as a function of the present state of the system, i.e. there exists a function $v: G \to U$ such that $u_x(t) = v(\xi_x(t))$ for $x \in G$. Therefore the optimal trajectories satisfy in G the differential equation

$$\dot{x} = f(x, v(x)). \tag{4}$$

Let us note that in many applications the ultimate goal of solving the optimal control problem is to find the function v, which is called the closed-loop optimal control, the optimal feedback law or the synthesis of optimal control.

Formally, one can consider (4) as an equation for optimal trajectories. In order to utilize it, it is important to know something about the properties of the function v. For example, for the classical existence and uniqueness theory of ordinary differential equations it would be useful if v were continuous. However, simple examples in which v can be constructed explicitly (cf. [**1**, Chapter III] or [**11**, Chapter 2]) show that due to unilateral constraints, which are typical for the optimal control theory, v is frequently discontinuous.

A deeper reason for studying the structure of v is the problem of sufficiency of the variational necessary conditions of optimality, in particular of the Pontrjagin maximum principle (PMP). Assume that for every initial state $x \in G$ there exists a unique control steering the system from x to x_1 and satisfying PMP, thus being the unique candidate for the optimal control. If we define $v(x) = u_x(0)$, we may ask whether u is the closed-loop optimal control, i.e. whether (4) yields optimal trajectories (and only optimal trajectories) as its solutions. As it is shown in [**1**], [**2**] this problem is closely connected with the problem of the sufficiency on the dynamic programming equation (which corresponds to the Hamilton–Jacobi equation of the classical calculus of variations).

When trying to resolve this question one is again confronted with the problem of the regularity of the behaviour of v. Bolt′anski observed that one can work also with a discontinuous synthesis, provided its set of discontinuities is sufficiently regular. This led him to introduce the concept of regular synthesis for the time-optimal control problem ($f^0 = 1$) (cf. [**1**], [**2**]). By a regular synthesis for the time-optimal control problem in a region G we understand a pair $(\mathcal{S}, v)$, where $\mathcal{S}$ is a locally finite partition of G into C^1 connected submanifolds of G (called cells), v is a function $G \to U$ satisfying the following conditions:

A. The set $\bar{G}'$ (where G' is the union of the cells of dimension $<n$) admits a stratification in G. (By a stratification $\mathcal{P}$ of a subset H of G we understand a locally finite partition of H into C^1 connected submanifolds of G (called strata) such that $P \cap \bar{Q} \neq \emptyset$ implies $P \subset \bar{Q}$ and $\dim P < \dim Q$ for any $P, Q \in \mathcal{P}, P \neq Q$.)

B. The function v is C^1 on each $S \in \mathcal{S}$ and can be extended to a C^1 function in some neighbourhood of S. The cells of $\mathcal{S}$ are of type I and type II. If S is of type I, then $f(x, v(x)) \in T_x S$ (the tangent space of S at x) for every $x \in S$ and there is a uniquely defined cell $\pi(S)$ such that every solution of (4) starting at any point $x \in S$ enters $\pi(S)$ transversally for some $\tau > 0$ (after staying in S on $(0, \tau)$) which is a continuous function of x. If S is of type II then $f(x, v(x)) \notin T_x S$ for all $x \in S$ and there is a unique cell $\Sigma(S)$ of type I such that v is C^1 on $S \cup \Sigma(S)$ and every solution of (4) starting in S lies in $\Sigma(S)$ for sufficiently small positive times.

C. Every trajectory $x(t)$ of (4) starting at some point $x \in G$ (which is by B uniquely defined until it stays in G) eventually reaches x_1 in finite time $T(x) \geqq 0$ passing through a finite number of cells only and together with the control $u(t) = v(x(t))$ satisfies PMP.

D. $T(x)$ is continuous in G. Let us note that this definition differs somewhat from Bolt′anski's one as well as from that of [3]. (For details, cf. [3] and the forthcoming Erratum to [3].)

In [2] (cf. also [1]) Bolt′anski proved that if $(\mathcal{S}, v)$ is a regular synthesis, then v is the closed-loop optimal control in the following sense:

The trajectory ξ_x (in the Carathéodory sense) on $[0, T(x)]$ of equation (4) starting at $x \in G$ is the optimal trajectory and $u_x(t) = v(\xi_x(t))$ is the optimal control.

Virtually in all the simple examples in which it has been possible to construct the synthesis explicitly, the latter has satisfied the conditions of regularity. However, except for some studies of the local structure of v near x_1 (cf. e.g. [14]) no attempt has been made to prove that a more general class of problems would globally admit a regular synthesis. Such a result has been made possible by Hironaka's theory of subanalytic sets [7], [9], [10]. It concerns linear control systems

$$\dot{x} = Ax + Bu \tag{5}$$

with

$$U = \text{co}\,\{w_1, \ldots, w_p\} \tag{6}$$

being a convex polytope. Such a problem is called normal if for every $i \neq j, k$,

$$\det\left(b_k(w_i - w_j), Ab_k(w_i - w_j), \ldots, A^{n-1} b_k(w_i - w_j)\right) \neq 0,$$

where $B = (b_1, \ldots, b_m)$. Let us note that normality is a generic property (cf. [11, Chapter 2, Theorem 11]).

THEOREM 1 [3]. *Assume that the control system defined by* (5), (6) *is normal and that* U *contains* 0 *in its interior. Then the time-optimal control problem with the target point* $x_1 = 0$ *admits a regular synthesis in the domain* G *of points that can be steered to* 0.

As mentioned above, the proof of this theorem makes use of the theory of subanalytic sets. A subset M of an analytic manifold is called subanalytic if it can be locally (in A) expressed as a finite union of sets of type $f(Y) \setminus g(Z)$, where Y, Z are analytic manifolds and f, g are analytic proper. By the central theorem of the theory of subanalytic sets, every subanalytic subset of A admits an analytic stratification, the strata of which are subanalytic (cf. also [13]).

The cells of the synthesis are obtained by an inductive construction. The sets of continuity of v are shown to be subanalytic and the synthesis cells are obtained by a sequence of partitions of these sets into connected analytic submanifolds. In addition to the standard theory of subanalytic sets one needs the following

LEMMA. *Let M be a subanalytic subset of an analytic manifold A and let $X_1, \ldots, X_r$ be analytic vector fields on A. Then M admits a locally finite partition $\mathcal{P}$ into connected analytic submanifolds of A, which are subanalytic in A, such that for every $P \in \mathcal{P}$ and $i=1, \ldots, r$, X_i is either everywhere or nowhere tangent to P.*

This lemma, an improved version of which has been proved by Sussmann, appears to be crucial also for other application of the theory of subanalytic sets in control theory (cf. [12]). From this theorem it immediately follows that the minimum steering time to x_1, $T(x)$, is analytic in G everywhere except for a stratified set (G') of dimension $n-1$ (=maximal dimension of the strata).

If one tries to extend the concept of regular synthesis to problems where PMP yields controls with corners which are not jumps (like time optimal control problems with control domains having piecewise analytic curvilinear boundaries, or linear-quadratic problems with linear constraints), one immediately sees that the transversality assumptions as well as the C^1 extendability of v to the neighbourhood of S in B cannot be required. Instead, one has to assume their consequences, namely that the time $\tau(x)$ at which $\xi_x(t)$ enters $\pi(S)$ for S of type I and $\pi(\Sigma(S))$ for S of type II, the trajectory $\xi_x(t)$ and the control $u_x(t)$ are C^1 functions of x, t for $x \in S$, $t \in [0, \tau(x))$ and can be extended to C^1 functions of x, t for $t \geqslant \tau(x)$ close to $\tau(x)$. With this difference, the definition of regular synthesis can be literally extended to control problems with other performance indices (T in D replaced by J, the performance index). Bolt'anski's proof can be extended easily to yield an extension of his sufficiency theorem to general performance indices.

Employing essentially the same induction techniques as in the linear time-optimal problem case, one can prove an abstract existence theorem. However, due to the lack of transversality mentioned above, in order to obtain the C^1 dependence of the required quantities one has to construct auxiliary partitions in the product space of the state and adjoint space. By suitable partitions one can achieve that the product flow of the system and its adjoint enters the cells in the product space transversally, thus yielding analyticity of the required quantities.

Because of lack of space we desist from introducing this theorem, which has a rather cumbersome formulation. This is due to technical assumptions, which are needed for the extendability of the solutions of certain vector fields to sufficiently long intervals. Rather we note that the most serious requirements (in addition to analyticity, of course) for a system to admit a regular synthesis in some region G are the following ones:

1. For every initial state $x \in G$, there has to be a unique control u_x satisfying PMP which steers the system from x to x_1.

2. The number of switchings (which are roughly speaking the points of non-analyticity) of the controls u_x has to be locally uniformly bounded.

The first requirement makes the range of applications of such a result rather limited. Indeed, although singular controls (not minimizing the Hamiltonian strictly), which are quite typical for nonlinear control problems, are not excluded in principle, when they appear the first requirement is usually not satisfied. On the other hand the second requirement, the validity of which is difficult to prove for more general classes of systems, is virtually always satisfied in particular problems.

The following theorem concerns a model class of problems in which these difficulties can be overcome—linear-quadratic optimal control problems with linear constraints.

THEOREM 2. *Consider the optimal control problem*

$$\dot{x} = Ax + Bu,$$

$$J = \int_0^T [x^*Qx + u^*Ru]\,dt \quad (R > 0,\ Q \geqslant 0),$$

$$U = \{u \in R^m \,|\, \langle l_j, u\rangle \leqslant m_j,\ j = 1, \dots, p\},$$

$x(T)=0$, *T fixed, and assume that this system is normal. Then the problem admits a regular synthesis.*

The normality assumption here consists in the non-vanishing of certain polynomials involving the entries of A, B, Q, l_j, m_j, as in the case of the linear time-optimal control problem it is a generic property.

Of course, this theorem has a similar impact on the regularity of the minimal value of the performance index as Theorem 1 had on the regularity of the minimal steering time.

Let us note that neither Theorem 1 nor Theorem 2 contribute anything to sufficient conditions of optimality (the sufficiency of PMP in both cases can be proved by other, simpler means). Their value lies rather in the insight they give into the structure of the closed-loop optimal control.

Finally let us note that in Bolt′anski's sufficiency results one understands the solutions in the classical Carathéodory sense. However, it has been demonstrated by several authors in the fifties that this concept is inadequate in the case of equations with discontinuities in the dependent variable. Because of the discontinuity of v this is the case for equation (4) in many control problems. Several concepts of solutions for such equations have been proposed, the most elaborate being that of Filippov [6]. Therefore it is natural to ask whether the optimal trajectories (which are the usual solutions of (4)) coincide with the Filippov trajectories or not. This problem is related to the problem of stability of the behaviour of the solutions of (4) with respect to perturbations (cf. [8], [4]). Using a slight improvement of Theorem 1 this question can be answered positively for the linear time-optimal control problem with $\dim u=1$ (cf. [3], [5]). However, the results of [4], where the problem is completely solved for the two-dimensional linear time-optimal control problem, show that there is a non-exceptional class of problems for which the optimal trajectories do not coincide with the Filippov trajectories of (4).

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