Nonlinear Analysis, Theory, Methods & Applications, Vol. 7, No. 2, pp. 167–176, 1983.
Printed in Great Britain.

0362–546X/83/020167–10 \$03.00/0

NOTES ON CHAOS IN THE CELL POPULATION PARTIAL DIFFERENTIAL EQUATION

PAVOL BRUNOVSKÝ
Institute of Applied Mathematics, Comenius University, 842 15 Bratislava, Czechoslovakia

(*Received in revised form* 15 *June* 1982)

Key words and phrases: First order partial differential equation, semiflow, chaos.

1. INTRODUCTION

IN [1], THE author investigates the differential equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = f(x, u), (t, x) \in D = [0, \infty) \times \Delta, \Delta = [0, 1]. \tag{1}$$

This equation describes the dynamics of growth of certain types of cell populations most prominent of which is the red blood cell population. It is shown in [1] that under certain natural conditions on c and f the equation (1) generates a semiflow $S_t, t \geq 0$ on $C_+(\Delta)$ (the space of nonnegative continuous functions on Δ) with an invariant set V_w on which the behaviour of the trajectories of S_t is chaotic in the sense of [2]. This means that S_t has a dense trajectory in V_w and each point of V_w is unstable (i.e. for each $v \in V_w$ there exists a neighbourhood U of $S_{(0,\infty)}v$ in $C(D)$ and a sequence $v_n \to v$ such that the trajectory of v_n leaves U for some $t \geq 0$).

The main purpose of this paper is to show that S_t exhibits also other features of chaos in V_w. Namely, there are periodic points of S_t of any basic period in V_w and the set of all periodic points of S_t is dense in V_w (Section 2).

For the proof a representation of S_t is employed which allows to prove the results on chaos of [1] in a more simple and transparent way. These proofs are presented in Section 3. Also, this technique helped to discover a small error in [1]. For the results on chaos of [1] to be true an additional (albeit also natural) assumption has to be added. We make this assumption in Sections 2 and 3. In Section 4 we discuss the modifications to be made if this additional assumption is dropped.

We keep all the notation of [1] in order to make it easier for the reader to relate the two papers. However, in order not to force the reader to look into [1] for every single concept or result we conclude this section by a list of assumptions and results of [1] used in the present paper.

Assumptions

A1. The functions c, f are continuously differentiable.
A2. $c(0) = 0$, $c(x) > 0$ for $x > 0$.
A3. There exists a $u_0 \in (0, 1]$ such that $f_u(0, u_0) < 0$, $f(0, u)(u - u_0) < 0$ for $u > 0$, $u \neq u_0$.
A4. $f(x, u) \leq k_1 u + k_2$ for some $k_1, k_2 \geq 0$ and all $x \in \Delta$, $u \geq 0$.
A5. $f(x, 0) = 0$ for all $x \in \Delta$.

Note that the assumptions **A1–A5** coincide with assumptions (16)–(18) in [1] with one difference:
A5 is somewhat sharper than the assumption
A5′. $f(x, 0) \geqslant 0$ for $x \in \Delta$ and $f(0, 0) = 0$
made in [1]. Also note that **A5** is satisfied if $f(x, u) = (p(x, u) - c(x))\, u$ as is the case if (1) models a reproductive, constantly differentiating cell population with proliferation rate p.

Results

Under the assumptions **A1–A4, A5′** the following results are proven in [1]:

R1. For $G \subset R^n$, $n > 0$, denote by $C_+(G)$, $C^1_+(G)$ the set of all nonnegative continuous and nonnegative continuously differentiable functions on G, respectively. For every $v \in C^1_+(\Delta)$, (1) has a unique solution u in $C^1_+(D)$ satisfying

$$u(x, 0) = v(x) \quad \text{for} \quad x \in \Delta. \tag{2}$$

A function $u \in C_+(D)$ is called generalized solution of (1) if it is a limit (uniform on compact subset of D) of solutions of (1). For each $v \in C_+(\Delta)$ there exists a unique generalized solution of (1) satisfying (2); henceforth we shall drop the adjective 'generalized'. The map $S: [0, \infty) \times C_+(\Delta) \to C_+(\Delta)$ defined by $S_t v(x) = u(t, x)$, where u satisfies (1), (2) is a continuous semiflow, i.e. $S_t: C_+(\Delta) \to C_+(\Delta)$ is continuous for each $t \geqslant 0$ and one has $S_0 = \text{id.}$, $S_t \cdot S_s = S_{t+s}$ for each $t, s \geqslant 0$.

R2. Along the characteristics of (1) which are the curves $x = \varphi(t; t_0, x_0)$ satisfying the ordinary differential equation

$$\frac{dx}{dt} = c(x) \tag{3}$$

and the initial condition $x(t_0) = x_0$, the solution $u(t, x)$ of (1) satisfies the ordinary differential equation

$$\frac{dy}{dt} = f(\varphi(t; t_0, x_0), y) \tag{4}$$

with initial condition

$$y(0) = v(\varphi(0; t_0, x_0)); \tag{5}$$

the solution of (4), (5) is denoted by $\psi(t, \varphi(0; t_0, x_0), v(\varphi(0; t_0, x_0))$. This means that the solution u of (1) and (2) can be expressed by the formula

$$u(t, x) = \psi(t; \varphi(0; t, x), v(\varphi(0; t, x))). \tag{6}$$

For $\varphi(t; 0, x)$, write also $\varphi_x(t)$. It follows from **A2** that $\varphi_0(t) = 0$, $\varphi_x(t)$ is strictly increasing both in t and in x for $x > 0$, $\varphi_x^{-1}(1)$ is well defined continuous and decreasing for $0 < x \leqslant 1$.

R3. There exists a unique solution $w_0(x)$ of the stationary equation

$$c(x)\frac{du}{dx} = f(x, u), \qquad x \in \Delta \tag{7}$$

satisfying $w_0(0) = u_0$. For each $v \in C_+(\Delta)$ such that $v(0) > 0$ one has $S_t v(x) \to w_0(x)$ for $t \to \infty$ uniformly in x.

R4. Let $V_0 = \{v \in C_+(\Delta): v(0) = 0\}$, $V_w = \{v \in V_0: v(x) < w_0(x) \text{ for } x \in \Delta\}$. The sets V_0, V_w are invariant for S_t and for each $v \in V_0$ there exists a $T_0 \geq 0$ such that $S_t v \in V_w$ for $t > T_0$.

We add two simple observations that will be used in the paper. Since $\varphi_0(t) = 0$ for all $t \geq 0$, it follows from (6) that a solution $u(t, x)$ of (1), (2) is well defined on $D^0 = [0, \infty) \times \Delta^0$ as soon as $v \in C_+(\Delta^0)$, where $\Delta^0 = (0, 1]$. In other words, the semiflow S_t can be extended to $C_+(\Delta^0)$; we denote this extended semiflow by S_t^0.

Further, since $u(t, x)$ is the solution of a first order ordinary differential equation along each characteristic, it follows from (6) that the semiflow S_t preserves ordering, i.e.

$$S_t v_1 \leq S_t v_2 \quad \text{for} \quad t \geq 0 \tag{8}$$

as long as $v_1 \leq v_2$, where $v_1 \leq v_2$ means $v_1(x) \leq v_2(x)$ for all $x \in \Delta$. This is true also for S_t^0.

2. EXISTENCE AND DENSITY OF PERIODIC POINTS

Throughout this and the following section assume **A1–A5**.

THEOREM 1. (a) For each $\tau \geq 0$ there is a continuum of periodic points of S_t in V_w of basic period τ. (b) The set of all periodic points of S_t is dense in V_w.

The basic tool of the proof of this theorem consists in the representation of S_t by the shift semigroup in $C_+[0, \infty)$. This representation is induced by the map $\Phi: C_+(\Delta) \to C_+[0, \infty)$ defined by

$$\Phi(v)(t) = (S_t v)(1).$$

Using (6) we can express Φ also by

$$\Phi(v)(t) = \psi(t; \varphi(0; t, 1), v(\varphi(0; t, 1))). \tag{9}$$

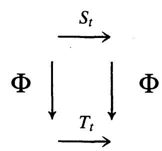
The family of shifts T_t, $t \geq 0$ defined by

$$(T_t g)(s) = g(t + s)$$

for $g \in C_+[0, \infty)$ is a semigroup and one has

$$T_t \Phi = \Phi S_t, \tag{10}$$

i.e. the diagram



commutes.

Indeed,

$$(T_t \Phi(v))(s) = \Phi(v)(s + t) = (S_{t+s} v)(1) = (S_s S_t v)(1)$$
$$= \Phi(S_t v)(s).$$

We can extend Φ to the map Φ_0 on $C_+(\Delta^0)$ by defining

$$\Phi_0(v)(t) = (S_t^0 v)(1).$$

Obviously, (10) holds with S_t, Φ replaced by S_t^0, Φ_0 respectively.

Let $g \in C_+[0, \infty)$. From (6) one immediately obtains $\Phi_0(v) = g$ if and only if

$$v(x) = \psi(-\varphi_x^{-1}(1); 1, g(\varphi_x^{-1}(1))) \quad \text{for} \quad x \in \Delta^0. \tag{11}$$

Using the argument leading to (8) one obtains from **A9** and (11)

$$v(x) \geq \psi(-\varphi_x^{-1}(1); 1, 0) = 0.$$

Thus we have

LEMMA 2.1. The map $\Phi_0 : C_+(\Delta^0) \to C_+[0, \infty)$ has an inverse which can be expressed by the formula (11).

Note that $\Phi_0^{-1}(g)$ is not necessarily in $C_+(\Delta)$ for an arbitrary $g \in C_+[0, \infty)$ since $\Phi_0^{-1}(g)$ may not have a limit for $x \to 0$.

As a consequence of **R3** one obtains immediately

LEMMA 2.2. Let $v \in C_+(\Delta)$ satisfy $v(0) > 0$. Then, $\Phi(v)(t) \to w_0(1)$ for $t \to 0$.

LEMMA 2.3. Let $g \in C_+[0, \infty)$ and let

$$g(t) \leq w_0(1) - \eta \tag{12}$$

for some $\eta > 0$ and each $t \geq 0$. Then $g \in \Phi(V_w)$.

Proof. Obviously, it suffices to prove

$$\lim_{x \to 0} \Phi_0^{-1}(g)(x) = 0 \tag{13}$$

since then $g = \Phi(v)$, where

$$v(x) = \begin{cases} \Phi_0^{-1}(g)(x) & \text{for } x \in \Delta^0 \\ 0 & \text{for } x = 0 \end{cases}$$

is from V_w. ■

To prove (13) we first introduce the following notation which will be used throughout the paper:

For any $c \geq 0$ we denote by $\mathbf{c}$ the constant function on Δ with value c and $h_c(t) = \Phi(\mathbf{c})(t)$.

Let now $\varepsilon > 0$. Since by lemma 2.2. $\lim_{t \to \infty} h_\varepsilon(t) = w_0(1)$, there exists a $t_0 > 0$ such that for $t > t_0$ one has

$$h_\varepsilon(t) > w_0(1) - \eta \geq g(t).$$

Let $x_0 = \varphi(0; t_0, 1)$. For $x < x_0$ one has $\varphi_x^{-1}(1) > t_0$, and, consequently, by (11).

$$\Phi_0^{-1}(g)(x) = \psi(-\varphi_x^{-1}(1); 1, g(\varphi_x^{-1}(1))) < \Phi_0^{-1}(h_\varepsilon(\varphi_x^{-1}(1))) = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary this proves (13).

Since for $g \in C_+[0, \infty)$ periodic with values in $[0, w_0(1))$ there is always an $\varepsilon > 0$ such that (12) holds we have

COROLLARY 2.1. The function $g \in C_+[0, \infty)$ with values in $[0, w_0(1))$ is periodic with prime period $\tau \geq 0$ if and only if $\Phi^{-1}(g)$ is a periodic point of S_t in V_w with basic period τ. In particular, all the solutions of the stationary equation (7) in V_w are obtained as pre-images of constant functions $< w_0(1)$ under Φ.

LEMMA 2.4. For each $0 < \varepsilon < \inf_{0 \leq x \leq 1} w_0(x)$ there exists a $\tau_\varepsilon > 0$ such that $h_\varepsilon(s + t) \leq h_\varepsilon(s)$ for each $s \geq 0$, $t \geq \tau_\varepsilon$.

Proof. By **R3**, there exists a $\tau_\varepsilon > 0$ such that $S_t\varepsilon > \varepsilon$ for all $t \geq \tau_\varepsilon$. Hence, for $t \geq \tau_\varepsilon$ we have

$$h_\varepsilon(s + t) = (T_t h_\varepsilon)(s) = \Phi(S_t\varepsilon)(s) \geq \Phi(\varepsilon)(s) = h_\varepsilon(s). \quad \blacksquare$$

Proof of theorem 2.1. Part (a) is an immediate consequence of corollary 2.1.

To prove (b) take any function v in V_w and choose an $\varepsilon > 0$. Denote $g = \Phi(v)$. Let $\delta > 0$ be such that $v(x) < \varepsilon$ for $x < \delta$, so

$$g(t) < h_\varepsilon(t) \quad \text{for} \quad t \geq t_1 = \varphi_\delta^{-1}(1). \tag{14}$$

Let $t_2 > \max\{t_1, \tau_\varepsilon\}$ be such that

$$h_\varepsilon(t) \geq \max_{0 \leq t \leq t_1} g(t), \tag{15}$$

for $t \geq t_2$, τ_ε being as in lemma 2.4.

From (14), (15) it follows that there exists a continuous function $\tilde{g} \in C_+[0, t_2]$ such that

$$\tilde{g}(t) = g(t) \quad \text{for} \quad 0 \leq t \leq t_1, \tag{16}$$

$$\tilde{g}(t) < h_\varepsilon(t) \quad \text{for} \quad t_1 \leq t \leq t_2, \tag{17}$$

$$\tilde{g}(t_2) = g(0).$$

Define $k \in C_+[0, \infty)$ by

$$k(t) = \tilde{g}(t - nt_2) \quad \text{for} \quad t \in [nt_2, (n + 1)t_2].$$

Then, k is periodic with period t_2 and, by lemma 2.3., there is a $z \in V_w$ such that $k = \Phi(z)$. From (14) and (15) we obtain

$$z(x) = v(x) \quad \text{for} \quad \delta \leq x \leq 1, \tag{18}$$

$$|z(x)| < \varepsilon \quad \text{for} \quad \varphi(0; t_2, 1) \leq x \leq \delta. \tag{19}$$

Let $n \geq 1$. For $nt_2 + t_1 \leq t \leq (n + 1)t_2$ we obtain by lemma 2.4 and (14)

$$k(t) = \tilde{g}(t - nt_2) \leq h_\varepsilon(t - nt_2) \leq h_\varepsilon(t); \tag{20}$$

for $nt_2 \leqslant t \leqslant (n+1)t_2$, (20) follows immediately from (15). Consequently, (19) extends to all $0 \leqslant x \leqslant \delta$ and we have

$$|z(x) - v(x)| \leqslant |z(x)| + |v(x)| \leqslant 2\varepsilon$$

for $0 \leqslant x \leqslant \delta$. This, together with (18), proves (b). ∎

3. EXISTENCE OF A DENSE TRAJECTORY AND INSTABILITY

Using the representation of S_t by T_t developed in Section 2 we now present an alternative proof of theorem 3 of [1]. That is, we prove

(a) every point $v \in V_w$ is unstable;

(b) there exists a $v \in V_w$ such that the orbit of v is dense in V_w.

Proof of (a). Let $v \in V_w$, $g = \Phi(v)$, $0 < a < w_0(1)$. Choose an $\varepsilon < 0$.

Let $\delta > 0$ be such that $v(x) < \varepsilon$ for $x \leqslant \delta$. Let $t_1 \geqslant \varphi_\delta^{-1}(1)$ be such that

$$h_\varepsilon(t) > a \tag{21}$$

for $t \geqslant t_1$.

We now construct a function $k \in C_+[0, \infty)$ as follows: We define

$$k(t) = g(t) \quad \text{for} \quad 0 \leqslant t \leqslant t_1$$

$$k(t_1 + j) = \begin{cases} a & \text{if} \quad g(t_1 + j) < \dfrac{a}{2} \\ 0 & \text{otherwise} \end{cases}, \; j = 1, 2, 3, \ldots$$

and we extend k to the interior of the intervals between the points $t_1 + j$ in such a way that k will be nonnegative continuous and its graph will lie below the graph of h_ε for $t_1 \leqslant t \leqslant t_1 + 1$ and below a for $t > t_1 + 1$. Then, we have

$$k(t) \leqslant h_\varepsilon(t) \quad \text{for} \quad t \geqslant t_1 \tag{22}$$

and

$$|k(t_1 + j) - g(t_1 + j)| \geqslant \frac{a}{2} \quad \text{for} \quad j = 1, 2, \ldots \tag{23}$$

By lemma 2.3., there exists a $z \in V_w$ such that $k = \Phi(v)$. Now, (23) can be rewritten as

$$|(S_{t_1+j}v)(1) - (S_{t_1+j}z)(1)| \geqslant \frac{a}{2}. \tag{24}$$

Also, we have from (18), (19)

$$z(x) = v(x) \quad \text{for} \quad \varphi(0; t_1, 1) \leqslant x < 1 \tag{25}$$

while

$$|z(x) - v(x)| \leqslant |z(x)| + |v(x)| \leqslant \varepsilon + \varepsilon = 2\varepsilon$$

for $0 \leqslant x < \varphi\,(0; t_1, 1)$. Since $\varepsilon > 0$ was arbitrary, (24)–(26) proves (a). ∎

Proof of (b). Let $\{v_n\}_{n=1}^{\infty}$ be a dense subset in V_w and let $\varepsilon_n \searrow 0$ for $n \to \infty$. Denote $g_n = \Phi(v_n)$. By lemma 3.2., there exists a sequence $\{t_n\}$ such that

$$t_1 = 0,\ t_{n+1} \geqslant t_n + 1. \tag{26}$$

$$h_{\varepsilon_j}(t_{n+1} - t_n) \geqslant \varepsilon_{j+1}(= h_{\varepsilon_{j+1}}(0)) \quad \text{for} \quad 0 \leqslant j \leqslant n \tag{27}$$

$$g_n(t) \leqslant h_{\varepsilon_j}(t + t_n - t_j) \quad \text{for all} \quad t \quad \text{and all} \quad 1 \leqslant j < n \tag{28}$$

$$g_n(t) \leqslant h_{\varepsilon_n}(t + t_n) \quad \text{for all} \quad t \geqslant 0. \tag{29}$$

First we note that a sequence of continuous functions $\tilde{g}_n \in C_+[0, t_{n+1} - t_n]$ can be found such that $\tilde{g}_n(t) = g_n(t)$ for $0 \leqslant t \leqslant t_{n+1} - t_n - 1$, $\tilde{g}_n(t_{n+1} - t_n) = g_{n+1}(0)$ and the inequalities (27)–(29) remain valid with g_n replaced by $\tilde{g}_n$ and t restricted to $t_{n+1} - t_n$ (we shall refer to them as ($\widetilde{27}$)–($\widetilde{29}$), respectively). We define

$$k(t) = \tilde{g}_n(t - t_n) \quad \text{for} \quad t_n \leqslant t \leqslant t_{n+1}.$$

Obviously, $k \in C_+[0, \infty)$ and $k(t) < w_0(1)$ for $0 \leqslant t < \infty$. Further, we have by ($\widetilde{29}$)

$$k(t) \leqslant h_{\varepsilon_n}(t) \quad \text{for } t_n \leqslant t \leqslant t_{n+1}$$

and, by (11),

$$\Phi_0^{-1}(k)(x) \leqslant \varepsilon_n \quad \text{for} \quad \varphi(0, t_{n+1}, 1) \leqslant x \leqslant \varphi(0, t_n, 1).$$

Consequently, $\lim_{x \to 0} \Phi_0^{-1}(k)(x) = 0$ and $k \in \Phi(z)$ for some $z \in V_w$.

Now, we have

$$(T_{t_n}k)(t) = g_n(t) \quad \text{for} \quad 0 \leqslant t \leqslant t_{n+1} - t_n - 1. \tag{30}$$

The inequalities ($\widetilde{27}$) and ($\widetilde{28}$) can be transcribed into

$$(T_{t_n}k)(t) \leqslant h_{\varepsilon_n}(t) \quad \text{for} \quad t \geqslant t_{n+1} - t_n - 1 \tag{31}$$

(($\widetilde{27}$) yields (31) for $t_{n+1} - t_n - 1 \leqslant t \leqslant t_{n+1} - t_n$ while ($\widetilde{28}$) yields (31) for $t \geqslant t_{n+1}$). From (30) and (31) we have

$$(S_{t_n}z)(x) = v_n(x) \quad \text{for} \quad (0; t_{n+1} - t_n - 1, 1) \leqslant x \leqslant 1) \tag{32}$$

$$(S_{t_n}z)(x) \leqslant \varepsilon_n \quad \text{for} \quad 0 \leqslant x \leqslant \varphi(0; t_{n+1} - t_n - 1, 1). \tag{33}$$

Also, from (27) we have

$$v_n(x) \leqslant \varepsilon_n \quad \text{for} \quad 0 \leqslant x \leqslant \varphi(0; t_{n+1} - t_n - 1, 1). \tag{34}$$

From (32)–(34) it follows

$$|(S_{t_n}z)(x) - v_n(x)| \leqslant 2\varepsilon_n \quad \text{for all} \quad x \in \Delta$$

which completes the proof. ■

Remark. It is easy to see that the function z giving the initial point of the dense trajectory in V_w can be constructed to be C^1 hence yielding a continuously differentiable solution of (1). This is true also for the functions z_n in part (a) and the periodic points of part (b) of theorem 2.1.

4. THE CASE $f(x, 0) \not\equiv 0$

Throughout this section we assume **A1–A4, A5′**. First we show that if **A5** is not satisfied there cannot be chaos in all of V_w.

PROPOSITION 4.1. Let $f(x_0, 0) > 0$ for some $x_0 \in \Delta$. Then V_w does not admit a dense trajectory.

LEMMA 4.1. For each $0 \leqslant t_1 \leqslant t_2$ one has

$$0 \leqslant S_{t_1}\mathbf{0} \leqslant S_{t_2}\mathbf{0}. \tag{35}$$

Proof. From (6) it follows

$$(S_t\mathbf{0})(x) \geqslant 0 \quad \text{for} \quad (t, x) \in D. \tag{36}$$

From (8), (36) and the semigroup property of S_t it follows

$$(S_{t_2}\mathbf{0})(x) = (S_{t_1}S_{t_2-t_1}\mathbf{0})(x) \geqslant (S_{t_1}\mathbf{0})(x). \quad \blacksquare$$

COROLLARY 4.1. Under the condition of proposition 4.1 there is a neighbourhood U of x_0 in Δ such that

$$(S_t\mathbf{0})(x) > 0 \quad \text{for each} \quad x \in U \quad \text{and} \quad t > 0. \tag{37}$$

Proof of proposition 4.1. Choose any $\tau > 0$ and denote $z = \frac{1}{2}S_\tau\mathbf{0}$. By corollary 4.1 we have $z \neq 0$. Assume $v \in V_w$ has a dense trajectory in V_w. Since $v \geqslant 0$, by (8) and lemma 4.1 we have

$$S_t v \geqslant 2z \quad \text{for all} \quad t \geqslant \tau. \tag{38}$$

Let $Z = \{w \in C_+(\Delta)\colon w(x) \leqslant z(x) \text{ for } x \in \Delta\}$. Since $z \neq 0$, $Z \neq \varnothing$. By (38), we have for all $\zeta \in Z$ and $t \geqslant \tau$

$$\sup_{x\in\Delta} |(S_t v)(x) - \zeta(x)| \geqslant \sup_{x\in\Delta} |2z(x) - z(x)| > 0.$$

Thus, in order that $S_{[0,\infty)}v$ be dense in V_w, $S_{[0,\tau]}v$ must be dense in Z. This, however, is easily seen to be impossible since $S_{[0,\tau]}v$ is compact in $C(\Delta)$ and does not contain all of Z. The compactness of $S_{[0,\tau]}v$ follows e.g. from the expression (6) from which one immediately concludes that the family of functions $\{S_t v\colon 0 \leqslant t \leqslant \tau\}$ is closed, uniformly bounded and equicontinuous.

Proposition 4.1 decides the question whether **A5** is necessary for the results on chaos to hold in their original form. Still, the results of [1] and Section 2 on chaos remain valid under **A5′** with V_w replaced by its invariant subset which we denote by W. To define W we need.

PROPOSITION 4.2. There exists a pointwise limit

$$w_1(x) = \lim_{t\to\infty} (S_t\mathbf{0})(x).$$

The function w_1 is a solution of the stationary equation (7) on Δ^0 satisfying

$$0 \leqslant w_1(x) \leqslant w_0(x) \quad \text{for} \quad x \in \Delta^0. \tag{39}$$

Proof. The existence of a pointwise limit w_1 of $S_t\mathbf{0}$ for $t \to \infty$ satisfying (39) is an immediate consequence of lemma 5. It remains to prove that w_1 is a solution of (7) on Δ^0. For the idea of this proof the author is indebted to J. Kačur.

Denote $u(t,x) = (S_t\mathbf{0})(x)$. For this proof we write (1), (7) in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(c(x)u) = q(x, u), \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(c(x)u) = q(x, u), \tag{41}$$

respectively, with $q(x, u) = f(x, u) + c'(x)u$.

Let $t \in [0, \infty)$, $x \in (0, 1]$. By integrating (40) we obtain

$$\int_1^x [u(t+1, \xi) - u(t, \xi)]\,\mathrm{d}\xi + \int_t^{t+1} [c(x)u(\sigma, x) - c(1)u(\sigma, 1)]\,\mathrm{d}\sigma$$

$$= \int_t^{t+1}\int_1^x q(\xi, u(\sigma, \xi))\mathrm{d}\xi\,\mathrm{d}\sigma. \tag{42}$$

Since $0 \leqslant u(t,x) \leqslant w_0(x)$ for all $(t,x) \in D$, by Lebesgue's convergence theorem we can pass to the limit for $t \to \infty$ in (42) to obtain

$$\int_t^{t+1} [c(x)w_1(x) - c(1)w_1(1)]\mathrm{d}t = \int_t^{t+1}\int_1^x q(\xi, w_1(\xi))\mathrm{d}\xi\,\mathrm{d}\sigma$$

and, consequently,

$$c(x)w_1(x) - c(1)w_1(1) = \int_1^x q(\xi, w_1(\xi))\mathrm{d}\xi. \tag{43}$$

From (43) it follows that w_1 is absolutely continuous on Δ^0. Thus, we can differentiate (43) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(c(x)w_1(x)) = q(x, w_1(x)) \tag{44}$$

which completes the proof. ■

Now, denote

$$W = \{v \in V_w\colon v(x) \geqslant w_1(x) \quad \text{for} \quad x \in \Delta\}. \tag{45}$$

One sees immediately that W is invariant. It is also attractive in V_0 but, unlike V_w, only in a 'pointwise' sense: the graphs of the upper and lower pointwise limits of $S_t v$ for $t \to \infty$ lie between the graphs of w_1 and w_0, for each $v \in V_0$. This follows immediately from **R4** and

$$\liminf_{t\to\infty} (S_t v)(x) \geqslant \liminf_{t\to\infty} (S_t\mathbf{0})(x) = w_1(x) \quad \text{for} \quad x \in \Delta.$$

The map Φ maps w_1 into the constant $w_1(1)$. If one replaces $C_+[0, \infty)$ by its subset of functions with values $\geqslant w_1(1)$, lemma 2.1 obviously holds true and one can repeat the arguments of Sections 2 and 3 almost literally to obtain.

THEOREM 4.1. The set W defined by (45) is invariant and pointwise attractive in V_0. Also, S_t is chaotic in W in the sense of theorem 3 of [1] and theorem 2.1.

It should be noted that the chaotic set W may very well be empty. Obviously, W is non-empty if and only if (7) has a non-negative solution w_2 on Δ satisfying $w_2(0) = 0$. Indeed, every non-negative solution of (7) on Δ^0 majorized by w_0 and different from w_0 vanishes at 0 (lemma 2.3.); if w_2 exists one has $w_1(x) \leqslant w_2(x) < w_0(x)$ for $x \in \Delta$. It follows that the question, whether W is empty or not, is decided by the local behaviour of f and c at $(0, 0)$.

For example, W is non-empty if $f(x, 0)$ vanishes in some right neighbourhood of 0. On the other hand, take $f(x, u) = x^2 + u^2$, $c(x) = x^2$ for $x \geqslant 0$, $u \geqslant 0$ small. All integral curves of the equation

$$x^2 \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 + x^2 \tag{46}$$

passing through points (x, u) with $x > 0$, $u \geqslant 0$ are given in parametric form by

$$x(s) = \mathrm{d} \exp[2.3^{-1/2} \arctan(3^{-1/2}(2s - 1))]$$

$$u(s) = sx(s) \quad (-\infty < s < \infty)$$

with $d > 0$. It can be readily seen that none of these curves approaches the point (0, 0), so (46) has no solution with $u(0) = 0$. Consequently, W is empty for any extensions of f, c satisfying **A1–A4**, **A5′**.

REFERENCES

1. LASOTA A., Stable and chaotic solutions of a first-order partial differential equation, *Nonlinear Analysis* **5**, 1181–1193 (1981).
2. AUSLANDER J. & YORKE J., Interval maps, factor of maps and chaos, *Tohoku math. J.* Ser. II, **32**, 177–188 (1980).