NOTES ON CHAOS IN THE CELL POPULATION PARTIAL DIFFERENTIAL EQUATION

PAVOL BRUNOVSKÝ
Institute of Applied Mathematics, Comenius University, 842 15 Bratislava, Czechoslovakia

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1. INTRODUCTION

In [1], the author investigates the differential equation
\[ \frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = f(x, u), \quad (t, x) \in D = [0, \infty) \times \Delta, \Delta = [0, 1]. \tag{1} \]

This equation describes the dynamics of growth of certain types of cell populations most prominent of which is the red blood cell population. It is shown in [1] that under certain natural conditions on \( c \) and \( f \) the equation (1) generates a semiflow \( S_t \) \( t \geq 0 \) on \( C_+ (\Delta) \) (the space of nonnegative continuous functions on \( \Delta \)) with an invariant set \( V_w \) on which the behaviour of the trajectories of \( S_t \) is chaotic in the sense of [2]. This means that \( S_t \) has a dense trajectory in \( V_w \) and each point of \( V_w \) is unstable (i.e. for each \( v \in V_w \) there exists a neighbourhood \( U \) of \( S(0, v) \) in \( C(D) \) and a sequence \( v_n \to v \) such that the trajectory of \( v_n \) leaves \( U \) for some \( t > 0 \)).

The main purpose of this paper is to show that \( S_t \) exhibits also other features of chaos in \( V_w \). Namely, there are periodic points of \( S_t \) of any basic period in \( V_w \) and the set of all periodic points of \( S_t \) is dense in \( V_w \) (Section 2).

For the proof a representation of \( S_t \) is employed which allows to prove the results on chaos of [1] in a more simple and transparent way. These proofs are presented in Section 3. Also, this technique helped to discover a small error in [1]. For the results on chaos of [1] to be true an additional (albeit also natural) assumption has to be added. We make this assumption in Sections 2 and 3. In Section 4 we discuss the modifications to be made if this additional assumption is dropped.

We keep all the notation of [1] in order to make it easier for the reader to relate the two papers. However, in order not to force the reader to look into [1] for every single concept or result we conclude this section by a list of assumptions and results of [1] used in the present paper.

Assumptions

A1. The functions \( c, f \) are continuously differentiable.
A2. \( c(0) = 0, c(x) > 0 \) for \( x > 0 \).
A3. There exists a \( u_0 \in (0, 1] \) such that \( f_u(0, u_0) < 0, f(0, u)(u - u_0) < 0 \) for \( u > 0, u \neq u_0 \).
A4. \( f(x, u) \leq k_1 u + k_2 \) for some \( k_1, k_2 \geq 0 \) and all \( x \in \Delta, u \geq 0 \).
A5. \( f(x, 0) = 0 \) for all \( x \in \Delta \).
Note that the assumptions A1–A5 coincide with assumptions (16)–(18) in [1] with one difference:
A5 is somewhat sharper than the assumption
A5'. \(f(x, 0) \geq 0\) for \(x \in \Delta\) and \(f(0, 0) = 0\)
made in [1]. Also note that A5 is satisfied if \(f(x, u) = (p(x, u) - c(x)) u\) as is the case if (1) models a reproductive, constantly differentiating cell population with proliferation rate \(p\).

**Results**

Under the assumptions A1–A4, A5' the following results are proven in [1]:

**R1.** For \(G \subseteq \mathbb{R}^n, n > 0\), denote by \(C_\ast(G), C_+^1(G)\) the set of all nonnegative continuous and nonnegative continuously differentiable functions on \(G\), respectively. For every \(v \in C_+(\Delta)\), (1) has a unique solution \(u\) in \(C_+^1(D)\) satisfying
\[
u(x, 0) = v(x) \quad \text{for} \quad x \in \Delta.
\]
A function \(u \in C_+(D)\) is called generalized solution of (1) if it is a limit (uniform on compact subset of \(D\)) of solutions of (1). For each \(v \in C_+(\Delta)\) there exists a unique generalized solution of (1) satisfying (2); henceforth we shall drop the adjective 'generalized'. The map \(S: [0, \infty) \times C_+(\Delta) \rightarrow C_+(\Delta)\) defined by \(S_t v(x) = u(t, x)\), where \(u\) satisfies (1), (2) is a continuous semiflow, i.e. \(S_t: C_+(\Delta) \rightarrow C_+(\Delta)\) is continuous for each \(t \geq 0\) and one has \(S_0 = \text{id.}, S_t \cdot S_s = S_{t+s}\) for each \(t, s \geq 0\).

**R2.** Along the characteristics of (1) which are the curves \(x = \varphi(t; t_0, x_0)\) satisfying the ordinary differential equation
\[
\frac{dx}{dt} = c(x)
\]
and the initial condition \(x(t_0) = x_0\), the solution \(u(t, x)\) of (1) satisfies the ordinary differential equation
\[
\frac{dy}{dt} = f(\varphi(t; t_0, x_0), y)
\]
with initial condition
\[
y(0) = v(\varphi(0; t_0, x_0));
\]
the solution of (4), (5) is denoted by \(\psi(t, \varphi(0; t_0, x_0), v(\varphi(0; t_0, x_0))\). This means that the solution \(u\) of (1) and (2) can be expressed by the formula
\[
u(t, x) = \psi(t; \varphi(0; t_0, x_0), v(\varphi(0; t_0, x_0))).
\]
For \(\varphi(t; 0, x)\), write also \(\varphi_x(t)\). It follows from A2 that \(\varphi_0(t) = 0\), \(\varphi_x(t)\) is strictly increasing both in \(t\) and in \(x\) for \(x > 0\), \(\varphi_x^{-1}(1)\) is well defined continuous and decreasing for \(0 < x \leq 1\).

**R3.** There exists a unique solution \(w_0(x)\) of the stationary equation
\[
\frac{c(x) du}{dx} = f(x, u), \quad x \in \Delta
\]
satisfying \(w_0(0) = u_0\). For each \(v \in C_+(\Delta)\) such that \(v(0) > 0\) one has \(S_t v(x) \rightarrow w_0(x)\) for \(t \rightarrow \infty\) uniformly in \(x\).
**R4.** Let $V_0 = \{ v \in C_+(\Delta) : v(0) = 0 \}$, $V_w = \{ v \in V_0 : v(x) < w_0(x) \text{ for } x \in \Delta \}$. The sets $V_0$, $V_w$ are invariant for $S_t$ and for each $v \in V_0$ there exists a $T_0 \geq 0$ such that $S_tv \in V_w$ for $t > T_0$.

We add two simple observations that will be used in the paper. Since $q_0(t) = 0$ for all $t \geq 0$, it follows from (6) that a solution $u(t, x)$ of (1), (2) is well defined on $D^0 = [0, \infty) \times \Delta^0$ as soon as $v \in C_+(\Delta^0)$, where $\Delta^0 = (0, 1]$. In other words, the semiflow $S_t$ can be extended to $C_+(\Delta^0)$; we denote this extended semiflow by $S_t^0$.

Further, since $u(t, x)$ is the solution of a first order ordinary differential equation along each characteristic, it follows from (6) that the semiflow $S_t$ preserves ordering, i.e.

$$S_tv_1 \leq S_tv_2 \quad \text{for } t \geq 0$$

as long as $v_1 \leq v_2$, where $v_1 \leq v_2$ means $v_1(x) \leq v_2(x)$ for all $x \in \Delta$. This is true also for $S_t^0$.

### 2. EXISTENCE AND DENSITY OF PERIODIC POINTS

Throughout this and the following section assume \textbf{A1–A5}.

**Theorem 1.** (a) For each $\tau \geq 0$ there is a continuum of periodic points of $S_t$ in $V_w$ of basic period $\tau$. (b) The set of all periodic points of $S_t$ is dense in $V_w$.

The basic tool of the proof of this theorem consists in the representation of $S_t$ by the shift semigroup in $C_+[0, \infty)$. This representation is induced by the map $\Phi : C_+(\Delta) \to C_+[0, \infty)$ defined by

$$\Phi(v)(t) = (S_tv)(1).$$

Using (6) we can express $\Phi$ also by

$$\Phi(v)(t) = \psi(t; \varphi(0; t, 1), v(\varphi(0; t, 1))).$$

The family of shifts $T_t$, $t \geq 0$ defined by

$$(T_tg)(s) = g(t + s)$$

for $g \in C_+[0, \infty)$ is a semigroup and one has

$$T_t\Phi = \Phi S_t,$$

i.e. the diagram

$$\begin{array}{c}
\Phi \\
S_t \\
\Phi
\end{array}$$

commutes.

Indeed,

$$(T_t\Phi(v))(s) = \Phi(v)(s + t) = (S_{t+s}v)(1) = (S_sS_tv)(1) = \Phi(S_tv)(s).$$

We can extend $\Phi$ to the map $\Phi_0$ on $C_+(\Delta^0)$ by defining

$$\Phi_0(v)(t) = (S_t^0v)(1).$$
Obviously, (10) holds with $S_t$, $\Phi$ replaced by $S'_t$, $\Phi_0$ respectively.

Let $g \in C_+[0, \infty)$. From (6) one immediately obtains $\Phi_0(v) = g$ if and only if

$$v(x) = \psi(-q_x^{-1}(1), 1, g(q_x^{-1}(1))) \quad \text{for} \quad x \in \Delta^0. \quad (11)$$

Using the argument leading to (8) one obtains from A9 and (11)

$$v(x) \geq \psi(-q_x^{-1}(1), 1, 0) = 0.$$

Thus we have

**Lemma 2.1.** The map $\Phi_0: C_+(\Delta^0) \to C_+[0, \infty)$ has an inverse which can be expressed by the formula (11).

Note that $\Phi_0^{-1}(g)$ is not necessarily in $C_+(\Delta)$ for an arbitrary $g \in C_+[0, \infty)$ since $\Phi_0^{-1}(g)$ may not have a limit for $x \to 0$.

As a consequence of R3 one obtains immediately

**Lemma 2.2.** Let $v \in C_+(\Delta)$ satisfy $v(0) > 0$. Then, $\Phi(v)(t) \to w_0(1)$ for $t \to 0$.

**Lemma 2.3.** Let $g \in C_+[0, \infty)$ and let

$$g(t) \leq w_0(1) - \eta \quad (12)$$

for some $\eta > 0$ and each $t \geq 0$. Then $g \in \Phi(V_w)$.

*Proof.* Obviously, it suffices to prove

$$\lim_{x \to 0} \Phi_0^{-1}(g)(x) = 0 \quad (13)$$

since then $g = \Phi(v)$, where

$$v(x) = \begin{cases} 
\Phi_0^{-1}(g)(x) & \text{for} \ x \in \Delta^0 \\
0 & \text{for} \ x = 0
\end{cases}$$

is from $V_w$. ■

To prove (13) we first introduce the following notation which will be used throughout the paper:

For any $c \geq 0$ we denote by $c$ the constant function on $\Delta$ with value $c$ and $h_c(t) = \Phi(c)(t)$.

Let now $\varepsilon > 0$. Since by lemma 2.2. $\lim_{t \to \infty} h_\varepsilon(t) = w_0(1)$, there exists a $t_0 > 0$ such that for $t > t_0$ one has

$$h_\varepsilon(t) > w_0(1) - \eta \geq g(t).$$

Let $x_0 = \varphi(0; t_0, 1)$. For $x < x_0$ one has $q_x^{-1}(1) > t_0$, and, consequently, by (11).

$$\Phi_0^{-1}(g)(x) = \psi(-q_x^{-1}(1), 1, g(q_x^{-1}(1))) < \Phi_0^{-1}(h_\varepsilon(q_x^{-1}(1))) = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary this proves (13).
Since for $g \in C_{+}[0, \infty)$ periodic with values in $[0, w_0(1))$ there is always an $\varepsilon > 0$ such that (12) holds we have

**COROLLARY 2.1.** The function $g \in C_{+}[0, \infty)$ with values in $[0, w_0(1))$ is periodic with prime period $\tau \geq 0$ if and only if $\Phi^{-1}(g)$ is a periodic point of $S_t$ in $V_w$ with basic period $\tau$. In particular, all the solutions of the stationary equation (7) in $V_w$ are obtained as pre-images of constant functions $<w_0(l)$ under $\Phi$.

**LEMMA 2.4.** For each $0 < \varepsilon < \inf_{0 \leq x < 1} w_0(x)$ there exists a $\tau_\varepsilon > 0$ such that $h_\varepsilon(s + t) \leq h_\varepsilon(s)$ for each $s \geq 0$, $t \geq \tau_\varepsilon$.

**Proof.** By R3, there exists a $\tau_\varepsilon > 0$ such that $S_t \varepsilon > \varepsilon$ for all $t \geq \tau_\varepsilon$. Hence, for $t \geq \tau_\varepsilon$ we have

$$h_\varepsilon(s + t) = (T_t h_\varepsilon)(s) = \Phi(S_t \varepsilon)(s) \geq \Phi(\varepsilon)(s) = h_\varepsilon(s).$$

**Proof of theorem 2.1.** Part (a) is an immediate consequence of corollary 2.1.

To prove (b) take any function $v$ in $V_w$ and choose an $\varepsilon > 0$. Denote $g = \Phi(v)$. Let $\delta > 0$ be such that $v(x) < \varepsilon$ for $x < \delta$, so

$$g(t) < h_\varepsilon(t) \quad \text{for} \quad t \geq t_1 = \varphi_0^{-1}(1).$$

Let $t_2 > \max\{t_1, \tau_\varepsilon\}$ be such that

$$h_\varepsilon(t) \geq \max_{0 \leq t \leq t_1} g(t),$$

for $t \geq t_2$, $\tau_\varepsilon$ being as in lemma 2.4.

From (14), (15) it follows that there exists a continuous function $\bar{g} \in C_{+}[0, t_2]$ such that

$$\bar{g}(t) = g(t) \quad \text{for} \quad 0 \leq t \leq t_1,$$

$$\bar{g}(t) < h_\varepsilon(t) \quad \text{for} \quad t_1 \leq t \leq t_2,$$

$$\bar{g}(t_2) = g(0).$$

Define $k \in C_{+}[0, \infty)$ by

$$k(t) = \bar{g}(t - nt_2) \quad \text{for} \quad t \in [nt_2, (n + 1)t_2].$$

Then, $k$ is periodic with period $t_2$ and, by lemma 2.3., there is a $z \in V_w$ such that $k = \Phi(z)$.

From (14) and (15) we obtain

$$z(x) = v(x) \quad \text{for} \quad \delta \leq x \leq 1,$$

$$|z(x)| < \varepsilon \quad \text{for} \quad \varphi(0; t_2, 1) \leq x \leq \delta.$$

Let $n \geq 1$. For $nt_2 + t_1 \leq t \leq (n + 1)t_2$ we obtain by lemma 2.4 and (14)

$$k(t) = \bar{g}(t - nt_2) \leq h_\varepsilon(t - nt_2) \leq h_\varepsilon(t).$$
for \( nt_2 \leq t \leq (n + 1)t_2 \), (20) follows immediately from (15). Consequently, (19) extends to all \( 0 \leq x \leq \delta \) and we have

\[
|z(x) - v(x)| \leq |z(x)| + |v(x)| \leq 2\varepsilon
\]

for \( 0 \leq x \leq \delta \). This, together with (18), proves (b). ■

3. EXISTENCE OF A DENSE TRAJECTORY AND INSTABILITY

Using the representation of \( S_t \) by \( T_t \) developed in Section 2 we now present an alternative proof of theorem 3 of [1]. That is, we prove

(a) every point \( v \in V_w \) is unstable;
(b) there exists a \( v \in V_w \) such that the orbit of \( v \) is dense in \( V_w \).

Proof of (a). Let \( v \in V_w \), \( g = \Phi(v) \), \( 0 < a < w_0(1) \). Choose an \( \epsilon < 0 \).
Let \( \delta > 0 \) be such that \( v(x) < \epsilon \) for \( x \leq \delta \). Let \( t_1 \geq q_\delta^{-1}(1) \) be such that

\[
h_\varepsilon(t) > a
\]

for \( t \geq t_1 \).

We now construct a function \( k \in C_+[0, \infty) \) as follows: We define

\[
k(t) = g(t) \quad \text{for} \quad 0 \leq t \leq t_1
\]

\[
k(t_1 + j) = \begin{cases} a & \text{if} \quad g(t_1 + j) < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, 2, 3, \ldots
\]

and we extend \( k \) to the interior of the intervals between the points \( t_1 + j \) in such a way that \( k \) will be nonnegative continuous and its graph will lie below the graph of \( h_\varepsilon \) for \( t_1 \leq t \leq t_1 + 1 \) and below \( a \) for \( t > t_1 + 1 \). Then, we have

\[
k(t) \leq h_\varepsilon(t) \quad \text{for} \quad t \geq t_1
\]

and

\[
|k(t_1 + j) - g(t_1 + j)| \geq \frac{a}{2} \quad \text{for} \quad j = 1, 2, \ldots
\]

By lemma 2.3., there exists a \( z \in V_w \) such that \( k = \Phi(v) \). Now, (23) can be rewritten as

\[
|(S_{t_1 + j}v)(1) - (S_{t_1 + j}z)(1)| \geq \frac{a}{2}.
\]

Also, we have from (18), (19)

\[
z(x) = v(x) \quad \text{for} \quad \varphi(0; t_1, 1) \leq x < 1
\]

while

\[
|z(x) - v(x)| \leq |z(x)| + |v(x)| \leq \epsilon + \varepsilon = 2\varepsilon
\]

for \( 0 \leq x < \varphi(0; t_1, 1) \). Since \( \epsilon > 0 \) was arbitrary, (24)–(26) proves (a). ■
Proof of (b). Let \( \{v_n\}_{n=1}^{\infty} \) be a dense subset in \( V_w \) and let \( \varepsilon_n \searrow 0 \) for \( n \to \infty \). Denote \( g_n = \Phi(v_n) \). By lemma 3.2., there exists a sequence \( \{t_n\} \) such that

\[
t_1 = 0, t_{n+1} \geq t_n + 1.
\]

(26)

\[
h_{\varepsilon_j}(t_{n+1} - t_n) \geq \varepsilon_j + 1 \text{ for } 0 \leq j \leq n
\]

(27)

\[
g_n(t) \leq h_{\varepsilon_j}(t + t_n - t_j) \text{ for all } t \text{ and all } 1 \leq j < n
\]

(28)

\[
g_n(t) \leq h_{\varepsilon_n}(t + t_n) \text{ for all } t \geq 0.
\]

(29)

First we note that a sequence of continuous functions \( \tilde{g}_n \in C_{+}[0, t_{n+1} - t_n] \) can be found such that \( \tilde{g}_n(t) = g_n(t) \) for \( 0 \leq t \leq t_{n+1} - t_n - 1 \), \( \tilde{g}_n(t_{n+1} - t_n) = g_{n+1}(0) \) and the inequalities (27)–(29) remain valid with \( g_n \) replaced by \( \tilde{g}_n \) and \( t \) restricted to \( t_{n+1} - t_n \) (we shall refer to them as (27)–(29), respectively). We define

\[
k(t) = \tilde{g}_n(t - t_n) \text{ for } t_n \leq t \leq t_{n+1}.
\]

Obviously, \( k \in C_{+}[0, \infty) \) and \( k(t) < w_0(1) \) for \( 0 \leq t < \infty \). Further, we have by (29)

\[
k(t) \leq h_{\varepsilon_n}(t) \text{ for } t_n \leq t \leq t_{n+1}
\]

and, by (11),

\[
\Phi_0^{-1}(k)(x) \leq \varepsilon_n \text{ for } \varphi(0, t_{n+1}, 1) \leq x \leq \varphi(0, t_n, 1).
\]

Consequently, \( \lim_{x \to 0} \Phi_0^{-1}(k)(x) = 0 \) and \( k \in \Phi(z) \) for some \( z \in V_w \).

Now, we have

\[
(T_{t_n}k)(t) = g_n(t) \text{ for } 0 \leq t \leq t_{n+1} - t_n - 1.
\]

(30)

The inequalities (27) and (28) can be transcribed into

\[
(T_{t_n}k)(t) \leq h_{\varepsilon_n}(t) \text{ for } t \geq t_{n+1} - t_n - 1
\]

(31)

((27) yields (31) for \( t_{n+1} - t_n - 1 \leq t \leq t_{n+1} - t_n \) while (28) yields (31) for \( t \geq t_{n+1} \)). From (30) and (31) we have

\[
(S_nz)(x) = v_n(x) \text{ for } (0; t_{n+1} - t_n - 1, 1) \leq x \leq 1
\]

(32)

\[
(S_nz)(x) \leq \varepsilon_n \text{ for } 0 \leq x \leq \varphi(0; t_{n+1} - t_n - 1, 1).
\]

(33)

Also, from (27) we have

\[
v_n(x) \leq \varepsilon_n \text{ for } 0 \leq x \leq \varphi(0; t_{n+1} - t_n - 1, 1).
\]

(34)

From (32)–(34) it follows

\[
|(S_nz)(x) - v_n(x)| \leq 2\varepsilon_n \text{ for all } x \in \Delta
\]

which completes the proof. 

Remark. It is easy to see that the function \( z \) giving the initial point of the dense trajectory in \( V_w \) can be constructed to be \( C^1 \) hence yielding a continuously differentiable solution of (1). This is true also for the functions \( z_n \) in part (a) and the periodic points of part (b) of theorem 2.1.
4. THE CASE $f(x, 0) \neq 0$

Throughout this section we assume $A1$–$A4$, $A5'$. First we show that if $A5$ is not satisfied there cannot be chaos in all of $V_w$.

**Proposition 4.1.** Let $f(x_0, 0) > 0$ for some $x_0 \in \Delta$. Then $V_w$ does not admit a dense trajectory.

**Lemma 4.1.** For each $0 \leq t_1 < t_2$ one has

$$0 \leq S_{t_1}0 \leq S_{t_2}0.$$  \hspace{1cm} (35)

**Proof.** From (6) it follows

$$(S_t0)(x) \geq 0 \quad \text{for} \quad (t, x) \in D. \hspace{1cm} (36)$$

From (8), (36) and the semigroup property of $S$, it follows

$$(S_{t_2}0)(x) = (S_{t_1}S_{t_2-t_1}0)(x) \geq (S_{t_1}0)(x). \hspace{1cm} \blacksquare$$

**Corollary 4.1.** Under the condition of Proposition 4.1 there is a neighbourhood $U$ of $x_0$ in $\Delta$ such that

$$(S_t0)(x) > 0 \quad \text{for each} \quad x \in U \quad \text{and} \quad t > 0. \hspace{1cm} (37)$$

**Proof of Proposition 4.1.** Choose any $\tau > 0$ and denote $z = \frac{1}{\tau}S_{\tau}0$. By Corollary 4.1 we have $z \neq 0$. Assume $\nu \in V_w$ has a dense trajectory in $V_w$. Since $\nu \geq 0$, by (8) and Lemma 4.1 we have

$$S_{\tau}\nu \geq 2z \quad \text{for all} \quad t \geq \tau. \hspace{1cm} (38)$$

Let $Z = \{w \in C_+(\Delta) : w(x) \leq z(x) \text{ for } x \in \Delta\}$. Since $z \neq 0$, $Z \neq \emptyset$. By (38), we have for all $\zeta \in Z$ and $t \geq \tau$

$$\sup_{x \in \Delta} |(S_{\tau}\nu)(x) - \zeta(x)| \geq \sup_{x \in \Delta} |2z(x) - z(x)| > 0.$$  \hspace{1cm} (39)

Thus, in order that $S_{[0, \tau]}\nu$ be dense in $V_w$, $S_{[0, \tau]}\nu$ must be dense in $Z$. This, however, is easily seen to be impossible since $S_{[0, \tau]}\nu$ is compact in $C(\Delta)$ and does not contain all of $Z$. The compactness of $S_{[0, \tau]}\nu$ follows e.g. from the expression (6) from which one immediately concludes that the family of functions $\{S_{\tau}\nu : 0 \leq t \leq \tau\}$ is closed, uniformly bounded and equicontinuous.

Proposition 4.1 decides the question whether $A5$ is necessary for the results on chaos to hold in their original form. Still, the results of [1] and Section 2 on chaos remain valid under $A5'$ with $V_w$ replaced by its invariant subset which we denote by $W$. To define $W$ we need.

**Proposition 4.2.** There exists a pointwise limit

$$w_1(x) = \lim_{t \to \infty} (S_t0)(x).$$

The function $w_1$ is a solution of the stationary equation (7) on $\Delta^0$ satisfying

$$0 \leq w_1(x) \leq w_0(x) \quad \text{for} \quad x \in \Delta^0. \hspace{1cm} (39)$$

**Proof.** The existence of a pointwise limit $w_1$ of $S_t0$ for $t \to \infty$ satisfying (39) is an immediate
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consequence of lemma 5. It remains to prove that \( w_1 \) is a solution of (7) on \( \Delta^0 \). For the idea of this proof the author is indebted to J. Kačur.

Denote \( u(t, x) = (S_t \theta)(x) \). For this proof we write (1), (7) in the form

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (c(x)u) = q(x, u),
\]

(40)

\[
\frac{d}{dx} (c(x)u) = q(x, u),
\]

(41)

respectively, with \( q(x, u) = f(x, u) + c'(x)u \).

Let \( t \in [0, \infty), x \in (0, 1] \). By integrating (40) we obtain

\[
\int_1^x [u(t + 1, \xi) - u(t, \xi)] d\xi + \int_t^{t+1} [c(x)u(\sigma, x) - c(1)u(\sigma, 1)] d\sigma
\]

\[
= \int_t^{t+1} \int_1^x q(\xi, u(\sigma, \xi)) d\xi d\sigma.
\]

(42)

Since \( 0 \leq u(t, x) \leq w_0(x) \) for all \( (t, x) \in D \), by Lebesgue's convergence theorem we can pass to the limit for \( t \to \infty \) in (42) to obtain

\[
\int_t^{t+1} [c(x)w_1(x) - c(1)w_1(1)] d\sigma = \int_t^{t+1} \int_1^x q(\xi, w_1(\xi)) d\xi d\sigma
\]

and, consequently,

\[
c(x)w_1(x) - c(1)w_1(1) = \int_1^x q(\xi, w_1(\xi)) d\xi.
\]

(43)

From (43) it follows that \( w_1 \) is absolutely continuous on \( \Delta^0 \). Thus, we can differentiate (43) to obtain

\[
\frac{d}{dx} (c(x)w_1(x)) = q(x, w_1(x))
\]

(44)

which completes the proof. \( \blacksquare \)

Now, denote

\[
W = \{ \nu \in V_w : \nu(x) \geq w_1(x) \text{ for } x \in \Delta \}.
\]

(45)

One sees immediately that \( W \) is invariant. It is also attractive in \( V_0 \) but, unlike \( V_w \), only in a 'pointwise' sense: the graphs of the upper and lower pointwise limits of \( S_t \nu \) for \( t \to \infty \) lie between the graphs of \( w_1 \) and \( w_0 \), for each \( \nu \in V_0 \). This follows immediately from \( \textbf{R4} \) and

\[
\liminf_{t \to \infty} (S_t \nu)(x) \geq \liminf_{t \to \infty} (S_t \theta)(x) = w_1(x) \text{ for } x \in \Delta.
\]

The map \( \Phi \) maps \( w_1 \) into the constant \( w_1(1) \). If one replaces \( C_+ [0, \infty) \) by its subset of functions with values \( \geq w_1(1) \), lemma 2.1 obviously holds true and one can repeat the arguments of Sections 2 and 3 almost literally to obtain.

**Theorem 4.1.** The set \( W \) defined by (45) is invariant and pointwise attractive in \( V_0 \). Also, \( S_t \) is chaotic in \( W \) in the sense of theorem 3 of [1] and theorem 2.1.
It should be noted that the chaotic set \( W \) may very well be empty. Obviously, \( W \) is non-empty if and only if (7) has a non-negative solution \( w_2 \) on \( \Delta \) satisfying \( w_2(0) = 0 \). Indeed, every non-negative solution of (7) on \( \Delta^0 \) majorized by \( w_0 \) and different from \( w_0 \) vanishes at 0 (lemma 2.3.); if \( w_2 \) exists one has \( w_1(x) \leq w_2(x) < w_0(x) \) for \( x \in \Delta \). It follows that the question, whether \( W \) is empty or not, is decided by the local behaviour of \( f \) and \( c \) at \((0, 0)\).

For example, \( W \) is non-empty if \( f(x, 0) \) vanishes in some right neighbourhood of 0. On the other hand, take \( f(x, u) = x^2 + u^2, c(x) = x^2 \) for \( x \geq 0, u \geq 0 \) small. All integral curves of the equation

\[
x^2 \frac{du}{dx} = u^2 + x^2
\]

(46)

passing through points \((x, u)\) with \( x > 0, u \geq 0 \) are given in parametric form by

\[
x(s) = d \exp\left[2.3^{-1/2} \arctan\left(3^{-1/2}(2s - 1)\right)\right] \\
u(s) = sx(s) \quad (-\infty < s < \infty)
\]

with \( d > 0 \). It can be readily seen that none of these curves approaches the point \((0, 0)\), so (46) has no solution with \( u(0) = 0 \). Consequently, \( W \) is empty for any extensions of \( f, c \) satisfying \( A1-A4, A5' \).

REFERENCES